Introductory economics

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A General Measure for Output-Variable Input Demand Elasticities[1/]

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I. Introduction

In recent years the development of new production function forms has given impetus to empirical work of measuring input demand and substitution elasticities in a variety of industries. The so-called "flexible" functional forms have given us a much richer set of tools to investigate these relationships, as compared to the familiar Cobb-Douglas and constant-elasticity-of-substitution functions. The majority of researchers have reported their results in terms of input parameters estimated under the assumption of fixed output. While this is appropriate for some questions, we argue in the next section that output-variable measures will often be more useful for the problem at hand. In section III we derive a general expression for the output variable price elasticity of input demand, of which the well-known expression of Allen is a special case. In the final section we discuss this measure in the context of several specific functional forms.

II. The Question of Output Variability

Although there has been a bewildering number and variety of input substitution and demand parameters put forth by researchers, perhaps the most widely used measure is the simple price elasticity of input demand:

$$\varepsilon_{ij} = \left.\frac{\partial \ln X_i}{\partial \ln P_j}\right|_{Q,P_k} \text{ for all } k \neq j,$$

where X_i is the quantity demanded of the i^{th} input, P_j the price of the j^{th} input, and Q refers to total output. To be consistent with demand theory this is the elasticity whose sign should determine whether an input pair

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are substitutes or complements. It may also be expressed in a slightly different form: $A_{ij} = \varepsilon_{ij}/S_j$, where S_j is the share of the j^{th} input in total cost. A_{ij} is the Allen-Uzawa partial elasticity of "substitution", so called despite the fact that it is simply a normalized price elasticity of demand.

The elasticity ε_{ij} calculated under the assumption that output is held constant can indicate the characteristics of particular production surfaces, that influence policy direction, i.e. the matter of substitutability and complementarity among inputs. Combinations of ε_{ij} can also be used to construct higher-order elasticities of substitution to study such properties as curvature of the surfaces.

For matters of public policy, however, the assumption of constant output is often a disadvantage. There we are usually concerned with measuring the consequence of particular actions; the effects of given subsidies on capital, or of limits on land inputs, or of increases in energy prices, for example. To be complete we must take into account both input substitutions along given isoquants and the effects of output changes on input demand. What are needed in this case are measures of total elasticity:

$$\eta_{ij} = \left.\frac{\partial \ln X_i}{\partial \ln P_j}\right|_{P_k \text{ for all } k \neq j}$$

In this case quantities of all inputs as well as output are allowed to adjust to input price changes.

The expression that results (equation 4 below) is similar to the Slutsky equation of consumer demand theory. Thus, the general measure is somewhat analogous to the difference between ordinary and compensated demand curves in consumer behavior. The ε_{ij} above are analogous to elasticities on the compensated demand curve, whereas for predicting real-world changes in consumption we wish to know the elasticities of the ordinary demand curves. For most goods it will not make much difference which measure is used because consumers will normally spread their incomes over a large number of goods. In recent studies of production, however this is not the case; most studies have used production functions containing only three or four inputs. For many of these inputs, therefore, the difference between elasticities with and without output effect could be considerable.

One output-variable input demand elasticity has already been made available. In the case of constant returns to scale (CRTS) in production but a downward sloping output demand function, changes in output are produced when the cost function shifts; the extent of the output change being related to the price elasticity of output demand. A total input demand elasticity in this case was provided by Allen (p. 508). Output effects can also be produced, however, even if output price is constant, if the production function is characterized by non-CRTS. In this case shifts in a sloping cost (supply) curve over a horizontal demand curve produce changes in output. Of course output effects could result from both a downsloping output demand function and non-CRTS in production. In the next section we derive a general expression for the output-variable elasticity of subsitution.

III. A General Expression

To derive a general expression for η_{ij}, i.e., one that permits both non-constant returns and non-constant output price, we make use of the following relationships:

production function $Q = f(X)$

dual minimum cost function $C = C(P,Q)$

demand function $Q = \phi(R)$,

where X and P are n-tuples of input quantities and prices, respectively, Q is output and R is output price. Market clearing requires that marginal cost equal output price, or

(1) $$C_q(P,Q) = \phi^{-1}(Q)$$

We use subscripts to denote partial differentiation with respect to that variable. By differentiating (1) totally, setting dP = o for all but the j^{th} factor price and rearranging, we get:

(2) $$\frac{\partial Q}{\partial P_j} = \eta \frac{C_{qj}Q}{(R-C_{qq}Q\eta)}$$

where $\eta = \partial \ell nQ/\partial \ell nR$, the price elasticity of demand for output.

According to Shepard's lemma, $\partial C/\partial p_i = C_i = X_i(P,Q)$, the cost-minimizing demand curve for input i. Differentiating this demand curve with respect to the j^{th} factor price gives

(3) $$\frac{\partial X_i}{\partial P_j} = C_{ij} + C_{iq}\frac{\partial Q}{\partial P_j}$$

Using (2) and substituting appropriately 1/ gives

(4) $$\eta_{ij} = \left.\frac{\partial \ell nX_i}{\partial \ell nP_j}\right|_{P_k,\ k \neq j} = S_j(A_{ij} - \eta\psi) = \varepsilon_{ij} - S_j\psi\eta$$

where $S_j = P_jX_j/C$, the share of the j^{th} input in total cost; $A_{ij} = C\, C_{ij}/C_iC_j$, the Allen-Uzawa elasticity of "substitution" expressed in terms of the cost function, and

$$\psi = \frac{C_{iq}C_{jq}Q^2}{C_iC_j(1-R^{-1}C_{qq}Q\eta}\ .$$

Suppose we have a downward sloping demand curve: $0<|\eta|<\infty$, and CRTS. In this case marginal cost is constant or $C_{qq} = 0$, giving

$$\psi = \frac{C_{iq}C_{jq}Q^2}{C_iC_j} = 1$$

since CRTS implies $C_i = C\,C_{iq} \;\forall\; i$. This gives the expression derived by Allen (p. 508):

$$\eta_{ij} = S_j\;(A_{ij} + \eta).$$

There are two cases where the expression gives constant-output elasticities, either CRTS ($C_{qq} \to \infty$) or a perfectly vertical demand curve ($\eta = 0$).

