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AN OPERATIONAL EXTENSION OF THE LEONTIEF DYNAMIC INPUT-OUTPUT MODEL

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ABSTRACT

Although conceptually superior to its static counterpart, the dynamic input-output model has failed to achieve popularity due to its virtual inoperability. Key extensions to the Leontief dynamic model facilitate specification using numerical and simulation techniques. The extended model's operability and capabilities are demonstrated using county-level data.

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In recent years static input-output models, estimated from primary or derived from secondary data, have been widely used in economic planning and impact analysis. The limitations of static models are well known. One of the more telling shortcomings is the inability of the static system to project the time path of local economic adjustment to changes in external conditions or internal structure and technology. The alternative approach involving estimation of Leontief-type dynamic input-output models is impractical, and incorporates behaviorally inconsistent assumptions. A modified Leontief dynamic model is presented below and applied to a rural economy in east-central Oregon: Grant County. Results suggest that the modified dynamic approach is superior to both the static and the Leontief dynamic formulations.

The Leontief Dynamic Input-Output Model

The desirability of dynamic representations of economic activity has long been recognized. Leontief suggested the practical importance of determining "...the empirical law of change of a particular economy". And further, "dynamic theory ... shows how certain changes in the variables can be explained on the basis of fixed, i.e., invariant, structural characteristics of the system" [Leontief, et. al., 1953, p. 53]. Leontief developed a dynamic extension to his static input-output model by explaining investment as a fixed response to changes in yearly output (production).

The static model is based on the identity

$$(1) \quad \vec{X} = A\vec{X} + \vec{Y},$$

which has the simultaneous solution

$$(2) \quad \vec{X} = [I - A]^{-1}\vec{Y},$$

where $\vec{X}$ is an nx1 vector of industry outputs, A is an nxn matrix of input-output (technical) coefficients, I is an nxn identity matrix, and $\vec{Y}$ is an nx1 vector of final demands. The Leontief dynamic model adjusts this identity by including the purchase or liquidation of capital stock

$$(3) \quad \vec{X}(t) = A\vec{X}(t) + \vec{Y}(t) + \dot{\vec{S}}(t)$$

where $\dot{\vec{S}}(t)$ is an nx1 vector of time derivatives of capital stock. The functional notation indicates that each element of these vectors is a continuous function of time. Specifically, $\dot{\vec{S}}(t)$ is the derivative of $\vec{S}(t)$, an nx1 vector of capital stocks related to output as follows:

$$(4) \quad \vec{S}(t) = B\vec{X}(t).$$

Here B is an nxn matrix of capital requirement coefficients such that b_{ij} is the stock of product i required to produce \$1.00 of product j, and

$$(5) \quad \dot{\vec{S}}(t) = B\dot{\vec{X}}(t).$$

Equation (3) can be rewritten as the first order differential equation

$$(6) \quad \vec{X}(t) = A\vec{X}(t) + \vec{Y}(t) + B\dot{\vec{X}}(t).$$ 1/

The solution to this system is

$$(7) \quad \vec{X}(t) = \hat{C}K \exp(t\vec{\lambda}) + \vec{L}(t),$$

where K is an nxn matrix and $\vec{\lambda}$ an nx1 vector of coefficients depending on the coefficients in A and B. $\hat{C}$ is an nxn diagonal matrix of constants reflecting the initial conditions (i.e., constants of integration). $\vec{L}(t)$ is an nx1 vector of relationships describing the time path of final demand for the n outputs [Leontief, et. al., 1953, pp. 76-82].

If the $\vec{L}(t)$ functions are completely determined then equation (7) represents a system of n equations in 2n unknowns -- the values of $\hat{C}$ which are determined by the n initial conditions ($\vec{X}(0) = \vec{X}^0$) and the n terminal conditions ($\vec{X}(T) = \vec{X}^T$). If either the original or terminal values of $\vec{X}(t)$ are specified, the others may be calculated for any value of t. If both $\vec{X}^0$ and $\vec{X}^T$ are given, then as many as n parameters of the $\vec{L}(t)$ function may be calculated -- that is, the final demand time paths required to achieve any terminal state from any original state may be deduced.

