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Risk - Mathematical model

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Risk-Efficient Production Plans Under Alternative Measures of Income Expectations

By

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ABSTRACT

Decision-making under uncertainty with income expectations conditioned on available information is contrasted with the standard risk modeling definition of expectations as mean income. Two MOTAD models are specified, a traditional MOTAD and one employing an ARMA model to develop conditional expectations. The analysis indicates that income variability may be reduced by conditioning expectations on relevant information.

Risk-Efficient Production Plans Under Alternative Measures of Income Expectations

Expected income-variance (E-V) frontiers are determined by finding the farm plans which yield the minimum income variance subject to fixed levels of expected farm income. Most methods of computing the efficiency locus estimate the variances and covariances of net income coefficients by assuming $E(g'x) = \bar{g}'x$, where $\bar{g}'$ is a row vector of average net incomes and x is a column vector of activities [Markowitz; Hazell; Anderson, Dillon, and Hardaker]. However, $\bar{g}'$ may be a poor representation of a rational manager's expectations of future net incomes. Using $\bar{g}'$ assumes the producer equally weighs all past net incomes and ignores any additional information about income variation. If additional information is available, it is unrealistic to believe that a rational manager would ignore such information which could explain some of the variability. More likely, the rational manager would implicitly use this information to reduce income variation when making decisions. Consequently, E-V efficiency functions derived by assuming $E(g'x) = \bar{g}'x$ may over-estimate income variances when information is available which could be used to reduce income variability.

For example, consider a production process with two activities, x1 and x2, which generate net incomes g1 and g2. If the producer uses the average net returns for both activities as the expectation of future revenues, then:

$$E(g'x) = \bar{g}_1 x_1 + \bar{g}_2 x_2$$

$$V(g'x) = x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2x_1 x_2 \sigma_1 \sigma_2 \rho_{12}$$

where: $\bar{g}i$ = average net revenue for activity i

σ_i = standard deviation of the net revenue of activity i

ρ_{ij} = correlation between net revenues of activities i and j

x_i = level of activity i.

The E-V efficiency locus may be found by parametrically varying L in the following programming model:

$$\text{(1)} \qquad \begin{aligned} \text{minimize} \quad V(g'x) &= x'W x \\ \text{subject to} \quad E(g'x) &= L \\ M x &\leq b \\ x &\geq 0 \end{aligned}$$

where:

g'= 1 x 2 vector of net revenue coefficients
x = 2 x 1 vector of activity levels
W = 2 x 2 covariance matrix of net revenues
M = 2 x 2 technical constraint matrix
b = 2 x 1 vector of constraint levels.

Now assume information I is available and the decision maker conditions revenue expectations on I. Further assume that the random vector $[g1\ g2\ I]'$ has a multivariate normal distribution with mean and covariance matrix:

$$\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu I \end{bmatrix}; \qquad \Sigma = \left[\begin{array}{cc|c} \sigma_1^2 & \sigma_{12} & \sigma_{1I} \\ \sigma_{12} & \sigma_2^2 & \sigma_{2I} \\ \hline \sigma_{I1} & \sigma_{I2} & \sigma_I^2 \end{array}\right] = \left[\begin{array}{c|c} W & \begin{array}{c}\sigma_{1I} \\ \sigma_{2I}\end{array} \\ \hline \sigma_{I1} \ \ \sigma_{I2} & \sigma_I^2 \end{array}\right].$$

As illustrated by Anderson (1958), the expectation of g1 and g2 conditional on I is:

$$E(g_i | I) = \mu_i + \sigma_{iI}/\sigma_I^2 (I - \mu_I); \quad i = 1,2$$

The covariance matrix of g1 and g2 conditional on I is:

$$\Psi = \begin{bmatrix} \sigma_1^2(1-\rho_{1_I}^2) & \sigma_1\sigma_2(\rho_{12}-\rho_{1_I}\rho_{2_I}) \\ \sigma_1\sigma_2(\rho_{12}-\rho_{1_I}\rho_{2_I}) & \sigma_2^2(1-\sigma_{2_I}^2) \end{bmatrix}$$

Using these covariances and expectations, a new programming model conditional on information I is:

$$\text{(2)} \qquad \begin{aligned} \text{minimize} \quad V(g'x|\ I) &= x'\ \Psi\ x \\ \text{subject to} \quad E(g'x|\ I) &= L \\ M\,x &\leq b \\ x &\geq 0 \end{aligned}$$

where g, x, M, and b are as previously defined, with the addition of I which is defined as a n x 1 vector of information, and:

$$E(g'x|\ I) = \{\mu_1 + \sigma_{1_I}/\sigma_I^2(I-\mu_I)\}x_1 + \{\mu_2 + \sigma_{2_I}/\sigma_I^2\ (I-\mu_I)\}x_2$$

$$V(g'x|\ I) = x_1^2\ \sigma_1^2(1-\rho_{1_I}^2) + x_2^2\sigma_2^2(1-\rho_{2_I}^2) + 2x_1x_2\sigma_1\sigma_2(\rho_{12}-\rho_{1_I}\rho_{2_I}).$$

