STOCHASTIC DOMINANCE OVER CORRELATED PROSPECTS

by

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ABSTRACT

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A difficulty occurs in stochastic dominance applications when alternatives are not mutually exclusive and mixed prospects may be formed. Exhaustive examination of all possible mixed prospects is often impractical. In this paper, rules are derived and tested for deciding when and to what extent mixed prospects should be examined.

STOCHASTIC DOMINANCE OVER CORRELATED PROSPECTS

Stochastic dominance procedures are powerful methods for evaluating decisions under uncertainty (as reviewed in Bawa; Cochran, Robison and Lodwick or Zentner et al). However, difficulties can occur when the individual prospects examined may be diversified; i.e., a convex combination or portfolio may be formed. In such cases, while one pure prospect may dominate another, diversified alternatives containing the dominated prospect may not be dominated by the dominant pure prospect. E.g., monoculture corn might dominate monoculture soybeans but fail to dominate cropping patterns including both crops. Such results indicate that stochastic dominance can yield misleading conclusions when the alternatives considered are not mutually exclusive. In this paper rules are developed indicating when dominance of one prospect over another implies dominance over all convex combinations of the two. The rules also identify cases when users of stochastic dominance analysis should systematically study diversified alternatives.

THEORETICAL DOMINANCE OF A PURE ACTIVITY OVER DIVERSIFICATION

Ordinarily the examination of diversification in stochastic dominance analysis would be difficult because explicit integration of the probability density function would be required. However, a number of moment based stochastic dominance rules have been derived (Pope and Ziemer give a partial review of the literature). Use of the moment-based rules in a study of the diversification problem requires knowledge of the distribution of the diversified combinations. This requirement led to the use of normal distributions for the theoretical investigations. Importantly, this restrictive assumption is relaxed in the empirical analysis.

The first theoretical result can be stated in the following theorem:

Theorem 1

Given two normally distributed prospects, x_1 and x_2, distributed with means μ_1 and μ_2, positive variances σ_1^2 and σ_2^2 and correlation coefficient ρ, where prospect x_1 dominates x_2 by SSD, then x_1 dominates all convex combinations of x_1 and x_2 by SSD if $\rho \geq \sigma_1/\sigma_2$.

Proof of this theorem is based on the moment-based SSD rule for normally distributed prospects. This rule says that x_1 dominates x_2 by SSD if $\mu_1 \geq \mu_2$ and $\sigma_1{}^2 \leq \sigma_2{}^2$, with at least one inequality holding strictly. The mean and variance of a convex combination (x_3) of x_1 and x_2 are given by

(1) $\mu_3 = \lambda\mu_1 + (1 - \lambda)\mu_2$ and

(2) $\sigma_3{}^2 = \lambda^2\sigma_1{}^2 + 2\rho\lambda(1 - \lambda)\sigma_1\sigma_2 + (1 - \lambda)^2\sigma_2{}^2$

where μ_3 and $\sigma_3{}^2$ are, respectively, the mean and variance of the convex combination, ρ is the correlation coefficient between x_1 and x_2, and λ is a real number such that $0 \leq \lambda \leq 1$.

The SSD condition on the means will be preserved over convex combinations since μ_3 will always be less than or equal to μ_1. For x_1 to dominate x_3 the variance condition must also be satisfied. This condition is

(3) $\lambda^2\sigma_1{}^2 + 2\rho k\ \sigma_1{}^2(\lambda - \lambda^2) + k^2\sigma_1{}^2(1 - 2\lambda - \lambda^2) \leq \sigma_1{}^2$

where k is equal to the ratio σ_2/σ_1 and must be greater than or equal to 1.

It can be shown that this condition is always satisfied only if

(4) $(\lambda - 1)\,[\lambda(k^2 - 2\rho k + 1) - (k^2 - 1)] \geq 0$

This is a convex function in λ and thus will only violate the inequality between the roots (i.e., those λ where the above function equals zero). One root is at $\lambda = 1$; the other must fall at or above 1 for the result to hold. The second root will always be greater than or equal to 1 whenever $\rho \geq 1/k$ or equivalently, $\rho \geq \sigma_1 / \sigma_2$, which is the result stated in theorem 1.

