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Sequential Decision Making in Production Models

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Single equation estimates of production models usually are justified by the assumption that production inputs are chosen as part of a one-period decision problem. Yet, most production decisions in agriculture are made sequentially. In this paper the farmer's optimal input choices are modeled as optimal controls in a stochastic control problem. A two-period Cobb-Douglas example is used to show that sequential solutions to production problems may yield models which require either single equation or simultaneous equation estimators. Functional separability, stochastic specification, and behavior under uncertainty are discussed in the context of dynamic production models.

Key words: dynamic models, production function estimation, stochastic control.

The literature abounds with agricultural production studies based on single-equation estimates of econometric production function models. The single-equation approach has been shown to be valid by Hoch (1958, 1962) and Mundlak and Hoch under the assumption that input decisions are based on "anticipated" output, and by Zellner, Kmenta, and Dreze under the assumption that input decisions are based on the maximization of the mathematical expectation of profit. These models are all based on the strong assumption that production inputs are chosen as part of a one-period decision problem. This view is inconsistent with most actual production decisions. Especially in agriculture, both short-run and long-run production decisions are based on a multiperiod, dynamic optimization problem because inputs are not all chosen or utilized simultaneously. Therefore, the farmer's optimal input choices may be regarded as optimal controls in a stochastic control problem.

The aim of this paper is to formulate a short-run, single-product production model within a stochastic control framework and to explore its implications for specifying and estimating econometric production models. The analysis demonstrates that sequential solutions generally result in input demand equations which differ from those of one-period solutions. In addition, sequential solutions may produce models which require either single-equation or simultaneous-equation estimation methods, depending on the assumptions about information used and data availability. In particular, it is shown that simultaneous equation estimators are not required if (*a*) decision makers do not feed back information about early stages' production to later input decisions, or (*b*) output and input data are available for each stage. Since neither of these conditions often occurs in agricultural production, the findings suggest that even though farmers choose inputs so as to maximize expected returns, single-equation estimates of agricultural production functions are generally subject to simultaneous equation bias.

If a production function is estimated without accounting for the sequential structure of the farmer's decision problem, the estimated marginal products are likely to be biased. Hoch showed that as a result returns to scale are biased towards 1 with the Cobb-Douglas production function. Also, Yotopoulos, Lau, and Lin found parameter estimates of a Cobb-Douglas production function to be very different when obtained directly from estimates of a profit function than from direct estimation of the production function. They suggest the dif-

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ferences may be due to simultaneity of inputs and outputs, biasing the direct production function parameter estimates. The results of this paper show that simultaneity between inputs and outputs is due to the farmer's sequential decisions. As a general principle, parameter estimates with desirable properties can be obtained only by specifying and estimating empirical production models consistent with the sequential structure of the production process and managers' solutions of input choice problems.

The paper's first section briefly describes the single-stage, two-input, Cobb-Douglas production models proposed by Marshak and Andrews and by Zellner, Kmenta, and Dreze. The second section extends the Cobb-Douglas example to two stages, defines various sequential solutions to the input choice problem, and discusses appropriate estimation methods. The third section shows a close connection between functional separability across production stages, production function error specification, and the implied relationship between inputs and production uncertainty.

Single-Stage Cobb-Douglas Models

The Cobb-Douglas production function provides an interesting special case because of its widespread use in theoretical and empirical research. It is also useful for illustrating issues of specification and estimation that arise in sequential models described later. The simple crop production model is defined as follows: the ith farmer chooses the amount of inputs L_{i1} and L_{i2} to use on a predetermined acreage, A_i. Output, Q_{i2}, is sold after harvest at price p_i, and input prices are w_{i1} and w_{i2}.

The Marshak-Andrews (MA) model is based on profit maximization in a single period. The theoretical model consists of the first-order conditions for profit maximization and the deterministic Cobb-Douglas production function, both in logarithmic form. The econometric model is obtained by adding random error terms to these equations. For our crop production example, the structural equations with parameters α_j, $j = 1, 2, 3$, are

$$\text{(1)} \quad \log Q_{i2} = \log \alpha_0 + \alpha_1 \log L_{i1} + \alpha_2 \log L_{i2} + \alpha_3 \log A_i + \epsilon_i$$

$$\log L_{it} = \log \alpha_t - \log \frac{w_{it}}{p_i} + \log Q_{i2} + u_{it}, \; t = 1, 2.$$

Here ϵ_i and u_{it} are independent random variables with zero means. The ϵ_i are random disturbances in production from weather, pests, etc. The u_{it} allow for nonsystematic errors in maximization by farmers. Adding ϵ_i to the production function transforms the deterministic theoretical model into a system of simultaneous equations with endogenous variables Q_{i2}, L_{i1}, and L_{i2}. Therefore, with sample data from $i = 1, \ldots, N$ farms, simultaneous equation estimators are needed to obtain consistent estimates of the parameters. In the MA model prices are known, nonstochastic variables.