The Leontief dynamic input-output system has a number of superior characteristics relative to its static counterpart. The major advantages of the dynamic system are easily enumerated. (1) It allows the use of expected time paths of exogenous changes (final demand) rather than simply cumulative changes. (2) The time path of endogenous variables rather than only their terminal levels are projected. (3) Investment is made endogenous to the system (and the accelerator effect of increased demand is recognized). (4) A system's initial conditions affect its performance.

On the other hand, and as Leontief recognized, the theory is not general since it incorporates only some of the relevant determinants of the dynamic economic adjustment process. Furthermore, certain important aspects of the model seem particularly limiting. (1) The model requires exact equation of production and consumption at each point in time, whereas in reality changes in inventory allow short-run independence between production and consumption. (2) The model is unstable because of the assumption of full capacity utilization [Petri, Sargan]. (3) The model allows for complete reversability of investment in capital stocks, when in reality excess capacity would occur during periods of falling demand rather than liquidation of capital stock. (4) The capital coefficients matrix is usually singular preventing certain types of solution procedures [Kendrick, Livesey]. (5) Certain solution procedures lead to results which are inconsistent with the initial conditions [Kendrick, Schinnar]. (6) The method often generates infeasible projections (negative levels of production or investment for example).

Extension of the Leontief Dynamic Input-Output Model

In the following discussion the Leontief dynamic system is generalized, with emphasis placed on removal of the limitations listed above. The balance equation is first amended to allow changes in inventories. This has the effect of making production and consumption independent in the short-run. Equations are then suggested which relate the level of output to supply and demand conditions, the level of capital stocks to changes in capacity, actual capacity to desired capacity, and desired capacity to changes in demand.

Consider the revised form of the balance equation (3),

$$(8) \quad \vec{X}(t) = A\vec{X}(t) + \vec{Y}(t) + \vec{I}(t) + \dot{\vec{N}}(t),$$

where $\dot{\vec{N}}(t)$ is an nx1 vector of changes in the inventories of each commodity, $\vec{I}(t)$ is gross investment and all other variables are as before. Rearranging,

$$(9) \quad \dot{\vec{N}}(t) = \vec{X}(t) - A\vec{X}(t) - \vec{Y}(t) - \vec{I}(t).$$

This framework allows $\vec{X}(t)$ to equal something other than the sum of $A\vec{X}(t)$, $\vec{Y}(t)$ and $\vec{I}(t)$ as before. Following Sargan, it is hypothesized that the level of $\vec{X}(t)$ is determined by the behavioral relationship,

$$(10) \quad \dot{\vec{X}}(t) = f(\vec{N}(t)), \text{ subject to } \vec{X}(t) \leq \vec{X}^{c}(t),$$

that is, the rate of change in $\vec{X}(t)$ is some function of the rate of change in inventories subject to the constraint that $\vec{X}(t)$ does not exceed capacity, $\vec{X}^c(t)$. If the relationships in equation (10) are assumed to be linear then they may be expressed as

$$(11)\quad \dot{\vec{X}}(t) = \Phi(-1)\dot{\vec{N}}(t), \quad \hat{\Phi} > 0,$$

where $\hat{\Phi}$ is an nxn diagonal matrix. If $\phi_{ii} = \phi_{jj} = \phi$ for all i, j = 1, 2, ..., n, then

$$(12)\quad \dot{\vec{X}}(t) = \phi(-1)\dot{\vec{N}}(t).$$

The next step is to develop an investment function which takes into account the criticisms listed above. Such a function may be of the form

$$(13)\quad \vec{I}(t) = B[\hat{D}\vec{X}^c(t) + \dot{\vec{X}}^c(t)], \text{ subject to } \dot{\vec{X}}^c(t) \geq (-1)\hat{D}\vec{X}^c(t)$$