Discussion of Theorem 1

In general when x_1 dominates x_2 by SSD, theorem 1 says x_2 can be dismissed from consideration in all convex combinations whenever their correlation coefficient is no less than the ratio of their standard deviations. This ratio, given that x_1 dominates x_2 by SSD, is known only to be positive and less than or equal to 1. This implies that the only value of ρ that will always satisfy the relationship is $\rho=1$. Thus to safely use SSD on potentially diversified strategies without considering diversified alternatives one must assume perfect correlation. In an empirical setting, however, sample estimates of ρ, σ_1 and σ_2 will be known and one can speculate that it is safe to disregard diversified strategies as long as rule 1 (theorem 1) holds (Clearly, this would be subject to sampling error). Rule 1 also implies that diversified strategies can never be ignored when x_1 and x_2 are negatively correlated since σ_1/σ_2 cannot be negative.

Dominance Based on Relaxation of the SSD Criterion

The SSD criterion is rather conservative, being adverse to low probability crossings. Thus, a rule will be developed based on expected utility theory. Freund has shown that under normality and constant risk aversion an equivalent way of evaluating whether one uncertain prospect has larger expected utility than another is to examine whether the following inequality holds:

$$(5) \qquad u_1 - \theta/2\ \sigma_1^2 \geq u_3 - \theta/2\ \sigma_3^2$$

where θ is the Pratt risk aversion parameter from the constant risk aversion utility function. Use of this condition leads to the following theorem.

Theorem 2

Given a) two normally distributed prospects x_1 and x_2, with means μ_1 and μ_2, variances σ_1^2 and σ_2^2 and correlation coefficient ρ;

b) that x_1 dominates x_2 by SSD such that $\mu_1 > \mu_2$ and $\sigma_1^2 \geq \sigma_2^2$

c) a value θ which is the Pratt risk aversion coefficient, and

d) the definition that dominance occurs between two distributions, x_1 and x_2, such that

1) $\mu_1 \geq \mu_2$ and 2) $\mu_1 - 0.5\theta\,\sigma_1^2 \geq \mu_3 - 0.5\theta\,\sigma_3^2$

then x_1 will dominate all convex combinations of x_1 and x_2 whenever

$$(6) \qquad \rho \geq \frac{\sigma_1}{\sigma_2} - \frac{(\mu_1 - \mu_2)}{\theta\,\sigma_1\sigma_2}$$

This theorem's proof relies upon the same approach outlined for theorem 1 but is omitted due to space limitations (see McCarl et al for the full proof).

Discussion of Theorem 2

Several observations can be made about this theorem. 1) As θ tends to infinity (extreme risk aversion) the theorem 2 rule reduces to that found in theorem 1. 2) The theorem 2 rule (rule 2) yields insight into the factors which are relevant in determining the critical correlation coefficient values. These factors are a) as the difference between the means decreases the minimum safe correlation coefficient increases; b) as the variances increase, the critical correlation must be larger; and c) the critical correlation coefficient is larger for more risk averse decision makers. 3) This rule is always less conservative than the rule given in theorem 1 and permits the critical correlation coefficients to be negative given large differences in means. 4) Both rules (1 and 2) are dependent on the normality assumption. The sensitivity of these rules to violations of normality is examined in the following section.

EMPIRICAL TESTING

The above rules provide criteria for deciding when to ignore mixed alternatives or portfolios in SSD analysis given normally distributed uncertain prospects. However, SSD applications often involve comparisons of data from unknown, possibly nonnormal, distributions. Thus, the usefulness of these rules in applied SSD analysis depends on their ability to identify relevant critical values when comparing uncertain prospects of various distributional forms. This section uses simulation to explore the reliability of the analytically derived rules in evaluating normally and nonnormally distributed data. Also, an empirical rule is estimated based on the study data.

Simulation Experiment Design

The reliability of the rules derived above was investigated in cases characterized by data from normal, right and left skewed beta and uniform distributions. Initially, three 1000 member distributions were created with means of 100 and variances of 15, 25, and 50 for the normal and uniform families. Positively and negatively skewed beta distributions with the same mean and variances were also created. The data used to approximate each distribution consisted of 1000 equally likely points corresponding with the .001 through .999 fractiles of the distributions. These twelve distributions were used as reference distributions in the experiment.

