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A MATHEMATICAL PROGRAMMING DECISION MODEL
FOR FARM MACHINERY REPLACEMENT: A CONCEPTUAL
SPECIFICATION

by

Garnett L. Bradford and Donald W. Reid*

Machinery

*Professor and Assistant Professor, respectively, Dept. of Agr. Economics, University of Kentucky, Lexington, 40546. Paper submitted for consideration as a selected paper at the AAEA Annual Meetings at Cornell University, August 5-8, 1984.

ABSTRACT

A MATHEMATICAL PROGRAMMING DECISION MODEL FOR FARM MACHINERY REPLACEMENT: A CONCEPTUAL SPECIFICATION

A multi-year integer programming model which accomodates production and investment decisions is specified. The model is shown to be a theoretically appropriate one by exploiting the dual model and demonstrating its relationship to a net future value criterion. Theoretical results from the dual model also address weaknesses in directly applying the net present value criterion of replacement.

A MATHEMATICAL PROGRAMMING DECISION MODEL FOR FARM MACHINERY REPLACEMENT: A CONCEPTUAL SPECIFICATION

The standard capital and financial theory approach to evaluating capital investments is net present value (NPV) analysis (Lutz and Lutz; Hirshleifer). Directly applying NPV criteria and related decision rules allows consideration of the time value of cash flows. The primary weakness of this analytical method for farm investments in machinery is that funds and other resource constraints are not explicitly considered, and production decisions cannot be determined simultaneously with investment decisions.

Some weaknesses of directly applying NPV analyses can be overcome with linear programming models for certain types of investments (Lorie and Savage; Weingartner; Boehlje and White). Still, the basic weakness of linear programming models in machinery investment is a failure to handle the discrete nature of the investment.

Recently, Danok, McCarl, and White (1978 and 1980) made significant contributions in machinery selection modeling with the use of mixed integer programming. They stressed the need for selecting machinery within a simultaneous investment-production framework. However, their models lack time considerations. That is, the solutions produced do not incorporate expectations about the future or time-value aspects.

Both expected changes and time-value considerations are important in all types of farm machinery investment analyses, but especially in replacement analysis. Without time considerations many important dimensions of the problem such as declining machinery resale values, increasing repair costs, opportunity costs of downtime, and expected changes in input and product prices cannot be considered. Accordingly, the purpose of this paper is to develop a conceptual programming model based on theoretically appropriate investment and replacement criteria.

Investment and Replacement Criteria for the Firm

Assuming the objective of a firm is to maximize its wealth gain, investments should be made to the point where the marginal rate of return on investments equals the marginal cost of funds. Hirshleifer demonstrated that the NPV criterion is a theoretically appropriate criterion for evaluating investment projects in order for the firm to achieve the most wealth gain, i.e., its highest present market value. The discrete time NPV criterion for an investment of a known duration is the familiar

$$(1) \quad G_j = \sum_{t=1}^{A} Q_{tj} (1+r)^{-t} - C_j + S_j (1+r)^{-A}$$

where

G_j = net present value of project j over a life of A years;

Q_{tj} = net operating cash flow (quasi-rent) from project j at time t;

C_j = the initial investment in project j;

S_j = the remaining market value of project j;

r = the specified discount rate;

A = the life of the project

This criterion evaluates the stream of costs and returns at the present, although any point in time is equally valid. Discounting (or compounding) is simply a method of considering the opportunity cost of invested funds. Thus, for independent projects the decision rule is to accept all projects with $G_j > 0$, i.e., if the rate of return on the project is greater than the opportunity cost of funds, then it is an investment which increases the value of the firm.

The replacement criterion differs from the criterion for investments of a known duration; however, the overall objective of maximizing the value of the firm is the same. Specifically, a project should be replaced at the age that maximizes the present value of that project when considered with the infinite series of replacements to follow. The appropriate discrete-form present value criterion for a project to be replaced with a project that has identical cash flows (i.e. a pure replacement) is

$$(2)\quad \text{Maximize:}\quad G_j = \frac{1}{1-(1+r)^{-A}}\left[\sum_{a=1}^{A} Q_{aj}\,(1+r)^{-a} - C_j + S_j\,(1+r)^{-A}\right]$$

where all symbols are the same as previously defined, except G_j is now the net present value of project j assuming replacement every A years. Quite a bit of confusion has surrounded this criterion, but it has been discussed thoroughly in the literature (e.g., Lutz and Lutz; Hirshleifer; Perrin).