The Zellner-Kmenta-Dreze (ZKD) model is also a one-period model. In contrast to the MA model, it assumes that firms recognize production to be stochastic. Firms, therefore, choose inputs to maximize the mathematical expectation of profit. Prices are viewed as independent random variables. Writing the stochastic production function as

$$Q_{i2} = \alpha_0 L_{i1}^{\alpha_1} L_{i2}^{\alpha_2} A_i^{\alpha_3} e^{\epsilon_i}, \; \epsilon_i \sim N(0,\sigma^2),$$

and letting a bar over a variable denote its expectation, the decision problem is

$$\max_{L_{i1},L_{i2}} E[\pi_i] = \bar{p}_i \bar{Q}_{i2} - \bar{w}_{i1} L_{i1} - \bar{w}_{i2} L_{i2}, \\ = p_i \alpha_0 L_{i1}^{\alpha_1} L_{i2}^{\alpha_2} A_i^{\alpha_3} e^{\sigma^2/2} - \bar{w}_{i1} L_{i1} - \bar{w}_{i2} L_{i2}.$$

The structural econometric model, in log form, consists of the first-order conditions and the production function:

$$\text{(2)} \quad \log Q_{i2} = \log \alpha_0 + \alpha_1 \log L_{i1} + \alpha_2 \log L_{i2} + \alpha_3 \log A_i + \epsilon_i,$$

$$\log L_{it} = \log \alpha_t - \log \frac{\bar{w}_{it}}{\bar{p}_i} + \log \bar{Q}_{i2} + u_{it}, \; t = 1,2,$$

where u_{it} is an independent random error added to the first-order conditions to represent nonsystematic maximization errors. For econometric estimation, the important difference between models (1) and (2) is that inputs depend on actual output, Q_{i2}, in the former and expected output, $\bar{Q}_{i2}$, in the latter. Since $\bar{Q}_{i2}$ is nonstochastic, inputs are independent of output as long as $E(u_{it}\epsilon_i) = 0$. The production function can be estimated with single-equation methods.

Two-Stage Cobb-Douglas Models and Sequential Decision Making

The two-stage Cobb-Douglas production function is defined as follows: before the first production stage, labor input L_{i1} is chosen, and during stage 1 the crop is planted and grown. Random events such as weather change occur during plant growth. The first-stage output, Q_{i1}, is the mature, unharvested crop:

$$(3) \qquad Q_{i1} = \beta_0 L_{i1}^{\beta_1} A_i^{\beta_2} e^{\epsilon_{i1}}$$

where ϵ_{i1} is a $N(0, \sigma_1^2)$ random error. In the second production stage, the crop Q_{i1} is harvested using labor input L_{i2}. Adverse weather may affect the harvest, so we write the second-stage production function as

$$(4) \qquad Q_{i2} = \gamma_0 Q_{i1}^{\gamma_1} L_{i2}^{\gamma_2} e^{\epsilon_{i2}},$$

where ϵ_{i2} is a $N(0,\sigma_2^2)$ random error term. Equations (3) and (4) comprise a system of recursive equations. Combining the two equations, we have

$$(5) \qquad Q_{i2} = \gamma_0 \beta_0^{\gamma_1} L_{i1}^{\beta_1\gamma_1} A_i^{\beta_2\gamma_1} L_{i2}^{\gamma_2}\ e^{(\gamma_1\epsilon_{i1}+\epsilon_{i2})}.$$

Note that final harvested output is a function of both ϵ_{i1} and ϵ_{i2}.

In order to estimate this model, we must carefully specify the production disturbances. The simplest assumption is that the ϵ_{it} are independently distributed across both firms and time so that

$$(6) \qquad E(\epsilon_{it}^2) = \sigma_t^2, \quad E(\epsilon_{it}, \epsilon_{i't'}) = 0,\ i \neq i',\ t \neq t'.$$

However, in agricultural production, the ϵ_{it} are likely to be correlated across time. Therefore, we also consider estimation under the assumptions

$$(7) \qquad \begin{aligned} \epsilon_{i2} &= \rho\epsilon_{i1} + v_i,\ |\rho| < 1,\\ E(v_i\epsilon_{it}) &= 0,\ t = 1, 2,\\ E(v_i\ v_{i'}) &= 0,\ i \neq i',\\ v_i &\sim N(0, \sigma^2). \end{aligned}$$

Often only observations of the final product Q_{i2} are possible or available. For example, only the quantity harvested may be known, but not the part of output attributed to each farming operation. With manufacturing or processing operations, in contrast, it may be possible to disaggregate production into separate stages each of which is observable. Because of this "observability" problem, we consider estimators based on the final product, Q_{i2} as well as Q_{i1} and Q_{i2}.