Additional distributions were created for comparison with these reference distributions. Each comparison distribution was from the same family as the reference distribution but with a different mean and variance. The data under the reference distribution (x_1) were altered to become that of the comparison distribution (x_2) by the following formula:

(7) $$x_{2i} = \frac{(x_{1i} - \mu_1)}{\sigma_1} \sigma_2 + \mu_2$$

The ratio of the means of the reference and comparison distributions (μ_1/μ_2) took 9 values from 1.05-2.5. The ratio of the standard deviations of the reference and comparison distributions (σ_1/σ_2) took nine values from 0.95-0.25. All combinations of these means and standard deviations were examined, producing 81 comparison distributions for each reference distribution. Stochastic dominance was checked by examining all 1000 points using numerical integration (Pope and Ziemer's empirical distribution function estimator). In all cases the reference distributions dominated the comparison distributions.

By construction the comparison distributions were perfectly correlated with the reference distributions. To test for the effect of correlations of less than one the observations of the comparison distributions were randomly reordered to obtain correlations from .99 to -.99 in intervals of .01 (the minimum correlation achieved for the beta distributions was -.86). In turn, given a set of correlated data, all convex combinations of the reference and comparison distributions between .05 of x_1 (the reference distribution) and .95 of x_2 (the comparison distribution) through 0.95 of x_1 and .05 of x_2, in .05 intervals, were tested for SSD using the empirical distribution function estimator. The lowest correlation coefficient at which the reference distribution dominated all convex combinations of the two distributions was noted. This procedure was repeated ten times for each comparison distribution, thus creating ten empirical minimum safe correlation coefficients for each pair of reference and comparison distributions. These empirical, minimum safe correlation coefficients provided a basis for evaluating the reliability of the analytically derived rules and for estimation of an empirical rule.

The observed safe correlation values can be compared with the values derived from normality-based rules in theorems 1 and 2 to discover the frequency with which they are correct. Information on the empirical power of the test may also be gained. However, in order to do this, values of θ are needed for use in the second rule. The values of θ used were based on the relationship of the Pratt risk aversion parameter and the risk premium. These were set so that $\theta = 2Z/\sigma_1$ where Z is the risk premium divided by σ_1

(as explained in McCarl and Bessler). Under this setup the Pratt risk premium for a bet with variance of σ_1^2 equals $Z\sigma_1$.

This choice of θ transforms rule 2 to

(8) $\rho \geq \sigma_1/\sigma_2 - (\mu_1 - \mu_2)/2Z\sigma_2$

Z values used were 1.5 and 2.5, indicating a risk premium for accepting a respective gamble 1.5 and 2.5 times the standard deviation of the gamble.

Experiment Results

The experiment results indicate that the analytically derived rules very reliably identify safe levels of correlation for the distributions studied. Rule 1 was violated (with an observed minimum safe correlation coefficient larger than specified by the rule) in only 1.5% of the 9,720 comparisons conducted. Rule 2 (equation 6), less conservative than rule 1, was violated in 6.5% of comparisons when Z=1.5 and 3.3% when Z=2.5. The reliability of the rules varied somewhat among the families of distributions tested. For the uniform distributions rule 1 was never violated while rule 2 was violated in less than 0.5% of comparisons. Rule 1 was violated in 3.3% of comparisons between normal distributions while rule 2 was violated in 8.1% and 14.7% of comparisons for Z levels of 2.5 and 1.5 respectively. The rules were violated with almost equal frequency for the two beta distributions. Rule 1 was violated in 1.4% of comparisons while rule 2 was violated in 2.5 and 4.1% of comparisons for Z values of 2.5 and 1.5 respectively.