where $\hat{D}$ is an nxn diagonal matrix of yearly capacity depreciation rates. Equation (13) implies that investment is made up of two parts -- replacement investment, $B\hat{D}\vec{X}^c(t)$, and net (induced) investment, $B\dot{\vec{X}}^c(t)$. Replacement investment is that portion of investment which exactly maintains capital stocks, while net investment is that portion of total investment which depends on (is induced by) the level of demand. The constraint allows net investment to assume negative values when demand falls, but limits this disinvestment to levels less than capacity depreciation. Hence gross investment, $\vec{I}(t)$, is constrained to non-negative values. This approach was suggested by Leontief [1966], but he was unable to satisfactorily incorporate it into his analytic model.

Next a behavioral relationship which predicts the actual capacity vector $\vec{X}^c(t)$, is required. The hypothesis made here is that the desired capacity at any given time is a function of the level of demand at that time. If this function is linear then

$$(14)\quad \vec{X}^{c*}(t) = \vec{\alpha} + \hat{\beta}[A\vec{X}(t) + \vec{Y}(t) + \vec{I}(t)],$$

where $\vec{X}^{c*}(t)$ is the desired capacity, $\vec{\alpha}$ is an nx1 vector of intercepts (representing a constant buffer of excess capcity), and $\hat{\beta}$ is a diagonal matrix of slopes (representing the ratio of capacity to demand). Capacity is in turn some delayed or lagged function of desired capacity. This lag may be discrete as in

$$(15)\quad \vec{X}^{c}(t) = \vec{X}^{c*}(t - \tau),$$

where τ is a constant structural lag or it may be a continuous, exponential type lag as in

$$(16)\quad \vec{X}^{c}(t) = \vec{X}^{c*}(t) + [\vec{X}^{c}(0) - \vec{X}^{c*}]\exp[-mt/k]\sum_{j=0}^{m-1} \frac{(mt/k)^j}{j!} .$$

Equation (16) is a generalized m^{th} order exponential lag (or smoothing) function which describes the time path of $\vec{X}^{c}(t)$ from its initial level, $\vec{X}^{c}(0)$ at $t = \infty$. The constant, k, determines the length of the delay; and the order of the exponential lag, m, determines the shape of the time path of capacity adjustment. The constant, k, can be replaced by a vector $\vec{k}$ if empirical evidence indicates that the lag differs among sectors. As the order of the delay increases, the lag function in equation (16) approaches the discrete lag in equation (15).

This lag structure has been suggested and interpreted by Allen and by Bargur. An alternative lag structure is offered by the logistics curve. While its economic interpretation is less evident, the logistics curve closely resembles the upper order exponential lags but has a number of practical advantages. The differential forms of the first order exponential lag and the logistics function are respectively:

$$(17)\quad \dot{\vec{X}}^{c}(t) = [\vec{X}^{c*}(t) - \vec{X}^{c}(t)](1/k), \text{ and}$$

$$(18)\quad \dot{\vec{X}}^{c}(t) = [\vec{X}^{c}(t) - [\vec{X}^{c}(t)]^2/\vec{X}^{c*}(t)](1/k).$$

This completes the model. By making appropriate substitutions this model (ignoring constraints) can be described by 4n simultaneous differential equations in 4n unknowns. The equations are:

$$
(19)\quad
\begin{aligned}
\dot{\vec{X}}(t) &= \phi[A\vec{X}(t) + \vec{Y}(t) + \vec{I}(t) - \vec{X}(t)],\\
\vec{I}(t) &= B[\hat{D}\vec{X}^{c}(t) + \dot{\vec{X}}^{c}(t)],\\
\dot{\vec{X}}^{c}(t) &= [\vec{X}^{c*}(t) - \vec{X}^{c}(t)](1/k), \text{ and}\\
\vec{X}^{c*}(t) &= \vec{\alpha} + \hat{\beta}[A\vec{X}(t) + \vec{Y}(t) + \vec{I}(t)].
\end{aligned}
$$