If both ρ_{1_I} and ρ_{2_I} are positive, then income variance of the conditional model (2) will be smaller than the variance in the unconditional model (1). Comparison of models (1) and (2) indicates that if information is available which is positively associated with net income and if the decision maker conditions net revenue expectations on this information, then income variability can be reduced.

If income variability may be reduced by conditioning expectations on additional information, comparison of the conditional and unconditional E-V frontiers would be informative. In order to compare a conditional E-V frontier which utilizes additional information with the E-V efficiency frontier which ignores this additional information, appropriate means and covariances must be used. For the programming model used to obtain the unconditional E-V function, the expected net revenue coefficients and covariances will be the means and covariances calculated as usual over the sample period. For the programming model used to compute the conditional E-V function, the means and covariances conditional on information I will be incorporated.

The definition of information I and how it relates to net revenue expectations is of crucial importance. A complete econometric model of the agricultural economy could be used but, for purposes of comparison, a simpler model is specified. The conditional model specified is one which conditions the current net revenue coefficients upon past values of the net revenue coefficients. This was done by assuming the net revenue series for the activities followed some form of an autoregressive-moving average time series process. In other words, the conditional expectation of the net revenue coefficient for activity i in time t could be represented as a (p,q) order autoregressive moving average model:

$$E(g_t | I) = \mu + \Phi_1 g_{t-1} + \dots + \Phi_p g_{t-p} + \Theta_1 \varepsilon_{t-1} + \dots + \Theta_q \varepsilon_{t-q}$$

Here the information I represents the actual net revenue coefficients from the p previous periods and the actual residuals from the q previous periods.

MOTAD

With the conditional expectation estimated, the quadratic programming model for both the conditional and unconditional scenarios can be estimated and compared. However, parametric quadratic programming is computationally difficult requiring iterative procedures which sometimes yield inconsistent results. Hazell (1971) proposed MOTAD which employs the mean absolute income deviation (A) as a surrogate of income variance (V) to determine an approximate efficiency locus of production plans in the E-A plane. The main advantage of the E-A criterion is that linear programming methods may be used to derive the efficiency frontier.

The MOTAD model's risk measure for a two activity production process is:

$$A = (1/s) \sum_{h=1}^{s} |(gh - E(g))'x|$$

where: A = mean absolute income deviation
s = the number of sample periods used to estimate A
x = 2 x 1 vector of levels of farm activities
gh = 2 x 1 vector of net income coefficients for hth period
E(g)= 2 x 1 vector of estimated net income coefficients

Also, new variables yh, yh+ and yh- are defined as:

$$yh = (gh - E(g))'x \qquad \text{for } h = 1 \text{ to } s$$

such that $|yh| = yh+ + yh-$

and $yh+$ and $yh- > 0.$

This is to ensure that yh+ and yh- are selected so that when one is zero, the other is positive. The set of efficient production plans for expected income and mean absolute deviation are then obtained by parametrically varying L in the following linear model:

(3) $$\text{minimize } sA = \sum_{h=1}^{s} (yh+ + yh-)$$

such that $(gh - E(g))'x - yh+ + yh- = 0$

and $E(g'x) = L$

$$M\,x \leq b$$

$$x,\ yh+,\ yh- \geq 0$$

Estimates of the E-A efficiency frontier will not likely be as reliable as the estimates of the E-V efficiency frontier. As noted by Hazell, the estimate of population standard deviation generated from the sample mean absolute deviation is approximately 88% efficient for sufficiently large samples. However, it is felt that the computational advantages of MOTAD outweigh the loss of reliability for the purposes of this research.

An Application

Consider a 400 acre irrigated farm in southwest Nebraska which produces sugar beets, dry edible beans, corn for grain and hay. Also assume 1000 hours of labor are available. If farm overhead costs are constant for the length of the planning horizon and the income distribution of a farm plan is totally specified by the net income distribution, then approximate optimal farm plans may be obtained by using a MOTAD programming model. Subsequently, the E-A efficiency frontiers may be derived by parametrically letting L vary and obtaining a sequence of solutions.