A Conceptual Programming Model of Investment and Replacement Decisions

The problem with directly applying the NPV criterion is that not all opportunity costs are known. Mathematical programming methods allow appropriate valuation of opportunity costs.

Two difficulties are encountered when using a programming method for investment and replacement analysis. First, the model should be structured so that it has a sound theoretical basis. The second difficulty concerns the fact that replacement decisions should consider an infinite chain of replacements, while a programming model can explicitly model only a finite number of periods.

Conceptual Replacement Model

Programming models for planning horizons of finite length, T, which maximize ending net worth are referred to as horizon models (Weingartner;

Boussard). Conceptually, an infinite horizon programming model may be constructed by using a horizon-type multiperiod mixed integer programming (MMIP) model for the first T periods, and a NPV model for periods T+1 to infinity. Because of the detail allowed in the constraint matrix of such a programming model, well-defined expectations in the periods close to the decision period can be modeled and evaluated in detail. For conditions and expectations in the periods beyond those modeled in the constraint matrix, modified NPV models may be used.

Using the foregoing notions, a general conceptual model of the firm's value may be represented by

$$(3) \quad W_T^Q = \sum_{t=0}^{T} Q_t \rho_t + \sum_{t=T+1}^{\infty} \frac{Q_t}{(1+r)^{t-T}}$$

where

W_T^Q = the value of the firm at time T;

Q_t = all cash flows in period t;

ρ_t = the implicit compounding rate of the MMIP model; and

r = some specified discount rate.

The terms through time T represent all cash flows compounded to time T by the implicit compounding rate. The value of these terms is determined by the MMIP portion of the overall model. The terms beyond T represent the value of the firm's activities beyond T, but valued at T. These terms are the NPV portion of the model.

The specific form of the NPV portion of the model depends on the asset or liability to be valued at T. Assets such as cash or market investments can be represented by their market values at T. This assumes that these assets earn the cost of equity for infinity. Debt also can be represented by its market value, assuming the debt earns the cost of debt capital for infinity. Land value can be included by capitalizing the annual returns to land. For assets requiring replacement, the value can be represented by using a modified NPV replacement formula.

The modified replacement models are the most difficult to formulate. A model of the replacement activity may be represented by

(4)(a) Maximize: $G_j = C_j \rho_o + Q_{1j} \rho_1 + Q_{2j} \rho_2 + \ldots + Q_{TJ} \rho_T$

(b) $$+ \frac{Q_{T+1,j}}{(1+r)} + \frac{Q_{T+2,j}}{(1+r)^2} + \ldots + \frac{Q_{A*,j}}{(1+r)^{A*-T}}$$

$$+ \frac{S_j}{(1+r)^{A*-T}}$$

(c) $$+ \left[\frac{1}{(1+r)^{A*-T}}\right]\left[\frac{1}{1-(1+r)^{-A**}}\right]$$

$$\left[\sum_{t=1+A*}^{A*+A**} \frac{Q_{tj}}{(1+r)^t} - C_j + \frac{S_j}{(1+r)^{A**}}\right]$$

where ρ_j is the implicit compounding factor for the MMIP model, and A* and A** are optimal replacement ages of the first and second machines, respectively. Note that expression 4 is a modification of the infinite-series replacement model given in expression 2. The terms in 4a represent the compounding of cash flows that are modeled within the constraint matrix of the MMIP model. Expression 4b represents the remainder of the cash flows associated with the first machine up to the time of the optimal replacement based on current anticipations. The terms in 4b are not modeled within the matrix of the MMIP; they are evaluated external to the MMIP model.

Conceptual MMIP Model of Production-Investment-Replacement

In order to cast the foregoing conceptual model into a programming framework, the values in the objective function of the MMIP model must include the value at T of cash flows subsequent to T for each possible replacement or decision alternative. This entails evaluating these cash flows outside of the programming portion of the model, then including them as objective function values. By doing this, the solution obtained from the MMIP calculations depends not only upon the expected values of cash flows up to time T, but also on those occurring subsequent to the periods specifically modeled by the programming model.

