ON AGGREGATE NOTIONS OF SHEPHARD'S AND HOTELLING'S LEMMAS

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ABSTRACT

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Duality theory forms the basis of many recent analyses of aggregate economic relationships, despite the fact that the theory was conceived in the context of firm-level behavior. This paper explores notions of aggregate cost and profit functions that lead to aggregate analogs of Shephard's and Hotelling's lemmas.

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I. Introduction

A relatively recent phenomenon in the agricultural economics profession is the widespread use of duality theory as a basis for empirical and theoretical analysis. The dual approach suggests that, for example, the profit function of a firm contains the same economically relevant information about the production technology as the firm's production function--under certain general conditions. Duality theory is complementary to traditional neoclassical firm theory approaches for analyzing economic relationships.

The analysis of aggregate economic behavior in agriculture has been a prominent theme in duality-based studies. However, duality theory, just as neoclassical firm theory, is strictly speaking a firm-level paradigm. Given the popularity of dual approaches at the aggregate level, it would seem that there should be detailed reviews of the extension of the micro-level dual constructs to that level. This does not appear to be the case and our purpose here is to make a contribution in this regard. Specifically, we provide the fundamentals of an aggregate notion of the cost function and the profit function for which Shephard's lemma and Hotelling's lemma provide direct aggregate analogs to firm-level derivations of supply and demand functions using the lemmas. In the process we establish a number of properties of the aggregate cost and aggregate profit functions that are analogous to the familiar properties possessed by firm-level cost and profit functions.

II. Aggregate Cost/Shephard's Lemma

In the discussion of the aggregate cost function (and the aggregate profit function to follow), we consider the case of firms producing a single homogeneous product using many productive inputs. We focus on the issue of aggregating across firms. Firm-level properties of production, cost and profit functions are fundamental to the aggregate properties defined, and we discuss these properties where appropriate.

Define the aggregate cost function as follows:

$$C(W,Q) = \min_{q_1,\ldots,q_m} \sum_{j=1}^{m} c_j(W,q_j) \quad \text{s.t.} \sum_{j=1}^{m} q_j = Q \tag{1}$$

where W is the nx1 vector of input prices, Q is the aggregate output level, and $c_j(W,q_j)$ is the jth firm's cost function (the minimum cost of producing output level q_j given W and the jth firm's production technology). The aggregate cost minimization problem in (1) can be rewritten in Lagrangian form as

$$\min_{q_1,\ldots,q_m,\lambda} L(q_1,\ldots,q_m,\lambda) = \min_{q_1,\ldots,q_m,\lambda} \sum_{j=1}^{m} c_j(W,q_j) - \lambda[\sum_{j=1}^{m} q_j - Q] \tag{2}$$

having first-order conditions given by

$$\frac{\partial L(\cdot)}{\partial q_j} = \frac{\partial c_j(W,q_j)}{\partial q_j} - \lambda = 0 , \qquad j = 1,\ldots,m \tag{3}$$

$$\frac{\partial L(\cdot)}{\partial \lambda} = Q - \sum_{j=1}^{m} q_j = 0 .$$

Assuming that the firm-level cost function $c_j(W,q_j)$ is twice continuously differentiable and strongly convex in q_j $\forall j$ (see Roberts and Varberg, p. 268) it can be straightforwardly shown that the Jacobian matrix of derivatives of (3) with respect to $(q_1,\ldots,q_m,\lambda)$ is nonsingular, implying by the implicit function theorem that solutions of the first order conditions are continuously differentiable and take the form[1]

$$q_j^* = q_j(W,Q) \quad , \quad j=1,\ldots,m, \tag{4}$$
$$\lambda^* = \lambda(W,Q).$$

The optimal solutions (4) define the aggregate cost-minimizing distribution of output across firms and the marginal cost of aggregate output, given W and Q. The aggregate cost function can then be defined as the indirect objective function

$$C(W,Q) = \sum_{j=1}^{m} c_j(W,q_j(W,Q)) = \sum_{j=1}^{m} c_j^*(W,Q) \tag{5}$$

where $c_j^*(W,Q)$ is the jth firm's contribution to the cost of producing aggregate output level Q given W.