Since the analytic rules appear conservative, can other, less conservative, criteria be adopted? One approach for developing an alternative criterion is to estimate the minimum safe correlation coefficient empirically. This was done using the data set created to test the analytic rules. Only a subset of the data, specifically the maximum value of the minimum safe correlation coefficients over the 10 replications for each comparison distribution, were used. Thus the empirical rule, estimated by ordinary least squares regression, provides a conservative estimate of the minimum safe correlation coefficient based on the simulated data. The form of the estimated empirical rule was derived based on the terms contained in rule 2 with the addition of terms for the third and fourth moments (see McCarl et al for details).

The resulting estimated empirical rule is

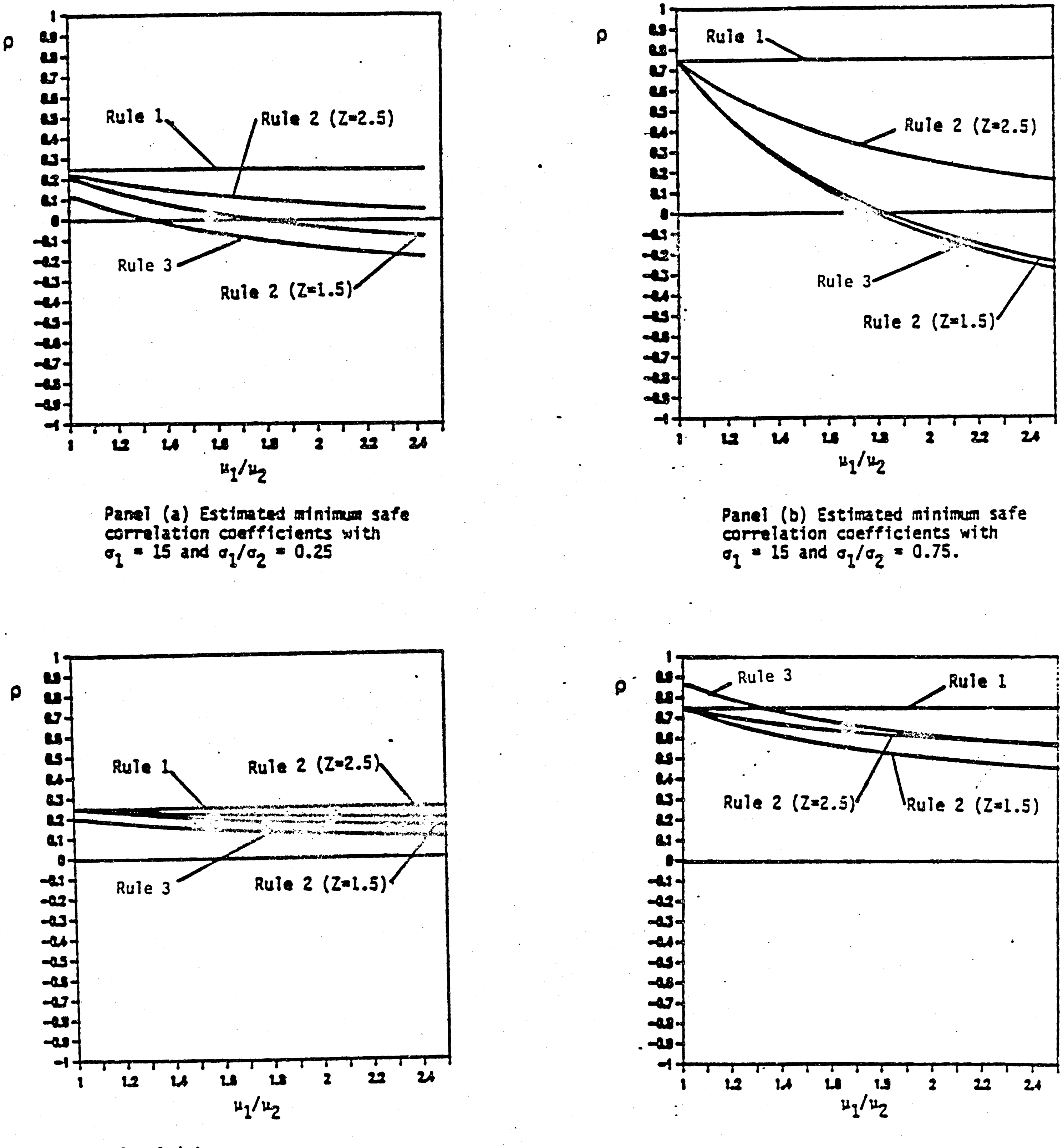
$$(9)\quad p = -0.134 + 1.282\,(\sigma_1/\sigma_2) - 0.636\,(\mu_1/\sigma_2) + 0.59\,(\mu_2/\sigma_2) - 0.143(\sigma_1 m_3/\sigma_2)$$
$$- 0.101(\mu_1 m_3/\sigma_2) + 0.047\,(\mu_1 m_4/\sigma_2) + 0.118(\mu_2 m_3/\sigma_2) - 0.039(\mu_2 m_4/\sigma_2)$$