The case that has not been considered before is that characterized by $\eta \to \infty$ and $C_{qq} > 0$, where the output effect is produced by the shifting of a sloping supply curve over a horizontal demand curve. In this case:

$$\lim_{\eta \to -\infty} (\eta\psi) = -\,A_{ij}\,\frac{C_{iq}C_{jq}}{C_{ij}C_{qq}}$$

giving:

(5)
$$\eta_{ij} = S_jA_{ij}\;(1 - \frac{C_{iq}C_{jq}}{C_{ij}C_{qq}}).$$

IV. Special Cases

It is of interest to consider the case of output price constant, quantity variable elasticity (equation (5)) in the case of specific production function forms. Several recent studies (Sidhu and Baanante; Yotopoulos, Lau and Lin) have used a non-CRTS Cobb-Douglas function: $\ell nQ = \sum_i \ell n X_i^{\alpha_i}$, with of course $A_{ij} = 1$, and $\Sigma\alpha_i = \mu < 1$.[2/] In this case, making appropriate substitutions into (5), and recognizing that $\alpha_j = S_j/\mu$,

$$\eta_{ij} = -\alpha_j \frac{1}{1-\mu} ,$$

a result that was derived originally by Lau and Yotopoulos. Note that, as long as decreasing returns to scale pertain (i.e., $\mu < 1$) all inputs will be judged "complements" ($\eta_{ij} < 0$) despite the fact that $A_{ij} = 1 \forall i, j$. This is another manifestation of the Cobb-Douglas inflexibility.

In the case of a multiple-output CES function:

$$Q = (\sum_i \alpha_i X_i^{-\beta})^{-\frac{\mu}{\beta}}$$

with a cost function of

$$C = Q^{\frac{1}{\mu}} \left(\sum_i \alpha_i^{\frac{1}{1+\beta}} P_i^{\frac{\beta}{1+\beta}} \right)^{\frac{1+\beta}{\beta}}$$

we have

$$\eta_{ij} = S_j \sigma \left(1 - \frac{1}{\mu}\right)$$

where $\sigma = \frac{1}{1+\beta}$.

Finally, suppose we have a translog cost function:

$$\ell n\, C = \ell n\, \alpha_0 + \sum_i \alpha_i \,\ell n\, P_i + \alpha_q \,\ell n\, Q + \tfrac{1}{2} \sum_i \sum_j \gamma_{ij} \,\ell n\, P_i \,\ell n\, P_j +$$

$$\sum_i \gamma_{iq} \,\ell n\, Q \,\ell n\, P_i + \tfrac{1}{2} \gamma_{qq} (\ell n\, Q)^2, \qquad i, j=1, \ldots, n.$$

In this case

$$\eta_{ij} = S_j A_{ij} \left[1 - \frac{(\gamma_{iq} + S_i S_q)\,(\gamma_{jq} + S_j S_q)}{(\gamma_{ij} + S_i S_j)\,(\gamma_{qq} + S_q^2)} \right]$$

where $S_i = \alpha_i + \sum_k \gamma_{ik} \,\ell n\, P_k + \gamma_{iq} \,\ell n\, Q \qquad i = 1, \ldots, n$

and $S_q = \alpha_q + \sum_k \gamma_{kq} \,\ell n\, P_k + \gamma_{qq} \,\ell n\, Q$

If factors i and j are substitutes, $(\gamma_{ij} + S_i S_j) > 0$. Furthermore, positive marginal cost, monotonicity and decreasing returns to scale imply, respectively $S_q > 0$, S_i, $S_j > 0$, and $\gamma_{qq} > 0$. Homotheticity requires that $\gamma_{iq} = 0 \; \forall \; i$, so in this case the output variable elasticity will be less than the output constant elasticity (i.e., $\eta_{ij} < S_j A_{ij}$). It need not be always negative, however, as is the case with the CD function. Also, if the function is sufficiently non-homothetic it could be that the "output" effect will lead to $\eta_{ij} > S_j A_{ij}$. Although, say, total output is decreased when the supply function shifts up (assuming decreasing returns to scale), non-homotheticity of the right type and magnitude (say γ_{iq} strongly negative while γ_{jq} close to zero) could give larger output-variable elasticities than output constant elasticities. In this case the "warping" of the isoquants is strong enough to offset the impact of the change in output.

FOOTNOTES

1/ We would like to thank James Houck and two anonymous referees for comments on an earlier version.

2/ This step makes use of the zero-profit condition C=RQ, and of the symmetry conditions $C_{ij} = C_{ji}$ and $C_{qi} = C_{iq}$.

3/ The non-CRTS functions in these studies referred to a subset of inputs from an overall function which is CRTS. A function which is not CRTS but for which in the short run, a subset of inputs are held fixed and the remainder are CRTS has short run elasticity measures identical to equation (36) except that the share S_j, the Allen partial elasticity A_{ij}, and the output elasticity, η, are based on the variable inputs alone. They could all be referred to as short run equivalents of the previous long run measures. Of course to analyze an overall production function in which some inputs are fixed and some variable would require a different analysis than that presented above.

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