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NECESSARY CONDITIONS FOR DSD EFFICIENCY OF MIXTURES OF RISKY ALTERNATIVES*

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"Necessary Conditions for DSD Efficiency of Mixtures of Risky Alternatives."

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Necessary conditions for DSD (decreasing absolute risk aversion stochastic dominance) efficiency of mixtures of risky alternatives are presented. A concave programming problem which is equivalent to these conditions is stated. A simple example is used to illustrate these ideas. An ex post analysis provides additional insight.

NECESSARY CONDITIONS FOR DSD EFFICIENCY OF MIXTURES OF RISKY ALTERNATIVES

Historically, mean-variance and mean-absolute deviations criteria have been used to find appropriate mixtures of risky investments, production activities and/or marketing alternatives. These criteria have been criticized because they are not always consistent with expected utility theory. More recently, methods more consistent with expected utility theory have been presented. These include the Target MOTAD model which was independently developed by Tauer and Watts, Held and Helmers as well as Porter's mean-target semivariance model. These models are useful for identifying selected subsets of the second (SSD) and third (TSD) degree stochastic dominance efficient sets. For some problems, they can identify all or major portions of the SSD and TSD efficient sets.

Although SSD and TSD are commonly used stochastic dominance criteria they are not the only criteria that could ever be of interest. A more stringent criterion is Vickson's decreasing (actually nonincreasing) absolute risk aversion stochastic dominance (DSD) criterion. TSD, which exploits a necessary condition for decreasing absolute risk aversion, is often used as a substitute for DSD. The use of TSD by agricultural economists suggests that the assumption of decreasing absolute risk aversion is sometimes accepted. Several studies (Hamal and Anderson; Hildreth and Knowles; Lins, Gabriel and Sonka; and Morin and Suarez) provide empirical evidence of decreasing absolute risk aversion.

Although Vickson's algorithm can be applied to any pair of probability distributions, using it to determine the DSD efficiency status of mixtures of risky alternatives would be tedious. An article by Dybvig and Ross provides the basis for an alternative approach. Although their article is most directly applicable to the SSD criterion, it provides ideas which can be applied to other criteria. This paper exploits these ideas to develop necessary conditions for DSD efficiency of mixtures of risky alternatives. Then, a method for determining when these conditions are satisfied is described. This method is illustrated by applying it to a simple example.

Basic Assumptions and Notation

Some of the assumptions used in this paper are similar to those adopted by others. A finite number, s, of states of nature is assumed. p denotes a row vector of probabilities associated with these states of nature. The elements of the column vector, y, are the (total) net returns associated with the various states of nature. This net returns vector is related to enterprise activity levels as follows.

(1) $y - Cx = 0$

x is a column vector of n activity levels. C is a matrix of per unit net returns associated with the activities and the states of nature. Specifically, C_{ij} is the net return per unit of activity j when the ith state of nature occurs.

Activity levels are restricted by resource and/or technical constraints as well as nonnegativity constraints.

(2) $Ax \leq b$

(3) $x \geq 0$

In (2), A is a matrix of resource or technical requirements and b is a column of resource levels.

Necessary Conditions for DSD Efficiency

The approach used to obtain necessary conditions for DSD efficiency is similar to that which Dybvig and Ross used to obtain necessary conditions for

SSD efficiency.[1/] Their conditions combine properties of optimal solutions to concave programming problems with properties common to the relevant class of utility functions.

The relevant class for Vickson's DSD criterion is the class of decreasing (nonincreasing) absolute risk aversion (DARA) utility functions. Vickson defines the DARA class of functions as those functions, u, for which

$$(4)\quad u'(y) = u'(a) \cdot \exp[-\int_a^y r_u(q)dq]$$

where u' is the derivative of u and r_u is a nonnegative, nonincreasing, piecewise smooth function. u'(a), the value of the derivative at a fixed net returns level, a, is (implicitly required to be) positive.