The aggregate cost function defined by (5) can be used to demonstrate an aggregate analog to Shephard's lemma. At the given point (W,Q), partially differentiate (5) with respect to W_i to obtain

$$\frac{\partial C(W,Q)}{\partial W_i} = \sum_{j=1}^{m} \left[\frac{\partial c_j(W,q_j^*)}{\partial W_i} + \frac{\partial c_j(W,q_j^*)}{\partial q_j^*} \frac{\partial q_j^*}{\partial W_i} \right] \tag{6}$$

From the first order conditions (3), $\frac{\partial c_j(W,q_j^*)}{\partial q_j^*} = \lambda^* \ \forall j$; and $\sum_{j=1}^{m} q_j^* = Q$ implies that $\sum_{j=1}^{m} \frac{\partial q_j^*}{\partial W_i} = 0$, so that (6) can be written equivalently as

$$(7) \qquad \frac{\partial C(W,Q)}{\partial W_i} = \sum_{j=1}^{m} \frac{\partial c_j(W,q_j^*)}{\partial W_i} + \lambda^* \sum_{j=1}^{m} \frac{\partial q_j^*}{\partial W_i}$$

$$= \sum_{j=1}^{m} \frac{\partial c_j(W,q_j^*)}{\partial W_i}$$

$$= \sum_{j=1}^{m} X_{ij}(W,q_j^*)$$

The last equality follows from the fact that $\frac{\partial c_j(W,q_j^*)}{\partial W_i}$ is the partial derivative of the jth firm's cost function with respect to input price i, evaluated at the point (W,q_j^*), which by Shephard's lemma applied at the firm level represents the level of input i utilized by firm j to produce output level q_j^* (the aggregate cost-minimizing output allocation to firm j) at input price W.[2] Defining the aggregate cost-minimizing input demand for input i, conditional on aggregate quantity Q, as

$$(8) \qquad X_i^+(W,Q) = \sum_{j=1}^{m} X_{ij}(W,q_j(W,Q)),$$

we finally write

$$(9) \qquad \frac{\partial C(W,Q)}{\partial W_i} = X_i^+(W,Q),$$

which represents the aggregate analog to Shephard's lemma stating that the partial derivative of the aggregate cost function with respect to input price i yields the aggregate constant output-input demand function for input i.

Partial differentiation of aggregate cost (5) with respect to Q yields

(10) $$\frac{\partial C(W,Q)}{\partial Q} = \sum_{j=1}^{m} \frac{\partial c_j(W,q_j^*)}{\partial q_j^*} \frac{\partial q_j^*}{\partial Q} .$$

From the first order conditions (3), we know that $\frac{\partial c_j(W,q_j^*)}{\partial q_j^*} = \lambda^* \; \forall j$, and the constraint $\sum_{j=1}^{m} q_j^* = Q$ implies that $\sum_{j=1}^{m} \frac{\partial q_j^*}{\partial Q} = 1$, so that

(11) $$\frac{\partial C(W,Q)}{\partial Q} = \lambda(W,Q),$$

representing the marginal cost of aggregate output.

The aggregate cost function inherits a number of familiar properties from the firm-level cost functions (see Varian, p. 44, for a review of firm-level properties):

C.1.) $C(W,Q)$ is nondecreasing in W.

<u>proof</u>. This follows from (7) and the fact that at the firm level, $\frac{\partial c_j(\cdot)}{\partial W_i} \geqq 0$.

C.2.) $C(W,Q)$ is homogeneous of degree 1 in W.

<u>proof</u>. Let γ be a positive number. It follows from (5) that

(12) $$C(\gamma W,Q) = \sum_{j=1}^{m} c_j(\gamma W, q_j(\gamma W,Q)).$$

It is evident from the first order conditions (3) and the linear homogeneity of $c_j(W,q_j)$ in W, $\forall j$, that $q_j(W,Q)$ is homogeneous of degree zero in W $\forall j$. Therefore

(13) $$C(\gamma W,Q) = \sum_{j=1}^{m} c_j(\gamma W, q_j(W,Q))$$

$$= \gamma \sum_{j=1}^{m} c_j(W, q_j(W,Q)) = \gamma C(W,Q).$$

C.3.) $C(W,Q)$ is concave in W.

proof. Given the twice continuously differentiable assumption pertaining to the firm-level cost functions, $\frac{\partial^2 c_j(W,q_j^*)}{\partial W \partial W'}$ is negative semidefinite by concavity of $c_j(W,q_j^*)$ in W. It follows from (7), and the fact that $\frac{\partial^2 c_j(W,q_j^*)}{\partial q_j \partial W_i} = \frac{\partial \lambda(W,q_j^*)}{\partial W_i}$ $\forall j$, that

$$(14) \qquad \frac{\partial^2 C(W,Q)}{\partial W \partial W'} = \sum_{j=1}^{m} \frac{\partial^2 c_j(W,q_j^*)}{\partial W \partial W'}$$

so that (14) is negative semidefinite, implying $C(W,Q)$ is concave in W.

C.4.) $C(W,Q)$ is continuous in W.

proof. Follows directly from (5) and the fact that both $c_j(W,q_j)$ and $q_j(W,Q)$ are continuous functions of their augments.

In summary, the concept of an aggregate cost function defined via (1) allows an aggregate analog of Shephard's lemma to be defined, and the aggregate cost function possesses the familiar properties attributed to firm-level cost functions. We highlight some caveats concerning the domain of definition and the empirical usefulness of $C(W,Q)$ in the concluding section of the paper.

