Low-Level Equilibrium and Fractional Poverty Traps

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Abstract:

For more than 200 years development and agricultural economists have sought to understand the dynamic relationships between population growth, land utilization, and agricultural productivity. Originating with Malthus, numerous scholars have pushed representations of low and high-level equilibrium traps. More recently the literature has explored asset dynamics and fractal poverty traps. In this paper we advance these models by introducing risk into a dynamic growth model using the stochastic calculus and Ito's Lemma. This approach does two important things. First it moves the discussion away from the idea of a stable short run equilibrium, to one in which the long-run economic attractor is an unknowable point in probability space. On this latter point we are able to show, via Monte Carlo simulation, that population growth is fractional and persistent with Hurst coefficient of around 7.0, while other measures such as output per capita are dynamically fractional with Hurst coefficients in the neighborhood of 0.3 to 0.4. This leads us to believe that poverty traps ought not be measured in a small time scale, but rather a longer time scale reflecting the frequency, duration, and intensity of below-subsistence excursion patterns. We make our case by simulating the Chines agricultural economy between about 1400 and 1900.

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1.1 Introduction

For more than 200 years development and agricultural economists have sought to understand the dynamic relationships between population growth, land utilization, and agricultural productivity. Originating with Malthus, numerous scholars in including Nurkse, Nelson, Elvin among others have pushed representations of low and high-level equilibrium traps. More recently the poverty trap literature has explored asset dynamics and fractal poverty traps (Swallow and Barrett). In this paper we advance these models by introducing risk into a dynamic growth model using the stochastic calculus and Ito's Lemma. This approach does two things which we feel are important. First it moves the discussion away from the idea of a stable short run equilibrium with a fixed attractor, to one in which the attractor is an unknowable point in probability space. This latter point becomes more important than the obvious, in that we are able to show, via Monte Carlo simulation that population growth is fractional and persistent with Hurst coefficient of around 7.0, while other measures such as output per capita are dynamically fractional with Hurst coefficients in the neighborhood of 0.3 to 0.4. This leads us to believe that poverty traps ought not to be measured in a small time scale, but rather a longer time scale and the frequency, duration, and intensity of below-subsistence excursion patterns. We make our case by simulating the Chines agricultural economy between about 1400 and 1900.

1.2 The Needham Puzzle

In terms of economic growth, China's agricultural economy might best be characterized as being historically perplexing. With the exception of minor adaptations to agricultural practices, it appears that China progressed little in terms of agriculture between the 14th century and the era of rural reconstruction which started about 1921 during the Republican era. Perhaps the most intriguing aspect of China's growth has been labeled the 'Needham Puzzle'[1] .

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[1] Lin, J. Y. (1995). The Needham puzzle: Why the industrial revolution did not originate in China. Economic development and cultural change, 43(2), 269-292.
Lin, J. Y. (2008). The Needham puzzle, the Weber question, and China's miracle: Long-term performance since the Sung dynasty. China Economic Journal, 1(1), 63-95.
Reference is to British sinologist Joseph Needham who published Needham, Joseph. Science and Civilization in China. Volume 1. Cambridge: Cambridge University Press, 1954 and multiple volumes thereafter. Posing the paradox established by Needham as a puzzle appears to have been asked by Needham in Joseph Needham, "Introduction," in China: Land of Discovery and Invention, by Robert K. G. Temple (Wellingborough: Patrick Stephens, 1986), p. 6.
Boulding, Kenneth. "The Great Laws of Change." in Anthony M. Tang, Fred M. Wsetfield and James S. Worley, eds. Evolution, Welfare and Time in Economics. Lexington: S.C. Heath Lexington Books, 1976.