To illustrate the essential differences between the one-period and sequential solutions we assume that farmers choose inputs to maximize expected returns and that prices are independently distributed. The maximum problem is

$$(8) \qquad \max_{L_{i1},L_{i2}} E[\pi_i] \text{ subject to (3), (4),}$$
$$E[\pi_i] = \bar{p}_i\bar{Q}_{i2} - \bar{w}_{i1}L_{i1} - \bar{w}_{i2}L_{i2}.$$

Sequential solutions to decision problems such as (8) may be differentiated from one-period solutions by the information utilized by the decision maker. The information pertains to three features of sequential solutions.

(*a*) Sequential dependence of decisions: decisions made earlier may affect those made later, so that the optimal choice of L_{i2} may be a function $L_{i2}^o(L_{i1})$. If the farmer takes this into account, then his optimal input choice in period 1 may depend on how it affects optimal inputs in period 2.

(*b*) Informaton feedback: information that becomes available during earlier stages may be utilized in subsequent decisions. The optimal choice of L_{i2} will depend on expected output $\bar{Q}_{i1}$ if there is no information feedback about first-period production. If there is feedback, L_{i2} depends on Q_{i1}. Thus, the farmer may use knowledge of the actual output, Q_{i1}, rather than original estimates of production, $\bar{Q}_{i1}$, to determine the optimal amount of labor to hire in the second stage.

(*c*) Anticipated revision: decisions made earlier may be revised later as new information becomes available. If the decision maker knows that information about Q_{i1} will become available in period 2, his choices in period 1 will depend on the conditional distribution $g_2(\epsilon_{i2}|Q_{i1})$ rather than the unconditional distribution $g_2(\epsilon_{i2})$. Thus, the farmer's planting decisions may be different if he knows he can revise harvest plans at harvest time rather than having to rely on initial expectations.

We consider four alternative sequential solutions to the input choice problem defined in (8) which utilize different information sets. We assume that, at the beginning of stage 1 when L_{i1} is chosen, each farmer knows wage rate w_{i1} and the probability distribution functions of ϵ_{i1}, ϵ_{i2}, p_i, and w_{i2}. This minimal information set is defined as I^o in table 1. In addition, the farmer may know that the optimal input in stage 2 is a function of the input chosen in stage 1. Augmenting I^o with this information, we have I^a in table 1, which incorporates the sequential dependence property (a). When choosing L_{i1}, the farmer may also know that he will be able to acquire information about Q_{i1} before choosing L_{i2}. This additional information is represented by replacing the unconditional distribution $g_2(\epsilon_{i2})$ with the conditional distribution $g_2(\epsilon_{i2}|Q_{i1})$. With this change, we obtain I^{ac} in table 1. In period 2, the farmer's choice of L_{i2} may be based only on the minimal information set I^o, or it may also depend on the additional information. When I^0 is updated with information about Q_{i1} and w_{i2}, we obtain I^b, table 1. Table 2 summarizes the information sets used in the four alternative solutions.

The Open Loop (OL) Control Solution

The OL solution uses the sequential dependence property (a) but not the information feedback property (b) or the anticipated revision property (c) of sequential solutions. The choice of L_{i1} is made with the knowledge that it may affect the optimal L_{i2}. Thus, it is based on I^a, but the information set is not updated in stage 2 so the choice of L_{i2} is conditioned on I^o. The OL solution implies that the farmer does not use what he learns during the growing season to choose the optimal harvest labor input. To calculate the OL solution, we proceed recursively from stage 2 to stage 1. The optimal L_{i2}, taking L_{i1} as given, is found by maximizing $E[\pi_i|I^o]$. Note that

$$E[\pi_i|I^o] = \bar{p}_i\gamma_0 L_{i2}{}^{\gamma_2}E(Q_{i1}{}^{\gamma_1}e^{\epsilon_{i2}}) - w_{i1}L_{i1} - \bar{w}_{i2}L_{i2}.$$

