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1986

Insurance, Crop

A Parametric Model of Stochastic Production

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presented @ AAEA 1986

February, 1986

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Abstract

A parametric model of stochastic production is proposed and demonstrated. The parametric model treats output as a random variable with a distribution that is conditional on inputs. Maximum likelihood estimation of the model is shown to produce consistent and asymptotically efficient estimates.

Just and Pope (1979) made a significant contribution to the study of stochastic production by explicitly modelling the dependence of the variance of a stochastic production function upon inputs, and by deriving a consistent estimation method for their model. Antle (1983) extended this work by constructing a model that expresses general moments of a stochastic production function as functions of inputs, providing a consistent method for estimating the parameters of the model, and providing a means for testing which moments are significantly influenced by inputs. This moment based model of stochastic production is a flexible tool that uses minimal assumptions about the probability distribution of output. In this paper a related model of stochastic production which uses stronger assumptions about the probability distribution of output is described.

Section one addresses several reasons why the model might be of use in production economics research. In section two the details of the model and its estimation properties are presented. And section three presents some results from an application of the model to the calculation of crop insurance premiums.

Some Reasons for the Model

The moment based model could be characterized as a non-parametric model of the probability distribution of output. It does not assume that output has a known probability distribution. The only assumption that is made about the probability distribution is that the moments of interest exist. This non-parametric character is a strength if there is no information for specifying a probability distribution and the moments of the distribution are of primary interest. However, if there is sufficient information for specifying a parametric probability distribution, or if an explicit expression for the distribution is needed, the nonparametric character of the moment method is a weakness.

The method proposed in this paper, which shall be called the parametric method, is intended for cases when there is sufficient information for specifying a distribution or when an explicit expression for the distribution is needed. Such cases are likely to be encountered in production economics research. For example, there may be sufficient prior information for specifying a specific parametric distribution for a crop yield. Or, a study on the value of information might require an estimate of the probability distribution of output for informed and uninformed traders.

The parametric method is a generalization of linear regression-based studies of stochastic production (eg. Wolgin). If output is expressed as a linear regression function of a collection of inputs, then output can be thought of as a random variable with a normal distribution that has a mean conditioned on the collection of inputs. This model can be generalized by treating output as a random variable with a distribution whose parameters are conditional on a collection of inputs. The strengths of this approach include the ability to express the entire probability distribution of output with a small number of parameters (the moment method could require an infinite number of moments in order to characterize an entire distribution), and the ability to derive estimates that are consistent and asymptotically efficient.

The empirical example in this paper is concerned with the probability distribution of corn yield. The parametric method is attractive for this problem because there is fairly good information for specifying a parametric probability distribution. One source of information is Day, who wrote:

> "I would suggest the skewed, bell shaped [Pearson] type I function as a reasonable hypotheses for further research, with the J shaped curve as an extreme limiting case that must be confronted only because of paucity of data. The bell shaped case may, of course, exhibit skewness in either direction."

There is other evidence that crop yields are significantly skewed and unimodal, suggesting that the parametric form recommended by Day might be appropriate.

Details of the Parametric Method

The development in this section will use Day's suggested hypothesis by treating output as a Pearson Type I, or Beta, random variable; the same development can be carried out for other parametric forms. Output will be specified to be distributed as a Beta random variable:

$$(1) \qquad p(y) = \frac{\Gamma(a+b)\; y^{(a-1)} (M-y)^{(b-1)}}{\Gamma(a)\; \Gamma(b)\; M^{(a+b-1)}} \qquad 0 \leq y \leq M$$

where y is output, M is the maximum possible output, a and b are the parameters of the distribution, and $\Gamma(.)$ is the gamma function. If M is known, this distribution is a member of the regular exponential family of distributions, and it is a standard statistical result that maximum likelihood estimates of the parameters are consistent, asymptotically normal, and asymptotically efficient (see Cramer, for example). If M is not known, then these results do not hold and it is not even known whether maximum likelihood estimates are consistent. The difference between these two cases is so great because when M is not known the support of the probability distribution is a random variable and the neighborhoods needed to prove consistency cannot be constructed. Thus, in order to get good statistical properties with this model it is necessary to specify a value for M, the maximum output. For many agricultural applications it should be possible to specify a reasonable value for M.

