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DESCRIBING AND DETERMINING THE COMPLETE SET OF TARGET MOTAD SOLUTIONS*

by

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Risk -- Mathematical models

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Abstract

"Describing and Determining the Complete Set of Target MOTAD Solutions."
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Tauer's Target MOTAD model can be used to compute stochastically efficient mixtures of risky alternatives. This paper suggests a way of describing the complete set of Target MOTAD solutions. It also presents an efficient method for determining the set of solutions. Hazell's example is used to illustrate these ideas.

DESCRIBING AND DETERMINING THE COMPLETE SET OF TARGET MOTAD SOLUTIONS

In a recent article, Tauer proposed Target MOTAD as a model for computing risk efficient mixtures of risky alternatives. In addition to its intuitive appeal, Target MOTAD possesses the attractive feature that all unique solutions are members of the SSD (second degree stochastic dominance) efficient set. Tauer used data from an Anderson, Dillon and Hardaker (pp. 209-10) example to demonstrate that some, but not all, standard MOTAD solutions are SSD efficient. He presented a few of the many possible Target MOTAD solutions and suggested that all Target MOTAD (and perhaps all SSD efficient) solutions could be obtained by computing the solutions associated with all possible target income (T) levels.

This paper extends Tauer's article in two ways. First, a way of describing the complete set of Target MOTAD solutions for any problem is presented. Second, an alternative method for determining this set is suggested. For most problems, the method suggested in this paper will involve less computational effort than the approach suggested by Tauer. The method is illustrated using data from an example described by Hazell. Although this example is small enough to be readily solved, its Target MOTAD solution set is more complex than the solution set for the Anderson, Dillon and Hardaker example. A similar or greater degree of complexity is likely to be exhibited by the Target MOTAD solution sets for most problems.

The Target MOTAD Model

The Target MOTAD model has the form:

(1) Max $\bar{c}'x$

subject to

(2) $Ax \leqq b$

(3) $-Cx - y \leqq -uT$

(4) $p'y = \lambda$

(5) $x, y \geqq o$

where $\bar{c}$ is an n element column vector of expected returns for the various activities, x is an n element column vector of activity levels, b is an m element column vector of resource or technical levels, A is an m by n matrix of resource or technical requirements, C is an s by n matrix of returns associated with the activities for various states of nature, y is an s element column vector of deviations from target income, u is an s element column vector of ones, T is target income, p is an s element column vector of probabilities associated with the various states of nature, λ is the absolute value of expected negative deviations from target income, n is the number of activities, m is the number of resource or technical constraints and s is the number of observations or states of nature. To apply Target MOTAD to the Hazell example it is assumed that each of the six states of nature (observations) is equally likely.[1/]

Describing the Complete Set of Target MOTAD Solutions

Tauer notes that the Target MOTAD model is a two attribute model. Any given solution is associated with one (or more) combination(s) of target income, T, and the absolute value of expected negative deviations from target income, λ. One way of describing the complete set of Target MOTAD solutions is as the set of solutions associated with all feasible combinations of T and λ.[2/] Fortunately, there is a more fruitful way of describing this set.

Since the Target MOTAD model is a linear program, the solution for any given combination of T and λ will ordinarily share a set of basis activities (or variables) with the solutions for a large number of other T and λ combinations. Although the number of Target MOTAD solutions is infinite, the number of bases associated with these solutions is finite.[3/] The number of extreme points associated with each basis is also finite and often small in number. Thus, a finite number of extreme points can be used to describe the complete set of Target MOTAD solutions. The subset of solutions associated with any basis is convex and consists of the convex combinations of its extreme points. The complete set of Target MOTAD solutions is the union of the subsets associated with the various bases.

Determining the Complete Set of Target MOTAD Solutions

There are several ways of identifying the bases and extreme points described above. For all methods, computational effort can be reduced by ignoring many combinations of T and λ. Some combinations can be ignored because the x components of their solutions are the same as those for other T and λ combinations. Other combinations of T and λ can be ignored because they do not permit feasible solutions.