To calculate the historical net revenue estimates, gross income per acre and costs per acre for each of the four included activities were required. Gross income per acre was estimated by obtaining prices received by producers for sugar beets, dry beans, corn and hay from 1954 to 1981 for Yuma County, Colorado and multiplying by the historical yields for the respective enterprises [Colorado Agricultural Statistics]. Yuma County, Colorado is adjacent to Chase County in Southwest Nebraska. Colorado prices were used due to lack of a data series of appropriate length for Chase County, Nebraska. All revenues and costs were converted to real (1972 = 100) dollars using the GNP price deflator.

Costs for the four activities were obtained from Estimated Crop and Livestock Production Costs assuming ditch irrigation technology in Chase County [Nebraska Cooperative Extension Service]. Annual per acre variable costs for each activity were obtained by deflating the 1980 nominal variable costs to the 1972 level. Historical activity costs were then calculated and the net revenue coefficients for each year were obtained.

Using MOTAD to calculate E-A efficiency functions requires estimates of expected net revenue coefficients for each activity. Following Hazell, when the decision maker does not base revenue expectations on additional information, $E(g) = \bar{g}$ is used in the MOTAD model where $\bar{g}$ is the vector of mean net revenue coefficients. If the decision maker does base net revenue expectations on additional information, net revenue expectations are hypothesized to follow an ARMA time series process. The exact ARMA specifications for each activity's net revenue for the years 1954 to 1981 was identified using Box-Jenkins (1976) methodology. All net revenue series were adequately modeled by an AR(1) process assuming stationarity:

$$E(g_{it} | I_{it-1}) = \mu_i + \Phi_i g_{it-1}$$

where g_{it} = net revenue coefficients for activity i in time t

I_{it-1} = information in time period t-1, here $I_{it-1} = g_{it-1}$

μ_i = overall series mean

Φ_i = autoregressive parameter

The expected net revenue for activity i in period t was estimated by conditioning on net revenue for period t-1. The unconditional expectation, $\bar{g}$ and conditional expectation, $E(gt|I)$, of net revenue for the years 1970 to 1981 were then substituted into the MOTAD model (3) and

E-A efficiency frontiers derived.

Results

Once conditional and unconditional expectations were estimated, covariance matrices of net revenue coefficients for the four activities were computed from residuals from each scheme. These covariance matrices are displayed in Table 1. All elements in the conditional covariance matrix were smaller than the elements in the unconditional matrix, indicating that conditioning on additional information should reduce income variance.

E-A efficiency frontiers were then derived. Table 2 shows optimum farm plans for different levels of expected income for both the unconditional and conditional models. Optimum activity levels from both models differ substantially. The efficiency frontiers shown in Figure 1 illustrates that the income variance from the conditional model was smaller than the income variance from the unconditional model through the shift upward to the left. There appeared to be a substantial decrease in income variance if revenue expectations were conditional on additional information.

Conclusions

MOTAD E-A efficiency frontiers are the loci of expected net incomes and mean absolute deviations associated with efficient farm plans. These E-A functions are estimated by assuming the decision maker's expected net revenue for each activity is the sample mean net revenue from historical data. In other words, the operator weighs all past net revenue coefficients equally and does not use any additional information in forming his expectations of future net revenue.

If additional information associated with future net revenue is available, the rational decision maker will use this information to reduce income variability. Consequently, using the sample mean net revenue to identify optimal short-run activity mixes may result in estimated E-A frontiers which indicate suboptimal portfiolios. Caution should be exercised when utilizing farm plans developed using unconditional MOTAD E-A efficiency frontiers which over estimate variance as a short-run planning mechanism.

Table 1. Income Variance-Covariance Matrices, Conditional and Unconditional Models

Unconditional Variances and Covariances

	Beets	Beans	Corn	Alfalfa
Beets	30856	27808	5669	-1180
Beans	27808	28559	5099	-2012
Corn	5669	5097	1293	-134
Alfalfa	-1180	-2012	-134	874

Conditional Variances and Covariances

	Beets	Beans	Corn	Alfalfa
Beets	27861	21611	4754	738
Beans	21611	21246	3904	655
Corn	4754	3904	1284	187
Alfalfa	738	655	187	214

det (cond cov)/det (uncond cov) = .725

Table 2. MOTAD Optimal Farm Plans
Conditional and Unconditinal Models

Farm Plan	Expected Income ($)	Absolute Deviation ($)	Sugar Beets (acres)	Dry Beans (acres)	Corn (acres)	Alfalfa Hay (acres)
Unconditional Model						
A	$39,018	$7,000	0	22	77	301
B	$65,971	$18,086	105	43	0	252
C	$85,674	$29,706	192	45	0	163
D	$104,334	$41,328	316	0	0	84
E	$122,877	$52,948	400	0	0	0
Conditional Model						
F	$23,960	$4,690	0	0	0	400
G	$64,462	$15,100	57	0	260	83
H	$89,827	$25,600	163	0	223	14
I	$111,789	$36,100	280	0	120	0
J	$132,996	$46,647	400	0	0	0