III. Aggregate Profit/Hotelling's Lemma

The conceptualization of the aggregate profit function has received some attention in the literature, usually in the context of general equilibrium analysis (See Debreu, 1959; Varian, 1984) although aggregate analogs to Hotelling's lemma do not appear to have been presented. We define aggregate profit as:

$$(15) \qquad \Pi(P,W) = \max_{Q,q_1,\ldots,q_m} P\cdot Q - \sum_{j=1}^{m} c_j(W,q_j) \qquad \text{s.t.} \sum_{j=1}^{m} q_j = Q$$

with P equalling the price of output and all other symbols are as previously defined. Again, we formulate the problem in the Lagrangian form

$$(16) \qquad \max_{Q,q_1,\ldots q_m,\lambda} L(Q,q_1,\ldots,q_m,\lambda) = \max_{Q,q_1,\ldots q_n,\lambda} P\cdot Q - \sum_{j=1}^{m} c_j(W,q_j) + \lambda[\sum_{j=1}^{m} q_j - Q]$$

The first-order conditions of the problem are:

$$(17) \qquad \frac{\partial L(\cdot)}{\partial Q} = P - \lambda = 0$$

$$(18) \qquad \frac{\partial L(\cdot)}{\partial q_j} = - \frac{\partial c_j(W,q_j)}{\partial q_j} + \lambda = 0 \qquad j=1,\ldots,m$$

$$(19) \qquad \frac{\partial L(\cdot)}{\partial \lambda} = \sum_{j=1}^{m} q_j - Q = 0$$

which imply that aggregate profit is maximized when each firm has maximized profits of producing q_j at prices (P,W) since (17) and (18) together imply that marginal cost equals marginal revenue for each firm. Following a non-singular Jacobian argument similar to the one used in the preceding discussion of the cost function, the implicit function theorem implies that solutions of (17)-(19) are continuously differentiable and of the form

$$(20) \qquad \overset{\circ}{q}_j = q_j(P,W) \qquad j=1,\ldots,m$$

$$\overset{\circ}{\lambda} = P$$

$$Q = \sum_{j=1}^{m} q_j(P,W).$$

The equations (20) define the aggregate profit maximizing distribution of production across firms (which in this case is simply the collection of supply function images of (P,W) for all firms), the marginal revenue as equal to P, and defines Q as the aggregate profit-maximizing output level.

Since aggregate profit is maximized when each individual firm has maximized profits, the aggregate profit function can alternatively be represented as

$$\Pi(P,W) = \sum_{j=1}^{m} \pi_j(P,W) \tag{22}$$

where $\pi_j(P,W)$ is the profit function of firm j. Using (22), an aggregate form of Hotelling's lemma is apparent. First partially differentiating $\Pi(P,W)$ with respect to P yields

$$\frac{\partial \Pi(P,W)}{\partial P} = \sum_{j=1}^{m} \frac{\partial \pi_j(P,W)}{\partial P} = \sum_{j=1}^{m} q_j(P,W) = Q(P,W) \tag{23}$$

since the partial derivative of jth firm's profit function with respect to output price yields the firm's supply function, $q_j(P,W)$. Thus the partial derivative of aggregate profit with respect to output price yields the aggregate supply function. Partially differentiating $\Pi(P,W)$ with respect to W_i, and then multiplying the result by -1 yields

$$-\frac{\partial \Pi(P,W)}{\partial W_i} = -\sum_{j=1}^{m} \frac{\partial \pi_j(P,W)}{\partial W_i} = \sum_{j=1}^{m} X_{ij}(P,W) = X_i(P,W) \tag{24}$$

since the negative of the partial derivative of the jth firm's profit function with respect to input price W_i yields the firm's input demand function for input i, $X_{ij}(P,W)$. Thus, the negative of the partial derivative of aggregate profit with respect to the price of the ith input yields the aggregate demand function for input i.

The aggregate profit function inherits all of the familiar properties attributed to firm-level profit functions (see Varian, p. 46, for a review of firm-level properties):

Π.1.) $\Pi(P,W)$ is nondecreasing in P and nonincreasing in W.

proof. Follows directly from (22) and the fact that the firm-level profit functions possess this property.

Π.2.) $\Pi(P,W)$ is homogeneous of degree 1 in (P,W).

proof. From (22), and the linear homogeneity of firm level profit function,

$$\Pi(\gamma P,\gamma W) = \sum_{j=1}^{m} \pi_j(\gamma P,\gamma W) = \gamma \sum_{j=1}^{m} \pi_j(P,W) = \gamma\Pi(P,W)$$

for $\gamma>0$.