where m_3 and m_4 are, respectively, the skewness and kurtosis of the dominant (reference) distribution. All estimated parameters were significant at the 0.01 level. The relationship between this equation and the analytic rules can be seen by noting that the second rule under the assumption on θ leading to equation (8) can be re-expressed as

$$(10)\qquad \rho \geq \sigma_1/\sigma_2 - (1/2Z)(\mu_1/\sigma_2) + (1/2Z)(\mu_2/\sigma_2)$$

Note that, since 1/2Z is a constant, the second, third and fourth terms in equation (9) are consistent with the right-hand side terms of equation (10). The remaining terms in equation (9) were included to account for the influence of the third and fourth moments of the distributions.

Figure 1 illustrates the relationship between the empirical rule (rule 3) and the analytic rules for the left-skewed beta distribution. These relationships are typical of the relationships for the other distributions studied. Panel (a) shows shows the three analytic rules and rule 3 when σ_1 = 15 and σ_1/σ_2 = 0.25. In this case the empirical rule is less conservative than the most conservative analytic rule (rule 2 with Z = 1.5) throughout the range of μ_1/μ_2 covered by the data. Panel (b) demonstrates that when the variances for the two distributions under comparison are more similar (σ_1/σ_2 = .75) the empirical rule becomes more conservative relative to the analytic rules and, in fact, in this case is almost identical with rule 2 with Z = 1.5. Panel (c),



Panel (a) Estimated minimum safe correlation coefficients with $\sigma_1 = 15$ and $\sigma_1/\sigma_2 = 0.25$

Panel (b) Estimated minimum safe correlation coefficients with $\sigma_1 = 15$ and $\sigma_1/\sigma_2 = 0.75$.

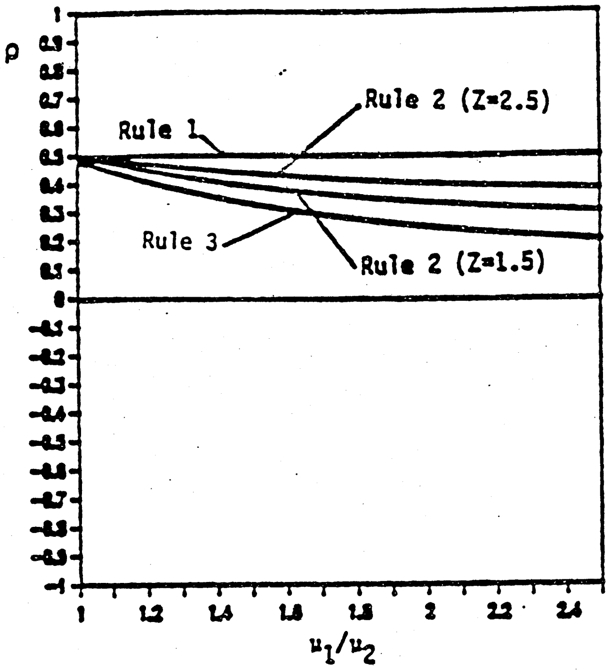
Panel (c) Estimated minimum safe correlation coefficients with $\sigma_1 = 50$ and $\sigma_1/\sigma_2 = 0.25$.

Panel (d) Estimated minimum safe correlation coefficients with $\sigma_1 = 50$ and $\sigma_1/\sigma_2 = 0.75$.