Relevant T Values

To determine the relevant range of T values, first find the x vector, x^*, which maximizes expected income subject only to the resource and nonnegativity constraints (2) and (5).[4/] One way to do this is to solve the Target MOTAD model using a combination of T and λ which does not restrict expected income. Any combination of very small T and/or a very large λ will satisfy that requirement.

After finding x^*, compute the vector Cx^*. The elements of this vector are the net returns which would be realized for the various states of nature if the strategy of maximizing expected income is adopted. When T is less than or equal to the smallest element of Cx^*, the x component of all Target MOTAD solutions will be x^* regardless of the (nonnegative) value of λ. When T is larger than or equal to the largest element of Cx^*, the x component of all feasible solutions will also equal x^*. Thus, only T values within the income interval bounded by the smallest and largest elements (incomes) of Cx^* need to be considered in subsequent computations. For the Hazell example, the bounds are \$37,558.82 and \$106,868.63.

An Upper Boundary for λ

For any T value within this interval, the λ parameter will restrict expected income only when it is smaller than some value, $\bar{\lambda}(T)$. This value can be computed from the following formula:

$$(7) \quad \bar{\lambda}(T) = \sum_{k=1}^{s} \max\,(0,\ p_k(T - w_k))$$

where p_k is the probability of state k occurring and w_k is the kth element of Cx^* (the income received if state k occurs). $\bar{\lambda}(T)$, the upper boundary of the set of relevant T and λ combinations, is a piecewise linear function of T. The T and λ values for points I through VI in Table 1 are the coordinates at which the slope of this function changes. The T values are elements of Cx^* (incomes associated with the various states of nature when $x = x^*$).

Although the upper boundary function was derived by considering the properties of the Target MOTAD model, it could have also been derived directly from stochastic dominance considerations. For any T, the slope of the function is the probability of receiving a return less than or equal to T when the strategy of maximizing expected income is adopted. Thus, $\bar{\lambda}(T)$ is the function F_2 associated with the SSD criterion. Since the strategy of maximizing expected income can be dominated by no other strategy, its F_2 function is a logical choice for an upper boundary.

TABLE 1 EXTREME POINTS USED TO DESCRIBE THE SET OF TARGET MOTAD SOLUTIONS

Point Number	Target Return T	Expected Deviations λ(T)	Crop Activity Levels Carrots	Celery	Cucumbers	Peppers	Expected Returns
	--------(Dollars)--------		----------(Acres)----------				(Dollars)
Upper Boundary Points							
I	37558.82	0	0	27.45	100	72.55	77958.12
II	80431.37	7145.43	0	27.45	100	72.55	77958.12
III	80492.16	7165.69	0	27.45	100	72.55	77958.12
IV	80513.73	7176.47	0	27.45	100	72.55	77958.12
V	81884.31	8090.20	0	27.45	100	72.55	77958.12
VI	106868.63	28910.46	0	27.45	100	72.55	77958.12
Lower Boundary Points							
1	37558.82	0	0	27.45	100	72.55	77958.12
2	47264.71	0	100	23.53	0	76.47	75145.05
3	51909.96	0	100	49.77	0	50.23	73224.85
4	55872.10	0	114.19	38.41	0	47.40	70324.60
5	60455.90	0	86.32	30.69	55.77	27.21	65277.86
6	80235.68	7012.25	5.69	26.68	94.77	72.85	77729.79
7	80455.11	7108.59	4.64	26.52	95.99	72.84	77734.51
8	80860.65	7355.21	3.90	27.30	96.10	72.70	77848.44
9	81884.31	8090.20	0	27.45	100	72.55	77958.12
10	106868.63	28910.46	0	27.45	100	72.55	77958.12
Interior Points							
A	55629.41	1394.12	100	23.53	0	76.47	75145.05
B	79631.90	6873.40	8.58	27.11	91.42	72.89	77716.79
C	80428.25	7134.59	.64	27.43	99.36	72.57	77940.19
D	59718.24	1627.19	100	38.73	0	61.27	74032.92
E	80502.42	7169.75	0	27.43	100.02	72.55	77955.89

The Lower Boundary for λ

For any value of T, feasible solutions to the Target MOTAD model can be obtained only when λ is larger than some value, $\underline{\lambda}(T)$. $\underline{\lambda}(T)$, the lower boundary, is a piecewise linear function of T. For any problem, this function can be determined by either maximizing $\bar{c}'x - M\lambda$ or minimizing λ subject to constraints (2), (3), (4) and (5), while parametrically varying T over the range of income values defined above.[5/] M is a positive number which is large enough to ensure that λ will be as small as possible for all values of T. The lower boundary is also related to the SSD criterion function F_2. However, rather than being the criterion function for a specific strategy, the lower boundary is an envelope.