Since

$$\begin{aligned} E(Q_{i1}{}^{\gamma_1}) &= (\beta_0 L_{i1}{}^{\beta_1}A_i{}^{\beta_2})^{\gamma_1}E(e^{\gamma_1\epsilon_{i1}}), \\ &= (\beta_0 L_{i1}{}^{\beta_1}A_i{}^{\beta_2})^{\gamma_1}e^{\gamma_1{}^2\sigma_1{}^2/2}, \\ &= \bar{Q}_{i1}{}^{\gamma_1}e^{(\gamma_1{}^2\sigma_1{}^2-\gamma_1\sigma_1{}^2)/2}, \text{ and} \end{aligned}$$

$$E(e^{\gamma_1\epsilon_{i1}+\epsilon_{i2}}) = e^{(\sigma_2{}^2+\gamma_1{}^2\sigma_1{}^2+\gamma_1\rho\sigma_1{}^2)/2},$$

we obtain

$$E[\pi_i|I^o] = \bar{p}_i\gamma_0\bar{Q}_{i1}{}^{\gamma_1}L_{i2}{}^{\gamma_2}e^{\omega} - w_{i1}L_{i1} - \bar{w}_{i2}L_{i2},$$

where $\omega = [\sigma_2{}^2 + \sigma_1{}^2(\gamma_1{}^2 + \rho\gamma_1 - \gamma_1)]/2$. Note that the expectation is taken over ϵ_{i1}, ϵ_{i2}, p_i, and w_{i2} because the only information assumed to be used in choosing L_{i2} is the farmer's knowledge of their distributions. The solution is

$$(9) \quad \log L_{i2}{}^o = \frac{1}{1-\gamma_2}[\omega + \log\gamma_0\gamma_2] - \frac{1}{1-\gamma_2}\log\frac{\bar{w}_{i2}}{\bar{p}_i} + \frac{\gamma_1}{1-\gamma_2}\log\bar{Q}_{i1}.$$

The OL solution for L_{i1} depends on the assumption that the decision maker knows $L_{i2}{}^o$ is a function of L_{i1} through $\bar{Q}_{i1}$. The optimal L_{i1} is obtained by maximizing $E[\pi_i|I^a]$. Because

$$E[\pi_i|I^a] = \bar{p}_i\gamma_o\bar{Q}_{i1}{}^{\gamma_1}E(L_{i2}{}^o)^{\gamma_2}e^{\omega} - w_{i1}L_{i1} - \bar{w}_{i2}E(L_{i2}{}^o),$$

the solution is a complicated nonlinear function given generally as

$$(10) \quad L_{i1}{}^o = L_{i1}{}^o(\bar{p}_i, w_{i1}, \bar{w}_{i2}, \sigma_1, \sigma_2, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2).$$

Because $L_{i1}{}^o$ and $L_{i2}{}^o$ are independent of the production function disturbances, ϵ_{i1} and ϵ_{i2}, the OL solution implies that a single-equation estimator of the production function's parameters is efficient and free of simultaneous equation bias. This result, also obtained with ZKD model, depends on the assumption that input choices are based only on information available before production begins. However, the functional form of the input equations derived from the OL solution differs from that of the ZKD model.

The Sequential Updating (SU) Solution

The SU solution exhibits only the information feedback property (b) of the sequential solution (Zellner). In each production stage, the information set is updated, but the effect of the current decision on future stages is ignored. Therefore, in period two, the labor input is chosen to maximize

$$(11) \quad E[\pi_i|I^b] = \bar{p}_i\gamma_0 Q_{i1}{}^{\gamma_1}L_{i2}{}^{\gamma_2}e^{\sigma_2{}^2/2} - w_{i1}L_{i1} - w_{i2}L_{i2}.$$

Note that the expectation in (11) is taken only with respect to ϵ_{i2} since, in stage two, Q_{i1} and w_{i2} are known. This information is used to choose the optimal L_{i2},

$$(12) \quad \log L_{i2}{}^o = \frac{1}{1-\gamma_2}\left[\frac{\sigma_2{}^2}{2} + \log\gamma_0\gamma_2\right] - \frac{1}{1-\gamma_2}\log\frac{w_{i2}}{\bar{p}_i} + \frac{\gamma_1}{1-\gamma_2}\log Q_{i1}.$$

To find the optimal L_{i1}, we take expectations of both ϵ_{i1} and ϵ_{i2} and maximize $E[\pi_i|I^o]$, ignoring that $L_{i2}{}^o$ is a function of L_{i1}. Solving the maximum problem gives

$$(13) \quad \log L_{i1}{}^o = \delta_0 + \delta_1\log A_i + \delta_2\log\frac{w_{i1}}{\bar{p}_i} + \delta_3\log E[L_{i2}{}^o|I^o],$$

Table 1. Information Sets Used in Sequential Solutions in the Cobb-Douglas Production Model

Information Set	Input and Price Information					Production Information			
	$g_p(p_i)$	w_{i1}	$g_w(w_{i2})$	w_{i2}	$L_{i2}^o(L_{i1})$	$g_1(\epsilon_{i1})$	Q_{i1}	$g_2(\epsilon_{i2})$	$g_2(\epsilon_{i2}\|Q_{i1})$
I^o	X	X	X			X		X	
I^a	X	X	X		X	X		X	
I^{ac}	X	X	X		X	X			X
I^b	X	X		X			X	X	