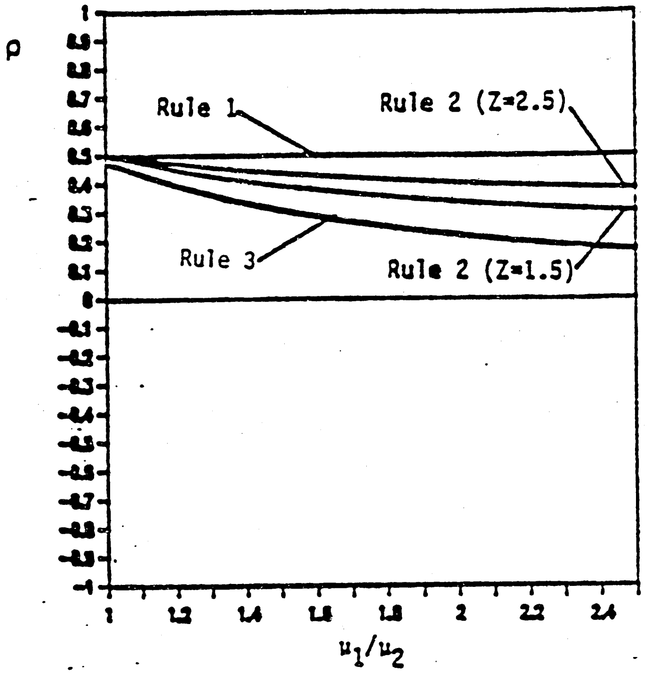
Figure 1. Three analytic rules and empirical rule (Rule 3) for the left skewed beta distribution with alternative values of σ_1 and σ_1/σ_2.

when compared with panel (a), demonstrates another important relationship among the rules. That is, when the dispersion (as measured by standard deviation) of the distributions under comparison increases ($\sigma_1 = 50$) all the rules converge toward rule 1. In this case, however, with $\sigma_1/\sigma_2 = 0.25$ the empirical rule is still less conservative than the analytic rules. Comparison of of panel (d) with panel (b) shows this same relationship but additionally demonstrates that when both σ_1 and σ_1/σ_2 are relatively large, the empirical rule is more conservative than rule 2 and is even more conservative than rule 1 when the means of the two distributions are quite similar ($\mu_1/\mu_2 < 1.4$). The results in panel (d) also show that errors can arise when normality-based rules are applied to nonnormal cases. The results show rule 3 to be above portions of all the rules illustrating the incidence of Type I error and indicating the importance of including higher moments in the empirical rule.

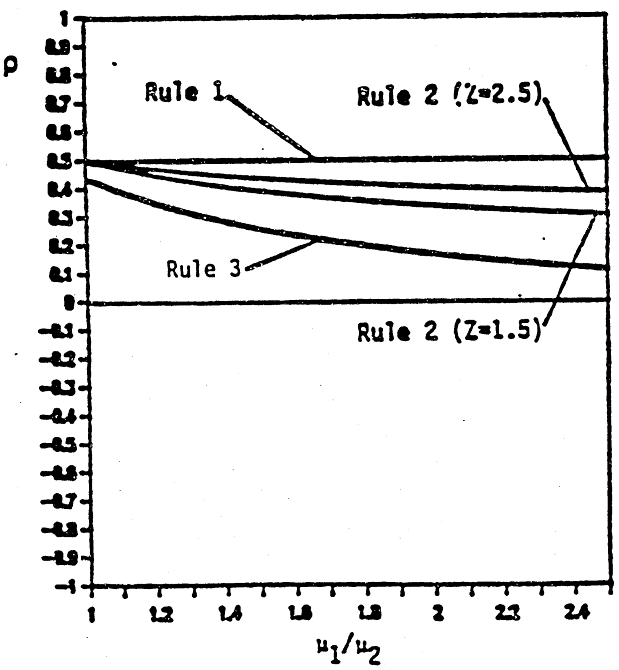
Note that the empirical rule, unlike the normality based analytic rules, takes the third and fourth moments of the distributions into consideration. The impact of this is illustrated in figure 2. Panels (a) through (d) show the empirical rule for the distributions studied when $\sigma_1=15$ and $\sigma_1/\sigma_2 = 0.5$. All panels include the analytic rules for reference purposes. Comparing pan-panels (a) and (b), illustrating rules for normal and uniform distributions reveals that a difference in kurtosis (3 for the normal distribution versus 1.8 for the uniform) has minimal effect on the minimum safe correlation coefficient estimated by the empirical rule. However, panels (c) and (d) show that skewness does significantly affect the empirical rule. The minimum safe correlation coefficients estimated by the empirical rule for the right-skewed beta distribution (panel (c)) is smaller than for any of the other distribution. Conversely, the estimated safe correlation coefficient for the left-skewed beta distribution is larger than for any other distribution and in fact is larger than that estimated by analytic rule 2 with Z = 1.5.