The coordinates of the ends of the line segments defining $\underline{\lambda}(T)$ are the T and λ values for points 1 through 10 in Table 1. Note that lower boundary points 1, 9 and 10 are the same as upper boundary points I, V and VI, respectively. Point 5 is a "minimax" solution; \$60,455.90 is the largest target income which can be obtained with certainty.

Determining the Subsets

Only combinations of T and λ which are below $\bar{\lambda}(T)$ and above $\underline{\lambda}(T)$ need be considered. Figures 1 and 2 show the relevant set of T and λ combinations. They also show the subsets associated with the twelve bases needed to define the complete set of Target MOTAD solutions. A two variable resource mapping procedure was used to identify the subsets. This procedure is analogous to the two variable price mapping procedure described by Heady and Candler (pp. 295-298). The specific procedure used involved computing one solution for each basis and then using a special type of post optimal analysis. By permitting simultaneous (rather than independent) variation of T and λ it simplified the task of determining the boundaries of the subsets.

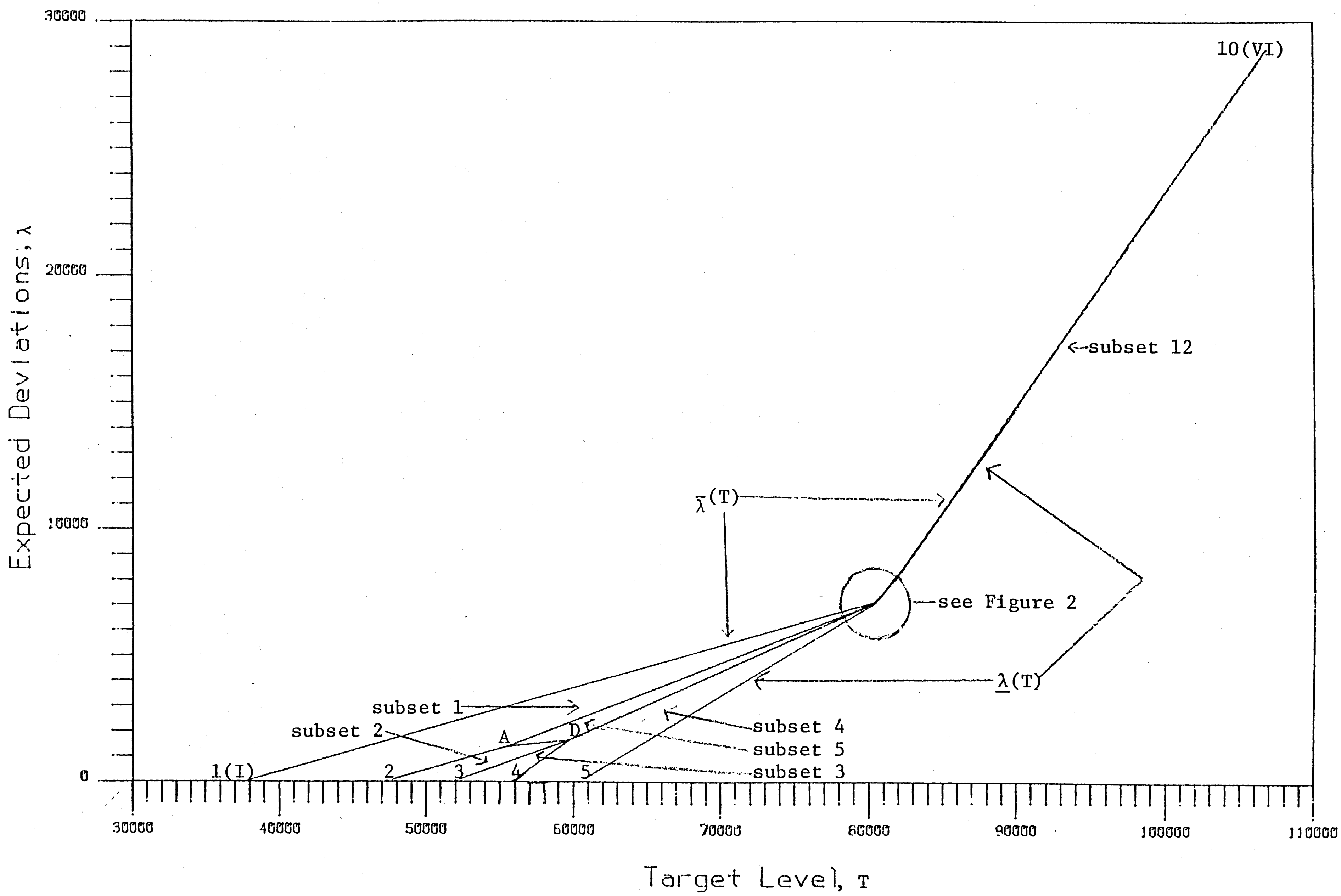

FIGURE 1. RELEVANT T, λ COMBINATIONS

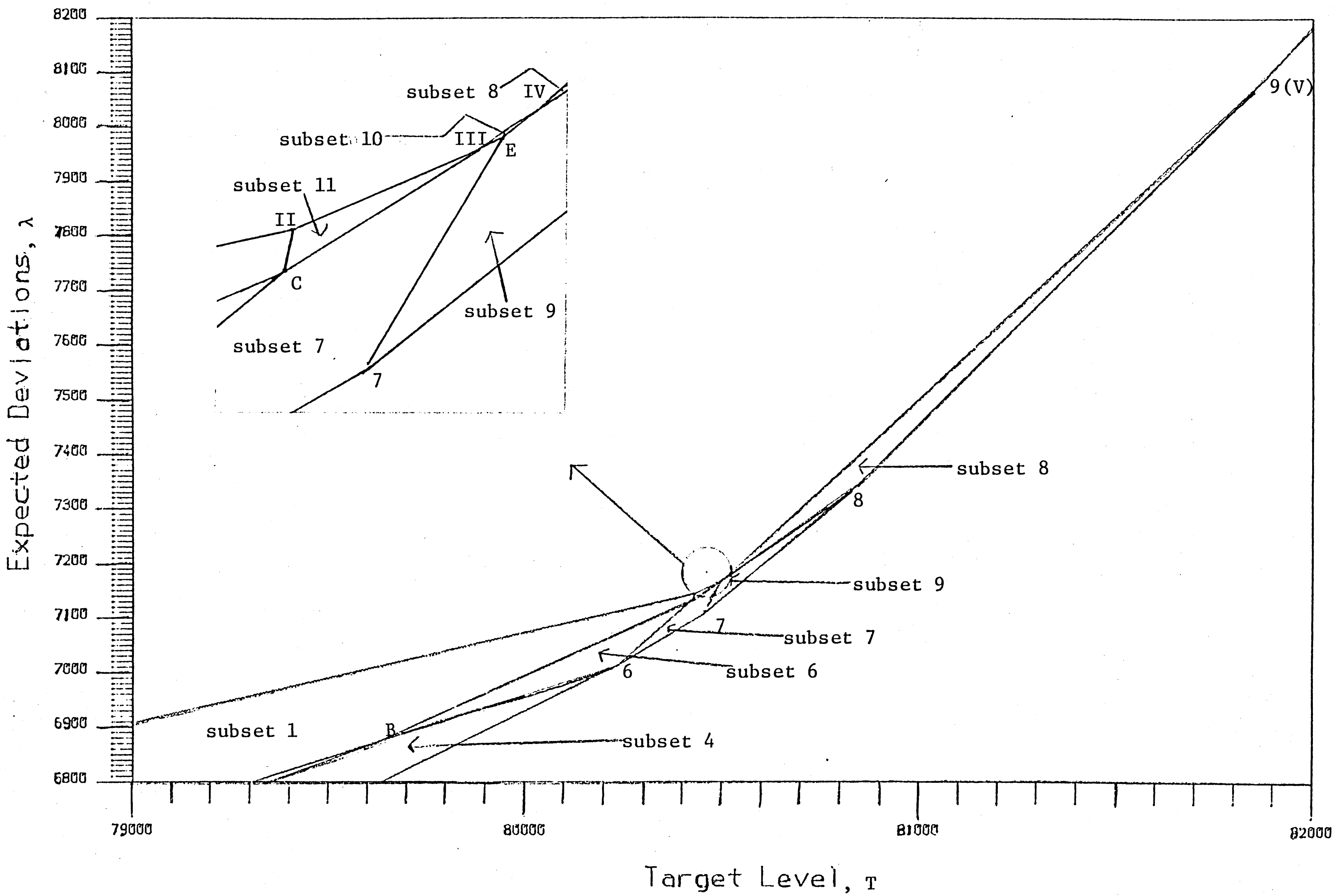

FIGURE 2. EXPANDED VIEW OF SOME T, λ COMBINATIONS

There are at least two other ways of determining these boundaries. Standard post optimal analysis (sometimes called range or range x) could be used but would require computing more than one solution for most bases. Tauer's parametric programming procedure could also be used. It involves examining vertical "slices" of the relevant set of T and λ values.