Note: Definitions are $g_p(p_i)$ is probability distribution of product price; w_{i1}, period 1 wage rate; $g_w(w_{i2})$, probability distribution of period 2 wage rate; w_{i2}, period 2 wage rate; $L_{i2}^o(L_{i1})$, optimal labor input in period 2; $g_1(\epsilon_{i1})$, probability distribution of period 1 production disturbance; Q_{i1}, actual production in period 1; $g_2(\epsilon_{i2})$, probability distribution of period 2 production disturbance; and $g_2(\epsilon_{i2}|Q_{i1})$, probability distribution of period 2 production disturbance conditional on Q_{i1}.

where δ_0, δ_1, δ_2, and δ_3 are functions of the production function parameters, σ_1 and σ_2. When information acquired in stage 1 about Q_{i1} is used to update the information for choosing L_{i2}, L_{i2}^o becomes a function of ϵ_{i1} through Q_{i1} and is correlated with Q_{i2}. However, L_{i1}^o is based on information set I^o and is not a function of ϵ_{i1} or ϵ_{i2}. Therefore, when decisions are sequentially updated, we obtain a simultaneous model consisting of equations (3), (4), (12), and (13) with properties similar to the MA model.

The Open Loop with Feedback (OLF) Solution

The OLF solution combines the sequential dependence and information feedback properties (*a*) and (*b*) of the OL and SU solutions. Therefore, it is generally superior to either. In stage 2, L_{i2} is chosen to maximize $E[\pi_i|I^b]$ as in the SU solution. Then, in stage 1, L_{i1} is chosen to maximize $E[\pi_i|I^a]$ as in the OL solution. Therefore, the OLF solution, like the SU solution, has L_{i2} as an endogenous variable. The full model consists of the production functions (3) and (4) and the input equations (10) and (12).

Table 2. Information Sets Used in Sequential Solutions

Solution	Information Set Used in Period 1	Information Set Used in Period 2
Open loop (OL)	I^a	I^o
Sequential updating (SU)	I^o	I^b
Open loop w/feedback (OLF)	I^a	I^b
Closed loop (CL)	I^{ac}	I^b

Note: See table 1 for definitions of the information sets.

The Closed Loop (CL) Solution

The CL solution utilizes properties (*a*), (*b*), and (*c*). It is similar to the OLF solution, except that the expectation in each stage is computed with the probability density conditioned on information available at that time as well as the knowledge that more information will become available in the future. This "closing" of the information loop distinguishes the OLF and CL solutions. Hence, the CL solution also possesses the simultaneity properties of the OLF and SU solutions and is based on maximization of $E[\pi_i|I^b]$ with respect to L_{i2} and maximization of $E[\pi_i|I^{ac}$ with respect to L_{i1}.

Sequential decision making, thus, has two distinct effects on the production model's form. First, optimal input choices are sequentially dependent. Sequential dependence generally leads to nonlinear input choice equations which are functions of production function parameters, prices, and previous inputs and outputs. Even for a simple, two-stage Cobb-Douglas model, the optimal first-stage input is obtained by solving a complicated polynomial equation. This result occurs for all but the simplest models with linear production functions and quadratic objective functions (Aoki 1967). Second, the information feedback causes inputs chosen in later stages to depend on previous stages' outputs. This may lead to simultaneity between inputs and outputs.

The OL model, with data for both Q_{i1} and Q_{i2} consists of the two production functions (3) and (4) and the two input equations (8) and (9). The L_{it}^o are nonstochastic, and the production functions may be estimated using single-equation methods. Since Cobb-Douglas functions are linear in logarithms, ordinary least squares estimates will be unbiased and efficient with error structure (6) because Q_{i1} in equation (4) is a predetermined endogenous variable. With error structure (7), the combination of a lagged dependent variable in equation (4) and autocorrelated errors causes least squares estimates to be biased and inconsistent. It is possible to utilize the instrumental variables technique with (6). A more efficient method is maximum likelihood estimation under appropriate distributional assumptions, (Theil, chap. 8). Another recursive estimation procedure is possible. Equation (5) shows that the final output can be expressed as a function of the exogenous variables alone. Therefore, the reduced-form parameters could be estimated efficiently using a single-equation estimator under either error structure, (6) or (7). However, it may not be possible to identify the parameters in each stage's function with this approach. For example, equation (5) shows that it is not possible to identify γ_0, β_0, γ_1, β_1, or β_2.