Following Dybvig and Ross, the necessary conditions for DSD efficiency can be obtained by considering the problem of maximizing the expected utility for a DARA function subject to (1), (2) and (3). Suppose y^o (= Cx^o) maximizes this function. Then, there must exist a support vector z^o which satisfies the inequality

$$(5)\quad z^{o'}y^o \geq z^{o'}y$$

for all y vectors satisfying (1), (2) and (3). z^o can be regarded as a vector of relative shadow prices for the net returns associated with various states of nature or as a generalized marginal expected utilities vector. Any element of z^o, z_i^o, is the product of p_i and the relative marginal utility of y_i^o which is denoted by w_i^o in this paper.

$$(6)\quad z_i^o = p_i w_i^o$$

The balance of the necessary conditions for DSD efficiency are based on characteristics common to the class of DARA functions associated with the DSD criterion. Two of these characteristics are obvious implications of (4). One is that marginal utility of net returns must be positive for all levels of net returns.

This means the marginal utility vector and the associated support vector must contain only positive elements. That is,

$$(7) \quad w^o,\ z^o > 0.$$

Since r_u is a nonnegative function, it is obvious that an additional necessary condition for DSD efficiency of y^o is

$$(8) \quad w_i^o \geq w_j^o \quad \text{if} \quad y_i^o < y_j^o.$$

A somewhat stronger necessary condition can also be derived from (4). It is[2/]

$$(9) \quad w_i^o/w_j^o \geq (w_j^o/w_k^o)^{(y_j^o - y_i^o)/(y_k^o - y_j^o)} \quad \text{if} \quad y_i^o < y_j^o < y_k^o.$$

Comparison with Necessary Conditions for TSD Efficiency

Analogous necessary conditions for TSD efficiency are the same as those stated above except that (9) is replaced by

$$(10) \quad \frac{w_i^o}{w_j^o} \geq 1 + \frac{(y_j^o - y_i^o)(w_j^o - w_k^o)}{(y_k^o - y_j^o) \quad w_j^o} \quad \text{when} \quad y_i^o < y_j^o < y_k^o.$$

It is possible to show that the DSD necessary condition (9) requires w_i^o/w_j^o to be strictly greater than the right hand side of (10) unless y_j^o equals y_i^o or w_j^o equals w_k^o. Thus, as would be expected, the necessary conditions for DSD efficiency are more stringent than those for TSD efficiency.

A Concave Programming Formulation of the Necessary Conditions

The necessary conditions pose a saddlepoint problem. The statement of an equivalent concave programming problem is simplified by assuming that the indices for the states of nature have been permuted so that[3/]

$$(11) \quad y_1^o < y_2^o < \ldots < y_s^o.$$

The concave programming problem is

$$(12) \quad \text{Minimize } b'v - z'y^o$$

subject to

(13) $A'v - C'z \geq 0$

(14) $z_i - p_i w_i = 0$ for $i = 1, 2, \ldots, s$

(15) $w_s = 1$

(16) $w_{s-1} - w_s \geq 0$

(17) $w_{i+1}(w_{i+1}/w_{i+2})^{(y^o_{i+1} - y^o_i)/(y^o_{i+2} - y^o_{i+1})} - w_i \leq 0$

for $i = 1, 2, \ldots, s-2$

(18) $v \geq 0$ (the signs of z and w are not formally restricted)

Relation to Necessary Conditions

The objective function can be related to (5) by noting that $b'v - z'y^o$ is the maximum (with respect to feasible y vectors) of $z'(y - y^o)$. This can be verified by remembering that, if the feasible set of y vectors is bounded, it can be defined as the set of convex combinations of extreme or "corner" y vectors.[4/] If this definition is used instead of (1) through (3), then b would be replaced by 1 and (13) would be replaced by

(19) $v - z'y^k \geq 0$ for $k = 1, 2, \ldots, t$

where y^k is the kth extreme or corner vector associated with the feasible set of y vectors and t is the number of extreme vectors.