Analysis of a specific MMIP model allows insight into how programming methods can modify NPV criteria to appropriately consider opportunity costs arising from constraints and multiple production opportunities. Only a few constraints are considered here because of the complexity of larger models. The example focuses on aspects which are important for farm machinery investment and replacement. The primal problem for such a model is:

(5)(a) Maximize: $$\sum_{j=1}^{N} g_j x_j + v_T + w_T$$

(b) Subject to: $$\sum_{j=1}^{N} a_{1j} x_j - \sum_{k=1}^{L} p_{1k} z_k + v_1 - w_1 \leq D_1$$

(c) $$\sum_{j=1}^{N} a_{tj} x_j - \sum_{k=1}^{L} p_{tk} z_k + v_t - (1+r) v_{t-1} - w_t + (1+r) w_{t-1} \leq D_t$$

for $t = 2, 3, \ldots, T$

(d) $$\sum_{j=1}^{N} - b_{tj} x_j + \sum_{k=1}^{L} c_{tk} z_k \leq C_t$$

(e) $$x_j = 0 \text{ or } x_j = 1$$

(f) $$w_t \leq B_t$$

(g) $$z_k,\ v_t,\ w_t \geq 0$$

where

x_j = the units of machine j;

z_k = the units of production project k;

v_t = the amount of funds loaned (market investments) in time t;

w_t = the amount of funds borrowed in time t;

a_{tj} = the cash flow associated with machine j (only directly associated with the machine, i.e., C_j and S_j in the NPV model);

p_{tk} = the cash flow associated with a unit of production project k (i.e., Q_{tj});

b_{tj} = the amount of capacity associated with a unit of machine j for the period t;

c_{tk} = the amount of machinery capacity required for a unit of production project k in period t;

r_L = the lending rate of interest (rate of return on market investment);

r_B = the borrowing rate of interest;

D_t = the funds endowed to period t;

C_t = the amount of endowed machinery capacity to period t;

B_t = the amount of funds available for borrowing in time t; and

g_j = the expected value of the cash flows associated with the infinite chain of the j^{th} asset subsequent to T and evaluated at T, i.e., g_j is the value of expressions 4b plus 4c.

Expressions 5b and c are funds constraints for period 1 and t, t = 2, 3, ..., T, respectively. These constraints separately consider the impacts of the cash throw-offs of the production projects, $p_{tk}z_k$, from cash associated with machinery purchases or sales. Constraint 5d ensures that the amount of machinery capacity required by the production activities, $c_{tk}z_k$, does not exceed the amount available from the machine, $b_{tk}x_j$, and any endowed capacity, C_t. Expression 5e allows only integer amounts of machinery units. With constraint 5f, only a fixed amount of funds, B_t, can be borrowed in period t. Non-negativity conditions are imposed on variables by constraint 5g.

Conceptual insights to the primal formulation are provided by interpreting the dual problem:

$$(6)(a)\quad \text{Minimize:} \quad \sum_{t=1}^{T} \rho_t D_t + \sum_{t=1}^{T} \alpha_t C_t + \sum_{j=1}^{N} \mu_j + \sum_{t=1}^{T} \beta_t B_t$$

$$(b)\quad \text{Subject to:} \quad \sum_{t=1}^{T} \rho_t a_{tj} - \sum_{t=1}^{T} \alpha_t b_{tj} + \mu_j \geq g_j$$

$$(c)\quad -\sum_{t=1}^{T} \rho_t p_{tk} + \sum_{t=1}^{T} \alpha_t c_{tk} \geq 0$$

$$(d)\quad \rho_T \geq 1$$

$$(e)\quad -\rho_T + \beta_T \geq -1$$

$$(f)\quad \rho_t - (1+r_L)\rho_{t+1} \geq 0$$

$$(g)\quad -\rho_t + (1+r_B)\rho_{t+1} + \beta_t \geq 0$$

$$(h)\quad \beta_t,\ \rho_t,\ \mu_j \geq 0$$

where

a_{tj}, p_{tk}, b_{tj}, c_{tk}, D_t, C_t, B_t, and g_j are defined as in the primal problem, (5);

ρ_t = the model determined compounding factor for period t, i.e., the opportunity cost in dollars at T of investing an extra dollar in period t;

α_t = the implicit value of the machinery and equipment capacity in time t;

β_t = the implicit value of a dollar of borrowing in period t; and

μ_j = the implicit value of fractional units of machine j.