The 4n endogenous unknowns are $\vec{X}(t)$, $\vec{I}(t)$, $\vec{X}^{c}(t)$, and $\vec{X}^{c*}(t)$. The exogenous variables include A, B, $\hat{D}$, $\vec{Y}(t)$, ϕ, k, $\vec{\alpha}$, $\hat{\beta}$, $\vec{X}(0)$, $\vec{X}^{c}(0)$, and t.

This system of equations has an analytic solution but it is complicated. Furthermore, a singular B matrix and the inequality constraints,

$$
(20)\quad
\begin{aligned}
\vec{X}(t) &\le \vec{X}^{c}(t), \text{ and}\\
\dot{\vec{X}}^{c}(t) &\ge (-1)\hat{D}\vec{X}^{c}(t),
\end{aligned}
$$

make an analytic solution impractical.

Fortunately, the solution of these equations can be greatly simplified by numerical integration and systems simulation techniques. Numeric estimates of the endogenous variables can be made as close as desired to their true analytic values. Since the matrices need not be inverted or their determinants calculated, the possible singularity of the B matrix is immaterial. The discontinuities created by the constraints are easily handled. Further, systems simulation allows the addition of stochastic elements to exogenous variables where they are subject to known or estimatable probability distributions. The following sections describe such a numeric technique.

An Application

The model portrayed in equations (19) and (20) forms the basis of the Dynamic Regional Economic Adjustment Model (DREAM) [Johnson, Obermiller, and Van Kooten]. The model is written in the GASP IV simulation language [Pritsker] and is currently being used on Oregon State University's Cyber 73 computer system. GASP IV is a FORTRAN based library of subroutines which provide the user with the time advance, numeric integration, random deviate generation, data computation and reporting, and other functions. The simulator generalizes the system in equations (19) by incorporating a number of alternative functional forms. For example, the user may choose between equation (17), (18) or Leontief's implicit assumption that

$$(21) \quad \vec{X}^c(t) = \vec{X}(t).$$

The simulator also allows the user to make investment exogenous, to specify the level of output of certain sectors, and to make a number of other assumptions which enhance the models' flexibility and applicability.

To demonstrate the features of the simulator, a sample simulation using equations (19) and (20) is presented. The model is fitted with data from the Grant County Input-Output Model [Obermiller and Miller] including input-output flow coefficients (A matrix), capital coefficients (B matrix), capacity depreciation rates ($\hat{D}$ matrix), capital/demand ratios ($\hat{\beta}$ matrix) and initial conditions ($\vec{X}(0)$, $\vec{Y}(0)$ and $\vec{X}^c(0)$). Other data are reported in Johnson.

The scenario chosen is the same as that used by Obermiller and Miller. An increase in allowable timber cut worth $12,647,000 in sales to the Wood Products industry is assumed. This increase is expected to materialize over a one-year period. All other final demand levels are assumed to remain constant.

The results of the simulation are illustrated in Figures 1 and 2. Figure 1 displays the level of output in selected sectors while capacity and output (including output foregone due to capacity constraints), investment response and final demand in the Wood Products sector are reproduced in Figure 2.

A number of interesting features of the model are highlighted in the two figures. First, the immediate movement of some variables away from the initial conditions is apparent. Since the initial conditions were 1977 static equilibrium levels, this movement indicates that static and dynamic equilibria are not necessarily equal, suggesting that the system was in a process of growth in 1977.

A second observation relates to the effects of capacity constraints on output levels. For example, the Wood Products industry produces at full capacity between 78.47 and 79.56. In contrast, the Timber Harvesting and Hauling industry is initially at capacity and continues to produce at full capacity in spite of a major investment program throughout the simulation period.