(14) and (17) are merely restatements of (6) and (9); (16) and (17) ensure that (8) is satisfied. (15) guarantees that marginal utility is always positive. A right hand side value of 1 was arbitrarily selected for (15); any other positive constant could have been chosen.[5/] (13) requires the imputed shadow prices for the resources to be large enough to guarantee that, at the margin, the value of the resources used by each activity (or enterprise) is at least as large as the expected marginal utility of the net returns distribution associated with that activity.

DSD Efficiency Test Criteria

If the optimal value of the objective function equals zero, then y^o satisfies the necessary conditions for DSD efficiency; otherwise it does not. When available, the solution to the dual of the concave programming problem provides a more sensitive indication of the efficiency status. The portion, x^*, of the dual solution vector associated with (13) can be interpreted as an enterprise mixture vector. Cx^* equals y^o when y^o (and x^o) satisfies the necessary conditions for DSD efficiency, but does not equal (and tends to be very different from) y^o if the necessary conditions are not satisfied.[6/]

Comparison with Vickson's DSD Algorithm

Vickson's algorithm was designed to determine whether two probability distributions can be ordered by the DSD criterion. Although the problem considered in this paper is more complex, it appears that a modified version of Vickson's algorithm could solve it. Regardless of the solution method adopted, at least one characteristic is shared by the solutions to the concave programming problem stated in this paper and Vickson's algorithm. In both cases, the function r_u is approximated by a step function.

An Example

Some of the ideas presented above are illustrated with data from Anderson, Dillon and Hardaker (pp. 209-210). The feasible set of crop mixes is portrayed in Figure 1. Selected feasible mixtures are presented in Table 1. A and G are two corners of the feasible set of crop mixtures. Mixtures B through F are convex combinations of these corner mixtures.

The set of mixtures which satisfy the necessary conditions for DSD efficiency must be a subset (perhaps improper) of the set of mixtures satisfying the necessary conditions for TSD efficiency. The mixtures which