The SU, OLF, and CL solutions differ from the OL solution because there is information feedback from previous stages' outputs to later stages' inputs and some inputs may be endogenous in the structural econometric model. To illustrate, consider the model from the OLF solution. With both Q_{i1} and Q_{i2} observed, it consists of the two production functions (3) and (4) and the two input equations (10) and (12). $L_{i1}{}^o$ is nonstochastic as in the OL solution but $L_{i2}{}^o$ depends on Q_{i1} and is stochastic. However, when Q_{i1} is observed equation (12) has no error term. Consequently, only the production functions need be estimated with problems identical to those encountered with the OL solution. With error structure (6), Q_{i1} and L_{i2} are predetermined in equation (4). Ordinary least squares may be applied to both production functions in log form. With error structure (7), autocorrelation causes bias in least squares estimates and must be accounted for.

When data for Q_{i1} is not available, equation (3) may be substituted into equations (4) and (12). The resulting "semi-reduced form" equations are

$$(14)\quad \log Q_{i2} = \log \gamma_0\beta_0{}^{\gamma_1} + \beta_1\gamma_1 \log L_{i1}{}^o + \beta_2\gamma_1 \log A_i + \gamma_2 \log L_{i2}{}^o + \gamma_1\epsilon_{i1} + \epsilon_{i2}, \text{ and}$$

$$(15)\quad \log L_{i2}{}^o = \frac{1}{1-\gamma_2}\left(\frac{\sigma_2{}^2}{2} + \gamma_1\log \beta_0 + \log \gamma_0\gamma_2\right) - \frac{1}{1-\gamma_2}\log\frac{w_{i2}}{\bar{p}_i} + \frac{\beta_1\gamma_1}{1-\gamma_2}\log L_{i1}{}^o + \frac{\beta_2\gamma_1}{1-\gamma_1}\log A_i + \frac{\gamma_1}{1-\gamma_1}\epsilon_{i1}.$$

Because ϵ_{i1} occurs in both equations, a simultaneous estimator must be utilized to obtain consistent estimates of the "semi-reduced form" parameters. Least squares estimates of equation (14) clearly would be biased in this case.

Thus, sequential solutions to the production problem can yield either single- or simultaneous-equation models. If the decision maker is assumed to update his knowledge with output information as production takes place, simultaneity between inputs and output is introduced. If input choices are sequentially dependent, the solution form differs from the nonsequential problem. Carefully specifying the sequential structure of the production problem leads to a better understanding of conventional production models. When considered in a sequential decision-making context, the MA model is logically inconsistent. In a one-period choice problem, inputs must be chosen before production begins. Yet, the MA model shows inputs to be functions of actual output which is not known until after inputs are chosen.[1] The SU solution produces a model which is similar in form to the MA model. However, its simultaneity stems from an explicit, sequential decision process. The ZKD model can be derived from a sequential solution of the input choice problem if the decision maker neither updates his information set nor takes into account effects of first-stage decisions on second-stage decisions.

These qualitative results with the Cobb-Douglas model can be generalized to models based on any production function and any number of production stages. Divide the production period into T stages, and let output of firm i in stage t be Q_{it}. With input vector $\mathbf{x}_{it}$, coefficient vector $\boldsymbol{\beta}_t$, and production disturbance ϵ_{it}, the stage production functions can be written

$$(16)\quad \begin{aligned} Q_{i1} &= f_1[\mathbf{x}_{i1}, \boldsymbol{\beta}_1, \epsilon_1] \\ Q_{it} &= f_t[Q_{i,\,t-1}, \mathbf{x}_{it}, \boldsymbol{\beta}_t, \epsilon_t], \end{aligned}$$

$$t = 2, \ldots, T;\ i = 1, \ldots, N.$$

[1] This criticism of the MA model is also valid for those of Hoch (1958, 1962) and Mundlak and Hoch. Their models, with endogenous input demand equations, specify input demands as functions of actual rather than expected output.

Assuming the final product Q_{it} is sold in period T at price p_{iT} and input prices are $\mathbf{w}_{it}$, profit is

$$(17) \qquad \pi_{iT} = p_{iT}Q_{iT} - \sum_{t=1}^{T} \mathbf{w}_{it}\mathbf{x}_{it}.$$

If firms maximize expected returns, the ith firm's objective is

$$\max_{\mathbf{x}_{i1},\ldots,\mathbf{x}_{iT}} E[\pi_{iT}] \text{ subject to (16), (17).}$$

This control problem is a terminal period problem and is a special case of a more general, multiperiod model in which output is sold in each period rather than just the final period. Solutions to this problem are generally nonlinear in the parameters, and probably distributions for Q_{it} are difficult to ascertain. When farmers make decisions sequentially and when each stage's output is not observed by the econometrician, then the structural production model is a system of nonlinear simultaneous equations. Estimation procedures for these models are available (Amemiya, Fair) but are very costly to use.