Most of the extreme points for the twelve subsets are on either the upper or lower boundary of the relevant set of T and λ combinations. Five extreme points, A through E, are not on either boundary. These interior points are presented in the bottom section of Table 1.

Table 2 defines each of the the twelve basis subsets in terms of their extreme points. For extreme points associated with both the upper and lower boundaries, the appropriate Roman numeral (upper boundary identifier) enclosed in parentheses following the Arabic number (lower boundary identifier).

While defining the subsets and identifying the extreme points associated with each subset, only their T and λ coordinates were considered. It is now appropriate to shift attention to the crop mix components of the extreme points. The values of the y variables can be ignored since their principal function is to link the crop mix to T and λ.

Examination of the crop mixes makes it obvious that a smaller set of extreme points can be used to describe the set of Target MOTAD solutions. Points I through VI as well as points 9 and 10 have the same crop mix as solution 1. Solution A has the same crop mix as solution 2. This means that at most twelve of the points in Table 1 are needed to describe the Target MOTAD solutions. It also simplifies the lists of extreme points for subsets 1, 2, 8, 10, 11 and 12. The crop mixes of some of the remaining extreme points for subsets 1 and 2 are convex combinations of the crop mixes of other extreme points for these same subsets. This permits additional

TABLE 2 SUBSETS OF TARGET MOTAD SOLUTIONS

Basis Subset	All Extreme Points[a/]	Essential Extreme Points
1	1(I), 2, A, B, C, II	1, 2
2	2, 3, D, A	2, 3
3	3, 4, D	3, 4, D
4	4, 5, 6, B, D	4, 5, 6, B, D
5	A, D, B	2, D, B
6	B, 6, C	B, 6, C
7	6, 7, E, III, C	6, 7, E, 1, C
8[b/]	IV, 8, 9(V)	8, 1
9	7, 8, IV, E	7, 8, 1, E
10[c/]	III, E, IV	E, 1
11[d/]	C, III, II	C, 1
12[e/]	9(V), 10(VI)	1

a/ The numbers and letters in this column refer to solutions presented in Table 1.

b/ The crop mixes in this subset also belong to subset 9.

c/ The crop mixes in this subset also belong to subsets 7 and 9.

d/ The crop mixes in this subset also belong to subsets 1 and 7.

e/ The crop mixes in this subset also belong to subsets 1, 7 and 9.

simplification. Finally, all of the crop mixes in subsets 11 and 12 belong to subset 1 and all crop mixes in subsets 8 and 10 belong to subset 9. This means that the complete set of Target MOTAD solutions is the union of seven subsets.

Concluding Remarks

This paper has suggested ways of describing and determining the complete set of Target MOTAD solutions. An example from Hazell was used to illustrate the ideas presented.

The complete set of Target MOTAD solutions has several characteristics. Some of these are summarized here since they were not all explicitly discussed in the body of the paper.