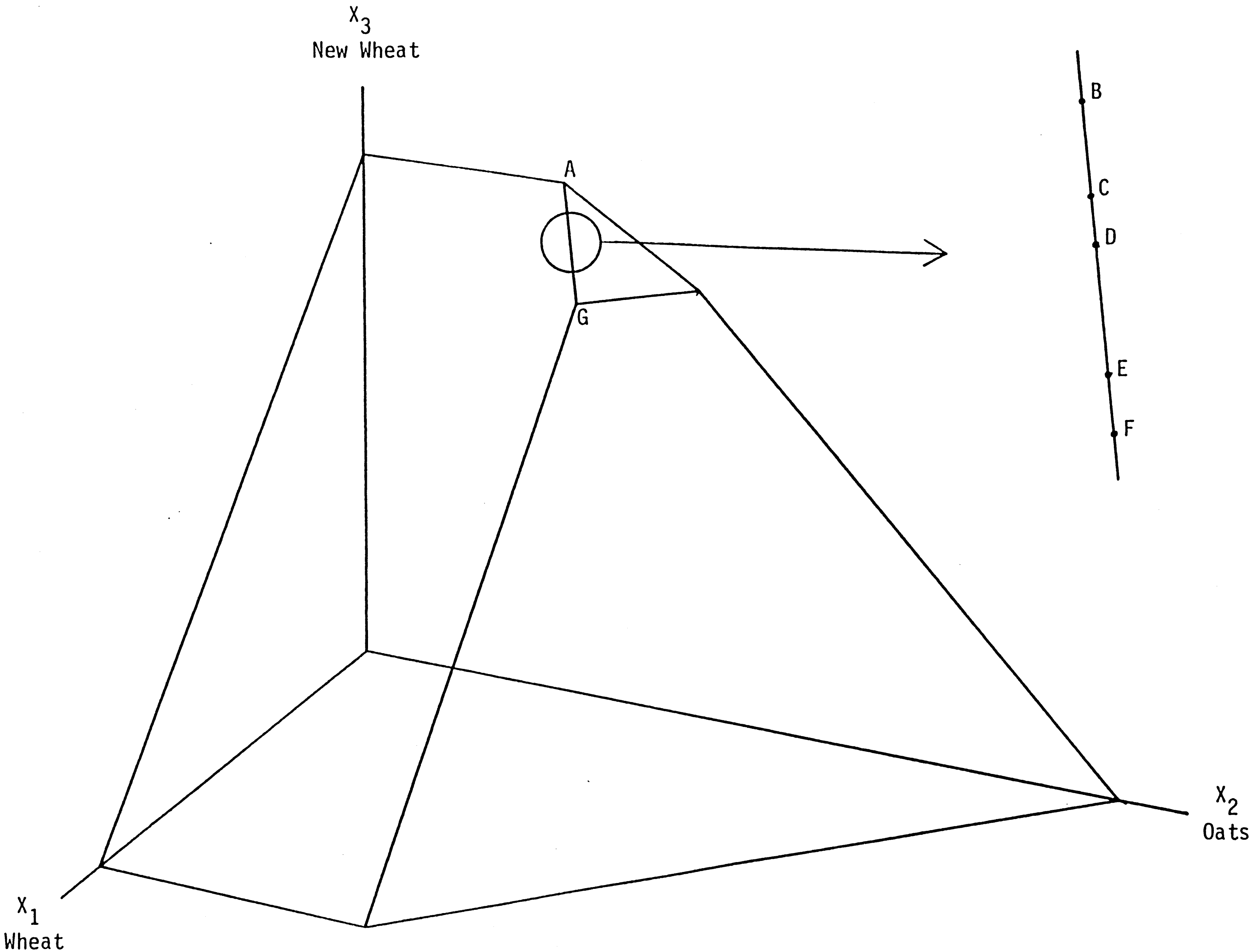

Figure 1. Feasible crop mixes

TABLE 1. Selected Crop Mixtures

Mixture	Wheat x_1	Oats x_2	New Wheat x_3	Expected Net Returns	DSD Test Criterion
	----------(hectares)----------			----------(dollars)----------	
A	0	3.2000	8.0000	1232.99	
B	.5500	3.5300	7.4500	1244.28	16.09
C	.5830	3.5498	7.4170	1244.96	15.41
D	.6000	3.5600	7.4000	1245.31	14.98
E	.6459	3.5875	7.3541	1246.25	0
F	.6659	3.5995	7.3341	1246.66	0
G	1.3333	4.0000	6.6667	1260.37	

satisfy the necessary conditions for TSD efficiency include all convex combinations of C and G. These mixtures also satisfy sufficient conditions for TSD efficiency.

The class of DARA utility functions includes the class of negative exponential utility functions. Thus, a sufficient condition for a crop mixture, x^o, to be DSD efficient is for its net returns vector, y^o, to be the only net returns vector which maximizes some negative exponential utility function subject to (1), (2) and (3). The set of such crop mixtures is called the NESD efficient set in this paper. It includes all convex combinations of E and G.

The necessary conditions for DSD efficiency were applied to several crop mixtures. Some of the results are included in Table 1. Given the considerations discussed above, it is not surprising that mixture F satisfies those conditions and mixture B does not. Only the efficiency statuses of mixtures between C and E are not known a priori. D is one such mixture. It is not DSD efficient. A series of tests on mixtures approaching E suggests that the DSD efficient set is little, if any, larger than the set of mixtures which maximize negative exponential utility functions.

The DSD efficient set is approximately 11 percent smaller than the TSD efficient set. Thus, the relative effectiveness of DSD was less than the average effectiveness, but well within the range of effectiveness percentages, observed by Vickson and Altmann.

Ex Post Analysis

The discussion above provides some insight into the nature of the method proposed in this paper. Additional insight can be provided by considering a characteristic of the necessary conditions, characteristics of the DSD criterion and the nature of the example.

A mixture will satisfy the necessary conditions for DSD efficiency only if it is not dominated by a mixture which is "close" to it.[7/] The fact that, for our example, all TSD efficient mixtures lie on a (single) line segment means that, effectively, only two "close" mixtures need be considered as alternatives when determining the DSD efficiency status of any mixture.