Error Specification, Functional Separability, and Uncertainty

Production economists have studied the relationship between production inputs and the stochastic characteristics of production processes (Day, Anderson, Roumasset, Just and Pope, and Antle). The error specification of the production function determines how inputs affect the probability distribution of output and the implied behavior of farmers toward production uncertainty. Dynamic production functions also may cause the probability distribution of each stage's output to be intractable. In general, when the stage functions f_t in (16) are nonlinear, nonseparable functions of $Q_{i,t-1}$, $\mathbf{x}_{it}$, and ϵ_{it}, the probability distribution of Q_{it} cannot be derived analytically. Maximum likelihood estimation cannot be used nor small sample inferences drawn. However, if the production function is either additively or strongly (nonadditive) separable, it is sometimes possible to obtain models with tractable distributions.

A production function which is additively separable in $Q_{i,t-1}$ and ϵ_{it} can be written

$$Q_{it} = \alpha_t Q_{i,t-1} + m_t[\mathbf{x}_{it}, \boldsymbol{\beta}_t] + \epsilon_{it},$$

where α_t is a parameter, and m_t is a concave function of $\mathbf{x}_{it}$. Substitution for $Q_{i,t-1}$, $Q_{i,t-2}$, etc., shows that the distribution of Q_{it} is a linear combination of the errors $\epsilon_{it}, \epsilon_{i,t-1}, \ldots, \epsilon_{i1}$. Therefore, if linear combinations of the ϵ_{it} have a known distribution, Q_{it} has a known distribution.

Additive separability of inputs across production stages is not usually a plausible hypothesis in agricultural production. Strong, nonadditive separability is a more reasonable assumpton. For example, additive separability in the crop production model implies that the marginal product of harvest labor is independent of the amount harvested. The strongly separable Cobb-Douglas function used in equation (4) shows that the marginal product of harvest labor L_{i2} depends on the amount harvested, Q_{i1}. A production function strongly separable in $Q_{i,t-1}$, m_t and ϵ_{it} can be written

$$Q_{it} = (Q_{i,t-1})^{\alpha_t} m_t[\mathbf{x}_{it}, \boldsymbol{\beta}_t]\epsilon_{it}.$$

Note that the logarithm of Q_{it} is a linear function of the logarithms of $Q_{i,t-1}$ and ϵ_{it} with this specification. If the ϵ_{it} has a log-normal distribution it can be shown that the output of each stage follows the lognormal distribution.

From these examples the following conclusion emerges concerning error specification and functional separability of inputs: tractable output distributions can be obtained if the production functions are additively separable and errors are additive, or if the production functions are strongly separable and errors are multiplicative. Otherwise, each stage's output is a nonlinear function of earlier stages' error terms and the probability distribution of output usually cannot be obtained analytically (Aoki 1967, chap. 2).

The relationship between error specification and functional separability also is relevant to the analysis of behavior under uncertainty. This is because additive and multiplicative errors have different implications for the probability distributions of output. Just and Pope showed that a multiplicative error specification restricts the relationship between input choice and output variance. More generally, not only the mean and variance but also higher moments of output may be functions of inputs (Day, Anderson, Roumasset). Antle shows that a model which does not impose restrictions on the relation between the inputs and the form of the output probability distribution can be specified and estimated with an additive error term. The above discussion shows that a tractable dynamic model with an additive error structure would have to be additively separable across production stages.

However, a model with desirable properties which is strongly separable across production stages also can be specified. First, define the production function as

$$Q_{it} = Q_{i,t-1}m_t[\mathbf{x}_{it}, \boldsymbol{\beta}_t]e^{\epsilon_{it}} = m_1[\mathbf{x}_{i1}, \boldsymbol{\beta}_1]m_2[\mathbf{x}_{i2}, \boldsymbol{\beta}_2] \ . \ . \ . \ m_t[\mathbf{x}_{it}, \boldsymbol{\beta}_t]e^{u_{it}},$$

where $E(u_{it}) = 0$, $u_{it} = \sum_{j=1}^{t} \epsilon_{ij}$. Second, assume the probability distribution of u_{it} is $g(u_{it}|\mathbf{x}_{i1}, \ldots, \mathbf{x}_{it})$, a function of inputs. Then the moments of u_{it} depend on inputs

$$\mu_{jit}(\mathbf{x}_{i1}, \ldots, \mathbf{x}_{it}) = \int_0^\infty (u_{it})^j g(u_{it}|\mathbf{x}_{i1}, \ldots, \mathbf{x}_{it})du_{it}.$$

Finally, note that $e^{u_{it}} = \sum_{j=0}^{\infty} \frac{u_{it}{}^j}{j!}$ and, therefore,

$$E(e^{u_{it}}) = 1 + \sum_{j=2}^{\infty} \frac{\mu_{jit}}{j!}.$$

Using this latter expression, it can be shown that the moments of output are functions of the inputs through the m_t and the μ_{jit}. Hence, this strongly separable production function yields a tractable output distribution. Moreover, it does not restrict the effects that inputs may have on the output distribution moments.