Table 3. Q. P. Optimal Farm Plans
Conditional and Unconditinal Models

Farm Plan	Expected Income ($)	Income Variance (^{2})	Sugar Beets (acres)	Dry Beans (acres)	Corn (acres)	Alfalfa Hay (acres)
Unconditional Model						
A	$39,018	8.25×10^7	0	30	61	289
B	$65,971	58.8×10^7	114	32	0	254
C	$85,674	158×10^7	231	0	0	169
D	$104,334	309×10^7	241	86	0	73
E	$116,570*	458×10^7	182	218	0	0
Conditional Model						
F	$23,960	2.18×10^7	0	0	84	182
G	$64,462	24.4×10^7	13	0	387	0
H	$89,827	111×10^7	156	0	244	0
I	$105,000	247×10^7	288	0	60	0
J	$107,000	285×10^7	319	0	7	0

* expected incomes differ from MOTAD model

Figure 1. E–A Frontiers,

Conditional and Unconditional Models

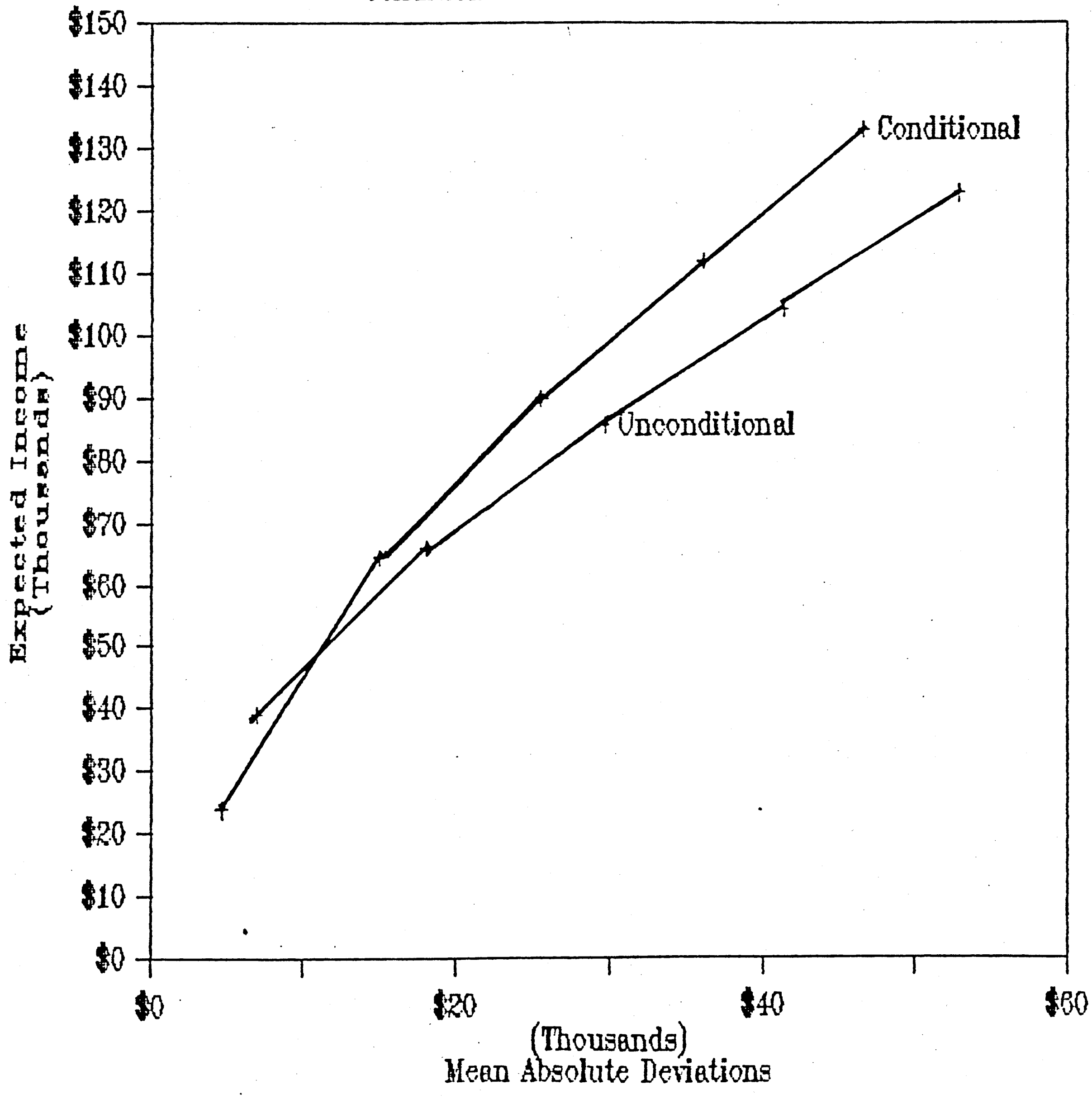
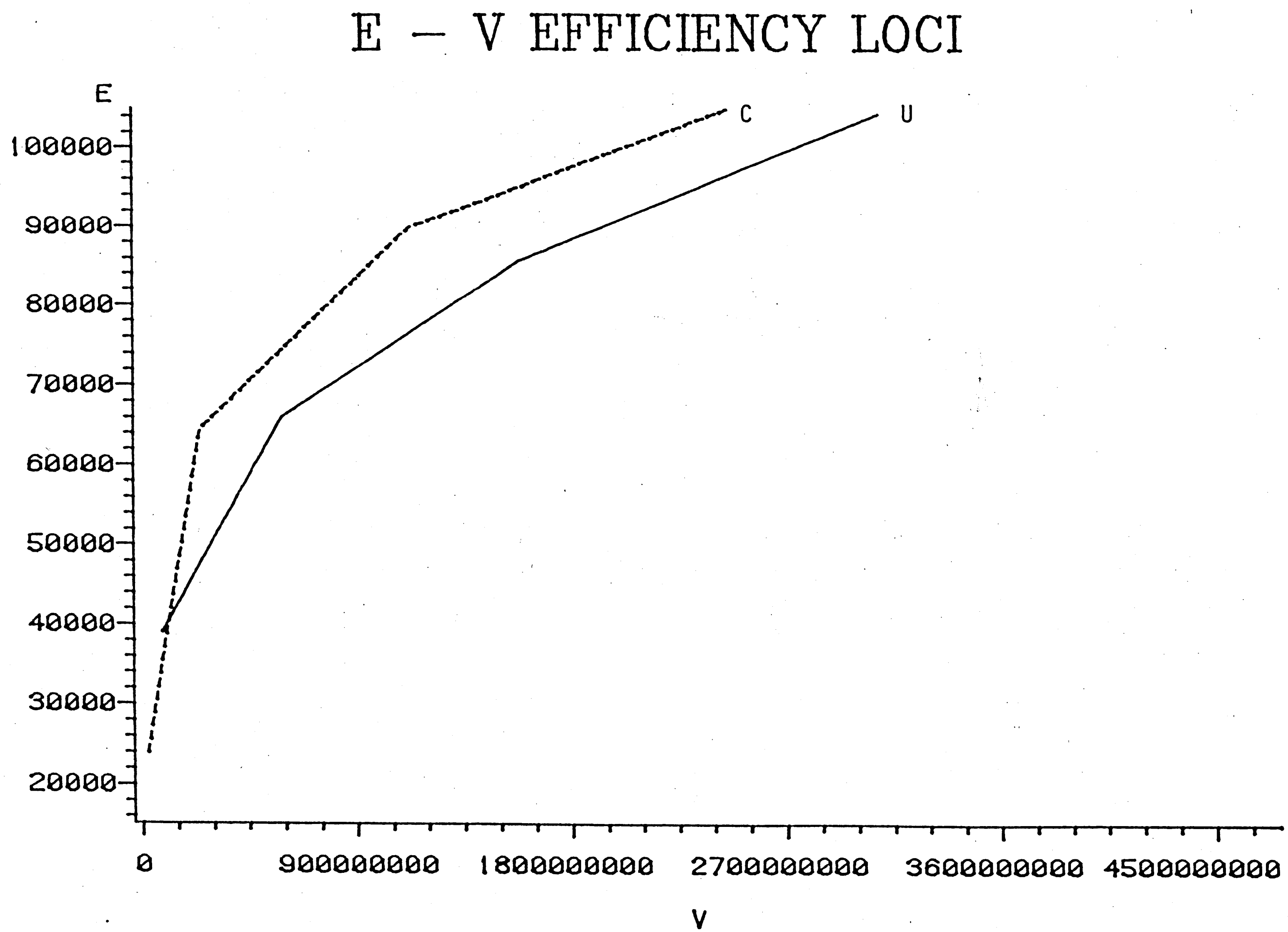


Figure 2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