Equation (1) must be extended in order to model stochastic production; it is necessary to introduce the affect of inputs on the distribution of output. This will be done by conditioning the parameters of the distribution on the inputs. The parameters, a and b, will be expressed as functions of the inputs. There are no strong a priori reasons for choosing any particular functional form for relating the inputs to the parameters, therefore many forms could be tried. In section three a and b will be represented as power functions of the inputs; these functions have the advantage of being easy to manipulate, and having a positive range.

Conditioning the parameters changes the distribution of y to a conditional distribution, so (1) becomes:

$$(2) \qquad p(y;x) = \frac{\Gamma[a(x)+b(x)]\; y^{(a(x)-1)}(M-y)^{(b(x)-1)}}{\Gamma[a(x)]\;\Gamma[b(x)]\; M^{(a(x)+b(x)-1)}} \qquad 0 \leq y \leq M$$

This model is closely related to linear regression models of production. The only difference is that the parameters of the Beta distribution are written as functions of the inputs instead of writing the mean of the normal distribution as a function of the inputs. The advantage of this generalization is that the Beta distribution is a more flexible distribution than the normal distribution, while remaining tractable. All of the moments of the Beta distribution exist and are rational functions of the parameters. An attribute of the distribution that is important for production analysis is that the skewness and kurtosis of the distribution are unrestricted.

Estimation of the parameters of this model by the method of maximum likelihood produces estimates with good statistical properties. The first order conditions of the maximum likelihood problem are a complicated system of 2n highly nonlinear equations, when n inputs are included in the model. Therefore explicit expressions for the maximum likelihood estimates are not derived, and the small sample properties of the estimates are not investigated. Rather, as is standard with maximum likelihood estimates, the asymptotic properties of the estimates are considered.

The first asymptotic property to be considered is consistency (ie. the convergence of the estimates toward the true parameters). Following the method of Cramer this property can be examined by expanding the first order conditions of the maximum likelihood problem in a first-order Taylor's series. If a(x) and b(x) are continuously differentiable with respect to the vector x and the observations are independent and identically distributed, then it can be shown that there exists a solution to the system of linear equations which converges in probability to the true parameter vector (see Cramer).

Asymptotic normality, unbiasedness, and efficiency can be verified in the same manner. The Taylor's series expansion can be rearranged to show that $\sqrt{n}\,(p^* - p0)$ is asymptotically distributed as a normal random variable with mean zero and covariance matrix equal to the inverse of the information matrix (ie. the negative of the Hessian of the log-likelihood function), where p^* is the MLE and p^0 is the true parameter vector. This result depends upon the same assumptions as those used to prove consistency, and it means that the maximum likelihood estimates for this problem are asymptotically normal, unbiased, and efficient.

Calculation of the maximum likelihood estimates must be accomplished by numerical maximization of the likelihood function, because analytic expressions for the maximum likelihood estimates do not exist. Given the current state of computer resources this is no hinderance to implementation of this approach. Most likelihood functions exhibit smooth behavior, so that convergence of nonlinear optimization routines usually can be achieved rapidly.

Before concluding this section, it should be noted that the parametric method can provide consistent estimates of the parameters of a stochastic production process, even when the parametric family of distributions is misspecified. This is because most parametric distributions can be estimated by maximum likelihood methods. And, following the work of Huber and White, maximum likelihood estimates are often consistent even when the likelihood function is misspecified.

Application to the Calculation of Insurance Premia

In order to demonstrate the usefulness of the parameter based approach to stochastic production, an application concerning the calculation of crop insurance premia will be described. This application investigates the affect of assumptions about the probability distribution of crop yields upon the calculation of premiums for crop insurance. The prevailing assumption used by the Federal Crop Insurance Corporation is that crop yields are normally distributed (Botts and Boles). In this application, conditional normal distributions (linear regressions) and conditional beta distributions were fit to data on individual farms in several Iowa counties. Then the probability of loss and the expected value of losses for several insurance coverage levels were calculated under the two distributional assumptions.