The foregoing dual formulation enables development of appropriate opportunity costs that should be used when evaluating replacement decisions. These opportunity costs include the opportunity of investment funds, i.e., the discount rate, opportunity cost of machine capacity, and opportunity cost of additional machine units.

Assuming that borrowing and lending cannot occur at T, $\rho_t = 1$, i.e., a dollar borrowed or lent at T simply would be worth one dollar at T. Now, constraint 6f shows that

$$(7) \qquad \rho_{T-1} = 1 + r_L$$

when lending is occurring. By substituting recursively, the result in 7 can be generalized to

$$(8) \qquad \rho_t = (1 + r_L)^{T-t}.$$

This result shows that when lending occurs in all t, the appropriate compound rate is the lending rate of interest. Constraint 6g shows that when borrowing occurs

$$(9) \qquad \rho_{T-1} = (1 + r_B) + \beta_{T-1}.$$

By substituting recursively, the result in 9 can be generalized to

$$(10) \qquad \rho_t = (1 + r_B)^{T-t} + \sum_{c=t}^{T} \beta_c (1 + r_B)^{c-t}.$$

This result shows that when borrowing occurs and is constrained in all t, the compound rate is the market borrowing rate plus a premium to reflect the appropriate opportunity costs of a borrowing constraint. When borrowing occurs for all T but is not constrained in any T, all $\beta_c = 0$ and the compound rate becomes the market borrowing rate.

The appropriate opportunity cost of machine production capacity helps determine if a machine should be replaced or if an additional machine is needed. The implicit value of machinery capactiy can be solved from expression 6c. When constraint 6c is met, it can be written as

$$(11) \qquad \sum_{t=1}^{T} \alpha_t C_{tk} = \sum_{t-1}^{T} \rho_t p_{tk}.$$

Thus, for any period t

$$(12)(a) \quad \alpha_t C_{tk} = \rho_t p_{tk}, \text{ or}$$

$$(12)(b) \quad \alpha_t = \frac{\rho_t p_{tk}}{C_{tk}}.$$

This result shows that the value of a unit of machine capacity depends on costs and returns from production activity k, the amount of capacity required for project k, and the implicit compound rate.

The implicit value of machine j can be solved from expression 6b. Substituting the value of α_t derived in 12b into expression 6b and when constraint 6b is met, it can be written

$$(13) \qquad \mu_j = \sum_{t=1}^{T} \rho_t \frac{p_{tk} b_{tj}}{C_{tk}} - \sum_{t=1}^{T} \rho_t a_{tj} + q_j.$$

The first component of expression 13 on the right side of the equality represents the beneficial value of machine j derived from the services supplied for production of project k for t = 1, 2, ..., T. The cost and maintenance of machine j is represented by $\Sigma \rho_t a_{tj}$ The component g_j reflects the remaining value of machine j plus the value of its replacement series valued at T.

An important theoretical aspect of expression 13 is that it forms a net future value model for replacement similar to the conceptual model delineated in expression 4. However, expression 13 evaluates appropriate costs by simultaneously determining opportunity costs. Therefore, the programming model specified in this paper is a theoretically appropriate one for determining optimal replacement decisions that will maximize the value of the firm at T, because it incorporates a theoretically correct criterion in a selection process.

Concluding Remarks

The MMIP model specification is shown to be theoretically appropriate by demonstrating its relationship to the standard NFV (or NPV) criterion. The reasons for developing a conceptual MMIP model are to address weaknesses in directly applying NPV criteria to problems of farm machinery investment and replacement decisions, with special emphasis on replacement.

The MMIP method simultaneously evaluates the various investment and production opportunitits. Therefore, this method potentially can determine (a) appropriate opportunity costs for machinery breakdowns and undercapacity; (b) appropriate compounding factors; (c) appropriate combinations of replacement, scale adjustment, and size adjustment of machinery; (d) appropriate income tax payment; and (e) many other aspects. Although the number of periods which can be modeled in the constraint matrix is limited this is not viewed as a severe limitation. This is because it is those periods nearest the decision period which have the most influence on the decision and need the most precise evaluation.

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