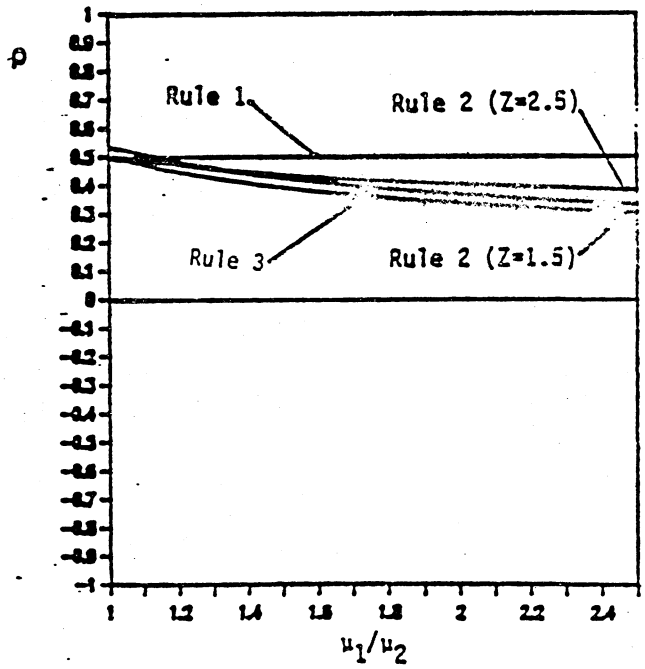
Panel (a) Normal distribution with $\sigma_1 = 15$ and $\sigma_1/\sigma_2 = 0.5$.



Panel (b) Uniform distribution with $\sigma_1 = 15$ and $\sigma_1/\sigma_2 = 0.5$.



Panel (c) Right skewed beta distribution with $\sigma_1 = 15$, and $\sigma_1/\sigma_2 = 0.5$.



Panel (d) Left skewed beta distribution with $\sigma_1 = 15$, and $\sigma_1/\sigma_2 = 0.5$.

Figure 2. Three analytic rules and empirical rule (Rule 3) for distributions from different families with equal means and variances but different third and fourth moments.

CONCLUDING COMMENTS

Stochastic dominance can give misleading results when comparing alternatives that are not mutually exclusive or perfectly correlated. The results presented here show that, given normally distributed prospects, if distribution 1 dominates distribution 2, then distribution 1 will dominate all convex combinations of the two as long as

1) the efficiency criterion is second degree stochastic dominance and the correlation coefficient satisfies $\rho \geq \sigma_1/\sigma_2$; or

2) the criterion is expected utility maximization under constant absolute risk aversion and the correlation coefficient satisfies:

   $\rho \geq \sigma_1/\sigma_2 - (\mu_1 - \mu_2)/(2\theta\sigma_1\sigma_2)$

   where θ is the Pratt risk aversion coefficient.

An empirical evaluation of these rules suggested that, although they were quite reliable, they were often conservative. Thus an empirical rule was estimated.

These results and rules can be used in two ways. First, given a set of data, one may compute the rule values and determine whether the comparisons among the alternatives will provide dominance results that are also applicable to convex combinations of the alternatives. Second, if convex combinations need to be considered, then the rules can be used to investigate whether the convex combinations are adequate; i.e., whether the correlation between two combinations is sufficient to ensure that dominance extends to all intermediate combinations.

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