Π.3.) $\Pi(P,W)$ is convex in (P,W).

proof. Let $\zeta = \begin{bmatrix} P \\ W \end{bmatrix}$. It follows from (22) that

$$\frac{\partial^2\Pi(P,W)}{\partial\zeta\partial\zeta'} = \sum_{j=1}^{m} \frac{\partial^2\pi_j(P,W)}{\partial\zeta\partial\zeta'} \tag{25}$$

and since each matrix in the sum is positive semidefinite by the convexity of firm-level profit functions, (25) is positive semidefinite and $\Pi(P,W)$ is convex in (P,W).

Π.4.) $\Pi(P,W)$ is continuous in (P,W).

proof. Follows directly from (22) and the continuity of individual firm-level profit functions.

In summary, the concept of an aggregate profit function defined by (15), which implies (22), allows an aggregate analog of Hotelling's lemma to be defined, and the aggregate profit function possesses the familiar properties attributed to firm-level profit functions. We now introduce a number of caveats concerning the domain of definition and empirical usefulness of $\Pi(P,W)$ and $C(W,Q)$.

IV. Concluding Remarks

The preceding discussion presented the basic ideas underlying an aggregate concept of Shephard's and Hotelling's lemmas. Our conceptualization essentially allows discussions of the firm-level theory to be interpreted in an aggregate context as well. While space does not permit a detailed discussion of refinements and extensions to the basic concepts presented, we mention some points worth pondering.

A caveat concerns the domain on which the application of Shephard's and Hotelling's lemmas to aggregate cost and aggregate profit functions is defined. The preceding development implicitly assumed that at the point of evaluation ((W,Q) in the cost function case, (P,W) in the profit function case), $C(W,Q)$ and $\Pi(P,W)$ are defined and continuously differentiable. A simple caveat then, which applies to firm-level applications as well but is often not stressed, is that the lemmas apply when the appropriate aggregate functions are defined and continuously differentiable. Depending on the properties assumed for the firm-level production functions, the application of the lemmas to the aggregate functions may be appropriate to the entire positive orthant of R^{n+1} - space, R_+^{n+1} (representing (P,W) or (W,Q) coordinates), or only to a subspace of R_+^{n+1}. For example, under conditions on the production function used by Lau (1978) in his discussion of firm-level profit functions, he shows that $\pi_j(P,W)$ is defined and continuously differentiable on all of R_+^{n+1}, so that the aggregate form of Hotelling's lemma presented heretofore would apply throughout R_+^{n+1}. Lau's assumptions are somewhat strong, where in particular he assumes strong concavity of the firm's production function on the interior of the input space, and that the range of the marginal product functions (i.e., the gradient of the production function) is all of R_+^n. It can also be shown under Lau's assumptions that $c_j(W,q_j)$ is defined and continuously differentiable on all of R_+^{n+1}, and thus the aggregate form of Shephard's lemma would also be applicable throughout R_+^{n+1}.

If the underlying firm-level production functions were only strictly quasiconcave, say, and exhibit strong concavity on only convex subsets of the input space with the range spaces for the gradient functions not equal to R_+^n, then the $\pi_j(P,W)$'s will not be continuously differentiable on all of R_+^{n+1}. Essentially, nondifferentiability of $\Pi(P,W)$ will occur at sets of (P,W) points on the boundaries of the price sets representing "shut down" prices for one or more firms in the aggregate. Similar nondifferentiability problems can occur for C(W,Q) when one or more firms are at a boundary solution in the cost minimizing distribution of production across firms in the aggregate. We stress again that these complications are not unique to the dual approach, but have analogs in the primal approach. For example, defining the aggregate supply function as the summation of segments of firm-level marginal cost curves above average variable cost produces an aggregate supply curve that can be discontinuous at a finite number of (P,Q) points. The discontinuities correspond to nonexistence of derivatives of $\Pi(P,W)$ with respect to P at certain points.

The results in the paper provide some justification for the practice in the literature of specifying cost or profit functions from firm-level arguments, and then applying them to the analysis of industry aggregates. It has been shown, in fact, that aggregate profit and cost functions inherit the common properties attributed to firm level profit and cost functions. Thus, functional forms used to model firm-level cost and profit functions are candidates for modeling aggregate cost and profit functions.

Additional research is needed on the linkages between the characteristics of firm-level technologies and the functional form appropriate for aggregate profit and cost functions. Also, methods for handling non-differentiability of aggregate profit and cost functions due to firm entry/exit require additional thought. It is hoped that the fundamental ideas presented in the paper encourage additional research into more rigorous theoretical interpretations of the use of dual constructs at the aggregate level.

FOOTNOTES

[1]It can also be shown that under the strong convexity assumption, the bordered Hessian has the appropriate negativity of all principal minors for the second order condition for a minimum to hold.

[2]The result (7) could also be established more directly, but less informatively, by a direct application of the Envelope theorem to (2) and (5).

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