1. The set of Target MOTAD solutions is generally not convex. For the Hazell example, consider the crop mix consisting of 50 acres of carrots, 38.61 acres of celery, 50 acres of cucumbers, and 61.39 acres of peppers. Even though this crop mix is the average of the crop mixes for points 1 and 3, it does not belong to the set of Target MOTAD solutions. This is consistent with, but not equivalent to, Dybvig and Ross's finding that the set of efficient portfolios need not be convex.
2. The set of solutions can include rather diverse mixtures. For Hazell's data, the ranges in acreages for two crops (carrots and cucumbers) are 100 acres. One of the objectives of stochastic dominance analyses is to reduce the set of choices. Even if the set of SSD efficient crop mixtures were identical to the set of Target MOTAD solutions, it would be clear that the efficient set for the Hazell example includes rather diverse crop mixtures.[6/]

3. Unfortunately, the complete set of SSD efficient solutions is not always identified by Target MOTAD. Another analysis has identified SSD efficient mixtures for the Hazell example which are not Target MOTAD solutions (McCamley and Kliebenstein).

4. The relation between a decision-maker's attitude toward risk and the Target MOTAD crop mix which he would choose is not yet very clear. Presumably, an extremely risk averse decision-maker would choose the "minimax" solution and a risk neutral decision-maker would choose the solution which maximizes expected income but it is not clear what choice would be made by a person having some other specific risk preference. For a given T value, the level of expected deviations, λ, which is acceptable may be inversely related to the degree of risk aversion, but it is not clear how the appropriate combination of T and λ (and the associated crop mix) would be chosen. Perhaps some other technique can be applied to the set of Target MOTAD solutions to order the solutions by degree of risk aversion.

5. Some standard MOTAD solutions are also Target MOTAD solutions. For the Hazell example, these standard solutions belong to subsets 1, 4 and 5.

Tauer's Target MOTAD article represents a valuable contribution to our professional literature. It is hoped that the ideas presented in this paper will be of benefit to those who use Target MOTAD.

Footnotes

1/ The Target MOTAD formulation of Hazell's example is:

Maximize

$$253x_1+443x_2+284x_3+516x_4$$

subject to

$$
\begin{array}{lcl}
x_1 + x_2 + x_3 + x_4 & & \leq 200 \\
25x_1 + 36x_2 + 27x_3 + 87x_4 & & \leq 10{,}000 \\
-x_1 + x_2 - x_3 + x_4 & & \leq 0 \\
-292x_1+128x_2-420x_3-579x_4 & -y_1 & \leq -T \\
-179x_1-560x_2-187x_3-639x_4 & -y_2 & \leq -T \\
-114x_1-648x_2-366x_3-379x_4 & -y_3 & \leq -T \\
-247x_1-544x_2-249x_3-924x_4 & -y_4 & \leq -T \\
-426x_1-182x_2-322x_3-5x_4 & -y_5 & \leq -T \\
-259x_1-850x_2-159x_3-569x_4 & -y_6 & \leq -T \\
 & y_1/6+y_2/6+y_3/6+y_4/6+y_5/6+y_6/6 & = \lambda
\end{array}
$$

$$x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$$

2/ As is the case with standard MOTAD, any of several alternative formulations of the Target MOTAD model can be used. One alternative would involve minimizing λ subject to a (parametric) restriction on expected income. The ideas presented in the paper are applicable, with appropriate modifications, to this and other alternative formulations.

3/ The term basis as used in this paper refers to a set of activities or variables. It should not be confused with the concept of a basic solution which refers to a set of particular values for the basis variables. It is important that this distinction be kept in mind because we will be describing subsets for which the solution changes as T and λ change but the basis remains the same.

4/ It is assumed that x* is unique. Nonuniqueness complicates the problem of computing the complete set of Target MOTAD solutions and clouds the SSD efficiency status of the solutions. Discussion of these important issues is beyond the scope of this paper.

5/ The only advantage of maximizing $\bar{c}'x - M\lambda$ rather than simply minimizing λ is that the x components of the solution vectors will be needed later. However, it is possible to compute these after the lower boundary has been identified.

6/ It appears that all of the Target MOTAD solutions are unique and, therefore, SSD efficient.

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