The data for this application comes from 15 counties in Iowa over the period 1961-1970. The data was collected by the Iowa Agricultural Experiment Station as part of a study on the effects of corn rootworm. Farms were sampled in years when they grew corn. In those years, detailed production information on soil characteristics, plant characteristics, fertilizer, and pesticides was collected. For this study, variables on nitrogen application, phosphate application, potassium application, soil slope, soil clay, and two dummy variables to represent the planting of nitrogen fixing crops in previous years were used as inputs. Other inputs were not included because the data on them was judged to be inadequate. Corn yield per acre was used as the output.

The sample consisted of 1263 observations. These observations cover farms in the study, for the years when they grew corn, between the years 1964 and 1969. Because observations were taken only in years when corn was grown, the sample is not a complete panel with time series observations for each year for each member of the panel. This irregularity in the data led to a decision against an attempt to exploit the cross-section, time series property of the data set. It was also decided that it would be inappropriate to pool all of the data because it is likely that farms over as wide a geographic area as all of Iowa would have different underlying random processes generating the probability distribution of crop yields.

Estimates of the normal model and the beta model were calculated for each of the 15 counties in the sample. The data consisted of pooled observations within each county. Thus, implicit in the construction of the data set is the assumption that farms within the same county have the same basic random processes influencing their production. This is similar to the assumption implicit in the Federal Crop Insurance Corporation practice of using counties as their basic geographic units for the calculation of insurance. There was an average of 82 observations per county.

The normal model that was estimated was a standard linear regression model. The beta model was estimated by maximizing the log-likelihood function:

$$(3)\quad L(p;y,X) = \sum_{i=1}^{N} \Big[\ln \Gamma(A\prod_j x_{ij}^{a_j}) - \ln \Gamma(B\prod_j x_{ij}^{b_j}) + [A\prod_j x_{ij}^{a_j} - 1]\ln y_i + [B\prod_j x_{ij}^{b_j} - 1]\ln(M - y_i) - [A\prod_j x_{ij}^{a_j} + B\prod_j x_{ij}^{b_j} - 1]\ln(M)\Big]$$

where p refers to the vector of parameters. The maximum possible output, M, was set to 200; the maximum observed output in the sample was 179.The function (3) was numerically maximized with the Modular In-Core Nonlinear Optimization System using analytical gradients of the objective function. A quasi-Newton algorithm which uses the analytical gradients and builds up information about the Hessian was used. The data was scaled so that all data values were between 0 and 2 in order to insure regular progression in the steps of the algorithm. Convergence was typically achieved in 9 cpu seconds on a CDC Cyber 205 computer. The analytical Hessian was constructed and checked at the solution values, and found to be negative definite at every solution.

Some intuitive checks on the parameters of the beta model were performed by examining changes in the mean and variance of output with respect to the inputs. The expression for the mean of output for this model is a relatively simple power function:

$$(4)\qquad Ey = 200[\,1 + (B/A)\prod_{j=1}^{7} x_j^{(b_j - a_j)}\,]^{-1}$$

From expression (4) it is easy to see that the change in mean output with respect to input j is positive (negative) if a_j is greater than (less than) b_j. Using this result, some general statements about the variables in the model can be made. Nitrogen and phosphates were found to increase mean yield in every county except one. Potassium was found to decrease mean yield in every county except one. This last result is perhaps due to the fact that uneconomic levels of potassium applications were being made. Concerning soil characteristics, it was found that increased slope decreased expected yield in every county.

There is no simple analytic expression like (4) for the variance of output. Therefore the change in variance with respect to change in a particular input was evaluated numerically in different counties at different input levels. One property that this numerical investigation revealed is that the change in variance is a smooth function of input levels; that is, as input levels were varied the change in variance was found to move smoothly without any abrupt jumps. The behavior of the change in variance was not found to be as regular as the change in the mean. In nine counties nitrogen and phosphates were found to increase the variance of yield (at the input values that were examined). In ten counties, potassium was found to decrease the variance of yield. In general, it appears that the fertilizer inputs either increase the mean of yield and increase the variance of yield, or decrease the mean and decrease the variance. This was considered to be an intuitively reasonable result which lends some credibility to the parameter estimates of the beta model.