A final observation involves the time path of sectoral adjustment to an initial economic shock, such as the 12.6 million dollar increase in Wood Products activity. The increase in exports begins at 77.5, and ceases at 78.5. However, its induced and indirect consequences continue throughout the simulation period. Investment peaks at about 80.04 followed closely by Construction activity at 80.17. Income peaks next at 81.04, followed by Automotive Sales and Services (81.08), Wholesale-Retail Trade (81.17) and Local Taxes (81.33).

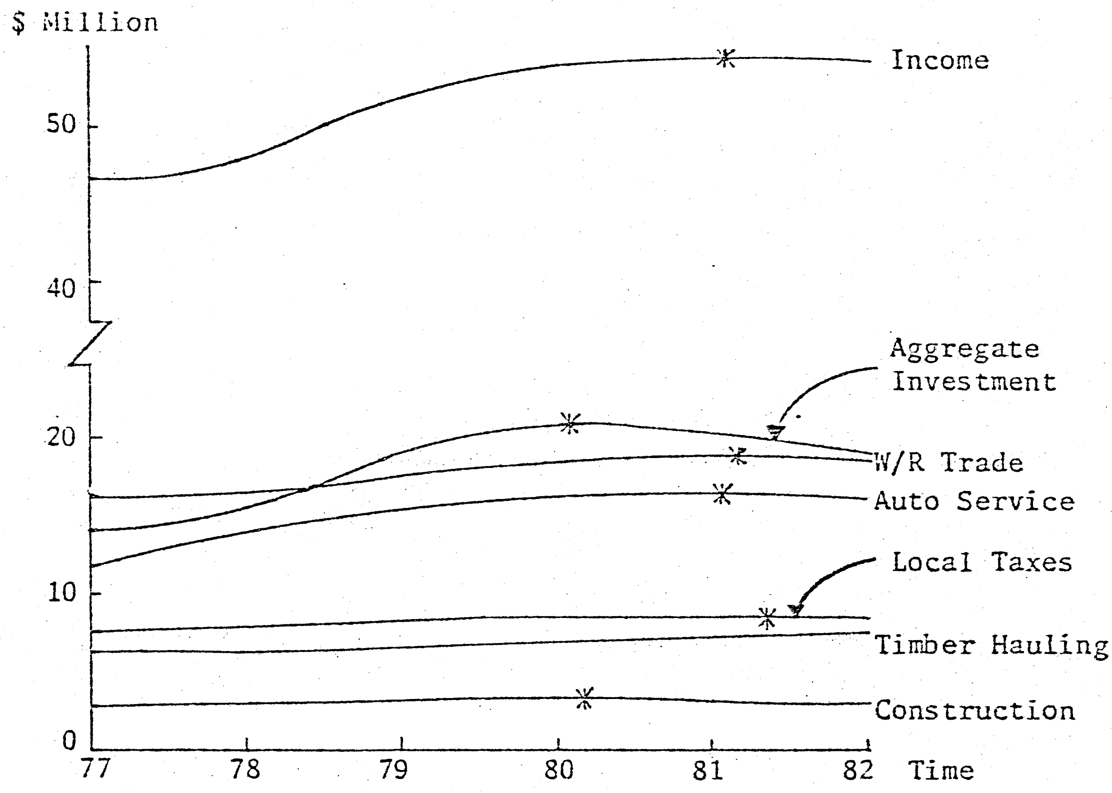

Figure 1. Selected Variables.
Note: * indicates local Max

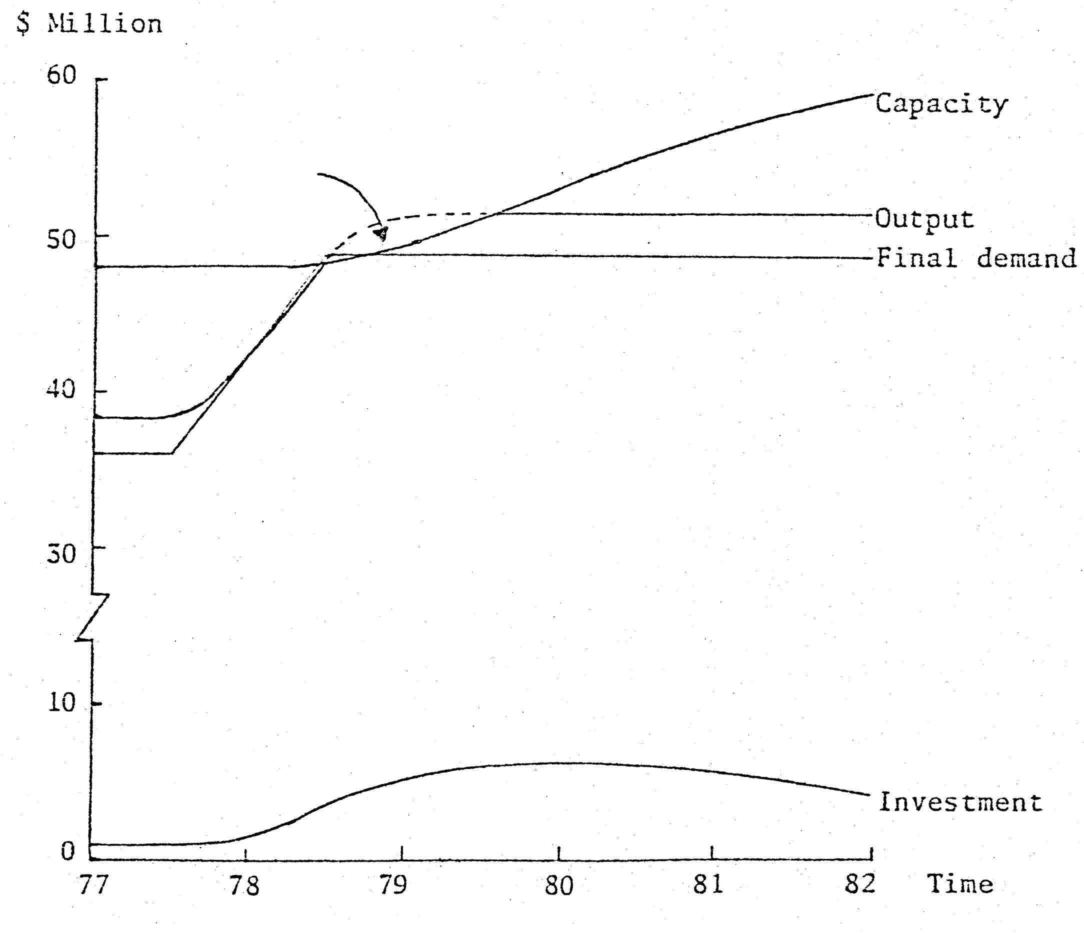

Figure 2. Output, Capacity, Final Demand and Investment in Wood Products Industry.

Conclusions

This paper demonstrates both the practicability and the empirical value of an extended Leontief dynamic input-output model. The various extensions transform an otherwise unrealistic and unworkable model into one exhibiting such desirable characteristics as stability, continuity, recursiveness, flexibility and efficiency. Still more extensions are possible. Such features as technological change, non-linear production and investment functions, factor supply constraints, and stochastic relationships could be incorporated. In addition, the model could be modified to accommodate various types of inputs and outputs and interaction between machine and user.

FOOTNOTES

1/ In subsequent articles Leontief 1966, 1970 introduced a discrete form of the dynamic model wherein the time derivative of $\vec{X}$ is replaced by $(\vec{X}_{t+1} - \vec{X}_t)$ where the subscripts indicate the time period to which the variables refer. Equation (6) becomes the first order difference equation

$$\vec{X}_t = A\vec{X}_t + \vec{Y}_t + B(\vec{X}_{t+1} - \vec{X}_t).$$

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