Conclusions

When a short-run input choice problem is solved sequentially, the resulting econometric production model generally differs in its functional form and stochastic structure from single-stage models. Because farm managers can be expected to utilize all available information in decision making, they will feed back information from earlier production stages to later input choices. In addition, only the final agricultural product usually is measured. This means that agricultural production models typically are systems of simultaneous equations. Single-equation estimates of production parameters will be subject to simultaneous-equation bias. Estimates with desirable properties can be obtained by formulating and estimating models that reflect the sequential structure of farm managers' input choice problems. However, in order to analyze multistage, sequential production problems, researchers must devise models which have desirable properties and are empirically tractable.

In assessing the practical importance of these findings, two points are relevant. First, the size of the simultaneous-equation bias due to input endogeneity remains to be ascertained. Second, as any applied production economist knows, a critical limiting factor is data availability. Most available production data do not contain information on inputs by production stage or operation. An important contribution to our understanding of both the simultaneity problem and the sequential structure of farm managers' decision making with stage-level production functions could be made by collection of production data by stages.

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References

Amemiya, T. "The Nonlinear Two-Stage Least Squares Estimator." *J. Econometrics* 2(1974):105–10.

Anderson, Jock. "Sparse Data, Climatic Variability, and Yield Uncertainty in Response Analysis." *Amer. J. Agr. Econ.* 55(1974):77–82.

Antle, J. M. "Testing the Stochastic Structure of Production: A Flexible Moment-Based Approach." *J. Bus. and Econ. Statist.*, in press.

Aoki, M. *Optimal Control and Systems Theory in Dynamic Economic Analysis*. Amsterdam: North-Holland Publishing Co., 1976.

———. *Optimization of Stochastic Systems*. New York: Academic Press, 1967.

Chow, G. C. *Econometric Analysis by Control Methods*. New York: John Wiley & Sons, 1981.

Day, R. H. "Probability Distributions of Field Crops." *J. Farm Econ.* 47(1965):713–41.

Fair, R. C. "The Estimation of Simultaneous Equation Models with Lagged Endogenous Variables and First-Order Serially Correlated Errors." *Econometrica* 38(1970):507–16.

Hoch, Irving. "Estimation of Production Function Parameters Combining Time-Series and Cross-Section Data." *Econometrica* 30(1962):556–78.

———. "Simultaneous Equation Bias in the Context of the Cobb-Douglas Production Function." *Econometrica* 26(1958):34–53.

Just, R. E., and R. D. Pope. "Stochastic Specification of Production Functions and Economic Implications." *J. Econometrics* 7(1978):67–86.

Marshak, J., and W. J. Andrews. "Random Simultaneous Equations and the Theory of Production." *Econometrica* 12(1944):143–205.

Mundlak, Y., and I. Hoch. "Consequences of Alternative

Specifications in Estimation of Cobb-Douglas Production Functions." *Econometrica* 33(1965):814–28.
Roumasset, J. A. *Rice and Risk: Decision Making among Low Income Farmers*. Amsterdam: North-Holland Publishing Co., 1976.
Theil, H. *Principles of Econometrics*. New York: John Wiley & Sons, 1971.
Yotopoulos, P. A., L. J. Lau, and W. Lin. "Microeconomic Output Supply and Factor Demand Functions in the Agriculture of the Province of Taiwan." *Amer. J. Agr. Econ.* 58(1976):333–40.
Zellner, A. *An Introduction to Bayesian Inference in Econometrics*. New York: John Wiley & Sons, 1971.
Zellner, A., J. Kmenta, and J. Dreze. "Specification and Estimation of Cobb-Douglas Production Function Models." *Econometrica* 34(1966):784–95.

ERRATUM

The editors apologize for a typographical error that appears in equation (3) of Michael Wyzan's February 1983 (p. 179) "Empirical Analysis of Soviet Agricultural Production and Policy: Reply." The equation should read as follows:

$$(3) \quad \frac{\partial^2 Q}{\partial B^2} = \frac{Q}{B} \cdot \frac{\partial^2 \ln Q}{\partial (\ln B)^2} + \left(\frac{B - Q}{QB}\right) \cdot \frac{\partial Q}{\partial B \cdot}.$$