The beta model was used to examine the effect of probability distribution assumptions on the calculation of insurance by first constructing normal and beta distributions for county average values of inputs. The results of this construction are shown in Table 1. Examination of this table reveals that the means of the two distributions are very close, but that the variance of the beta distribution usually tends to be slightly smaller than the normal variance. This might be due to the fact that the normal distribution imposes symmetry on the data, while this data exhibited some significant skewness. The skewness in the data is reflected in the parameters of the beta distribution. The relationship between a and b

Table 1
Distributions for Premium Calculations

County	Normal distribution mean	Normal distribution variance	Beta distribution mean	Beta distribution variance	a	b
Crawford	103.67	116.05	103.71	97.40	52.65	48.88
Fayette	115.28	197.94	115.23	161.46	34.28	25.22
Hamilton	120.02	208.51	120.00	171.58	32.97	21.98
Howard	87.25	413.55	86.84	422.14	9.44	12.30
Linn	122.80	116.82	122.85	121.48	47.31	29.71
Muscatine	127.14	102.40	127.19	90.40	64.51	36.93
Woodbury	97.12	346.15	97.06	285.14	16.52	17.52

determines the skewness of the beta distribution: $a < b$ implies positive skewness, $a = b$ implies symmetry, and $a > b$ implies negative skewness. Five of the seven beta distributions reported in the table exhibit negative skewness. This means that above average yields are more likely than yields which are significantly below average.

A comparison of crop insurance premiums derived from these two distributions is presented in Table 2. These premiums are the expected losses under three coverage levels of 50 percent, 65 percent, and 75 percent of average yield. The numbers in the table are expressed in units of bushels per acre. For example, if the normal probability model was used then a farmer in Woodbury county would have to pay approximately one-fourth bushel per acre to obtain coverage of 65 percent of average yield. To convert the numbers to dollar units any price per bushel could be used. It should be noted that the premiums in the table are much smaller than current premiums for two reasons. First, a small data set was used to calculate the distributions; it is probably less variable than a larger data set would be. Second, no loading factors for catastrophes of infinitesimal probability or for accumulation of capital reserves have been added to the premia. Thus Table 2 should not be used for comparison with current premia, only for

Table 2
Insurance Premiums from Alternative Distributions

	Normal			Beta		
County	50%	65%	75%	50%	65%	75%
Crawford	.0	.0011	.0291	.0	.0002	.0123
Fayette	.0	.0021	.1040	.0	.0008	.0563
Hamilton	.0011	.0076	.0990	.0	.0029	.0584
Howard	.113	.576	1.43	.072	.545	1.484
Linn	.0	.0004	.0179	.0	.0002	.0115
Muscatine	.0	.0	.0024	.0	.0	.0017
Woodbury	.026	.247	.834	.0044	.1135	.5395

comparison between the normal and beta distributions. The most apparent and significant difference between the normal and beta models is that premiums from the beta model are consistently smaller than premiums from the normal model. This is a reflection of the skewness of the distributions. The premiums depend upon the amount of probability mass in the lower tails of the distributions. Negatively skewed distributions have less mass in the lower tail than symmetric distributions. Most of the beta distributions that were calculated from the data are significantly negatively skewed.

In many cases the magnitude of the premium difference is substantial, suggesting that use of the beta density function could significantly reduce the premiums charged to farmers for crop insurance. For many of the entries

In Table 2, the premium based on the normal distribution is more than twice the premium based on the beta distribution. The magnitude of this difference suggests that there are potentially serious implications to the Federal Crop Insurance Corporation practice of using the normal distribution as a maintained hypothesis. It appears that the symmetry forced on the distribution of crop yields by the normal distribution causes the probability of significantly below normal yields to be overstated, and this causes insurance premia to be higher than they should be.

In this application it was necessary to have an explicit expression for the probability distribution of output, so that the probability of losses could be calculated. It would have been very difficult to obtain the necessary information with the moment based approach to stochastic production. Whereas the parameter based approach provided estimates of the distributions which could be used directly. And, as the example demonstrates, the ability to model the distribution of output as something other than normal can have a significant impact upon the results of a study. It would appear that the technique of modelling output as a random variable with a distribution conditioned on inputs can contribute to research on the economics of agricultural production.

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