Vickson has shown that several conditions can limit the ability of the DSD criterion to order TSD efficient probability distributions. When the means of two probability distributions are the same, the DSD criterion is equivalent to the TSD criterion. Since no two TSD efficient mixtures share the same mean, that potential barrier to the effectiveness of the DSD criterion was not present for our example. A mixture cannot be dominated by one which yields a smaller mean net return. For our example, this means that no TSD efficient mixture can be dominated by one closer to C.

A mixture cannot be dominated by another mixture if the smallest element in the y vector associated with the first mixture is larger than that for the second mixture.[8/] The fact that the smallest element of y increases as the mixture approaches F from C means that this sort of relationship was not a barrier to the effectiveness of the DSD criterion for mixtures associated with the line segment CF. By the same token, the fact(s) that the smallest element of y decreases while the mean income increases as the mixture moves from F to G means that all mixtures on FG are both NESD and DSD efficient.

When two probability distributions are compared, the number of times which the cumulative probability distributions cross determines the relative effectiveness of DSD and other criteria (Vickson). If the cumulative functions do not cross or cross only once, then DSD is equivalent to FSD and SSD. In such cases, DSD would also be equivalent to TSD. When the cumulative distribution functions cross twice, then the NESD criterion described earlier is equivalent to DSD. When there are at least three crosses, the NESD criterion is likely to be more stringent.

On line segment CE, all pairs of mixtures are associated with pairs of cumulative probability functions which cross three times. This suggests that the DSD efficient set may be larger than the NESD efficient set. The fact that the difference in these two sets is, at most, negligible may be attributed to the relationship between the set of risk aversion coefficients needed to make mixture E DSD efficient and the (single) risk aversion coefficient needed to make it NESD efficient. The implicit NESD risk aversion coefficient associated with mixture E is approximately .0485. A similar risk aversion coefficient is implied by the DSD criterion for net returns smaller than $978.44 while a risk aversion coefficient of zero is implied for larger net returns. Both criteria assigned marginal utilities to the smaller net return levels which are very large relative to those assigned to larger net return levels.

Conclusion

This paper has presented necessary conditions for DSD efficiency of mixtures of risky alternatives. The solution to a concave programming problem reveals whether these conditions are satisfied for specific mixtures. A simple example was used to demonstrate the application of these ideas.

Footnotes

1/ Actually, Dybvig and Ross' Theorem 1 states necessary and sufficient conditions for an income vector, y^o, to be stochastically efficient. Stochastic efficiency is a necessary condition for SSD efficiency.

2/ Condition (9) can be derived by noting that the ratios of marginal utilities associated with any three income levels ($y_i^o < y_j^o < y_k^o$) involve integrals of the function r_u over the intervals (y_i^o, y_j^o) and (y_j^o, y_k^o). Expressions for the ~~upper~~ lower limit of w_i^o/w_j^o and the ~~lower~~ upper limit of w_j^o/w_k^o can be obtained by substituting the unknown parameter $r_u(y_j^o)$ for r_u in the integrals. Eliminating this parameter yields (9).

3/ The modifications which are required to deal with "ties" among y^o vector elements can be obtained from the authors.

4/ If the set of feasible y vectors is not bounded, arbitrary bounds can be added. As noted later, a vector y^o is DSD efficient only if it is not dominated by a feasible y vector "close" to it. Thus, any set of bounds which does not exclude these "close" feasible y vectors can be added without affecting the validity of the argument.

5/ If the right hand side of (15) were some positive number, h, other than 1, the optimal value of the objective function would merely be h times its optimal value when the right hand side of (15) equals 1. Since, as noted later in the text, the crucial question is whether this optimal value is zero, the choice of h does not affect any conclusions about DSD efficiency status.

6/ In this paper, a mixture, x^o, is assumed to be DSD efficient if the income vector, y^o, associated with it is DSD efficient. The approach used in the text allows for the possibility that a DSD efficient income vector, y^o, might be associated with several feasible mixtures.

7/ Although the necessary conditions do have desirable "global" properties, they are useful largely because of an equivalence between local and global properties.

8/ This is sometimes referred to as a "left-tail" problem.

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