The Needham puzzle refers to a question posed by Joseph Needham as to why up to the past 3 or 4 centuries, China led the Europeans in so many aspects of science and technology yet failed to lead in the industrial revolution. Needham put this paradox in the form of two challenging questions: first, why had China been so far in advance of other civilizations; and second, why isn't China now ahead of the rest of the world?[2] By the 14th century China was cosmopolitan, technologically advanced, and economically powerful, so much so that in relation the West was essentially agrarian, poor and underdeveloped[3]. The marvels of agricultural innovation at the turn of the 14th century are not in dispute, nor is the observable decline in agricultural and related innovations. Perkins, for example notes that starting around the 14th century there were no dramatic changes in farming techniques or in rural institutions. As evidence he notes that in three separate handbooks date 1313, 1628, and 1742 included virtually the same list of 77 implements. So at one scale it is entirely possible to observe what Huang defines as 'involution' and a diminishing marginal productivity of labor[4], while at another scale observing gains in aggregate output to keep up with population increases, even at the level of subsistence. Nonetheless, Lin divides the many explanations into two categories, those which hypothesize that the explanation is due to a failure of demand for technology and those based on a failure of supply of technology. .Mark Elvin[5] notes that the falling land to labor ratio in the medieval period led to the use of multiple cropping to increase per-hectare yields, quick ripening rice was imported from Vietnam allowing two crops of rice in the south and a dual crops of wheat and rice in the north[6]. A practice of staggering crops throughout the year was employed to reduce risks from natural disaster and to smooth out the use of labor; and *"…it seems that a number of fine discoveries are due to bizarre experiments and simple caprice, or even to negligence and mistakes…The little extra efforts and knacks, inventions and discoveries, resources and combinations, which have caused people to exclaim at miracles in gardens, have been transported on a large scale out into the fields and have done marvels[7].*"

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[2] Lin, J.Y. (1995) Op Cit., page 271
[3] Lin, J.Y. (1995) Op Cit., page 270. Cf. Mark, page 177,
Elvin, Mark (1973) The Pattern of the Chinese Past .Stanford, Calif.: Stanford University Press, 1973
[4] Huang, P. (1985). The Peasant Economy and Social Change in North China. Stanford University Press, Stanford CA
[5] Elvin, M. (1982) "*The Technology of Farming in Late-Traditional China*", Chapter 2 in "The Chinese Agricultural Economy", Edited by R. Barker, R. Sinha and B. Rose, Westview Press, Boulder Colorado
[6] Reference is to Champa rice which originated in India and found its way to China via Vietnam during the reign of emperor Zhenzong (992-1022). See Barker, R. (2011) "The Origin and Spread of Early-Ripening Champa Rice: It's Impact on Song Dynasty China" *Rice* 4:184-186.
[7] Elvin M. (1982) *Op cit.* Page 13, cited from unnamed missionary in "*Mémoires Concernant les Chinois* (Paris, Nyon, 1776-1814) (Sic)

Duhalde's observations in 1741 that "*there is no part of China that can properly be said to be barren; and some parts are naturally so fruitful, that hey yield a crop twice in a year; and others again owe their fruitfulness to the indefatigable toil of the husbandmen.. But as the quantity of land proper to be cultivated is not very great in mountainous provinces, it is no wonder that hose which are more fruitful, should scarcely be sufficient for the maintenance of such a multitude of inhabitants* "[8]. He noted the varnish tree that produced a sap used by artisans, the tong-chu tree which provided a varnish used in preserving wood, a tallow-tree whose pulp is used to make candles, and a white-wax tree onto which worms affix themselves to produce a wax more durable and valuable than bee's wax. In Duhalde's 1772 report he describes the land pressures then already existing on the plains that "*neither hedge nor ditch is to be seen, and but few trees, so much are they afraid of losing an inch of ground*" [9]. He records the terracing of mountainsides already underway, and the methods of loosening stones on rocky mountains and using them to make little walls within which they level with good soil. He notes that laboring of constructing reservoirs to feed rice fields below and the invention of a 'hydraulic engine' to move water up from canals into the fields or terraces. He observes the use of salt, lime, ashes and natural animal and human dung and urine, and the practice of burying balls of hogs-hair and indeed human hair to invigorate the land, although they had yet to discover mineral fertilizers such as marl.

But it was Duhalde's description of population and land that Malthus latched on to in his (1798-1816) Principle of Population. With a largely agrarian economy the land is continually subdivided and distributed to successive heirs so that even the wealthiest of landowners would see their heirs reduced to poverty within three generations[10] .

Not surprisingly, the conditions of poverty in China in the late 18th century differed little from conditions in the early 20th century. Malthus reporting of conditions then would not materially be different from newspaper accounts at the turn of the 20th century. "*It is well known that extreme misery impels people to the most dreadful excesses… that mothers destroy or expose many of their children; that parents sell their daughters for a trifle; that there should be such a number of*

[8] DuHalde, P. (1742) The General History of China; Containing a Geographical, Historical, Chronological, Political and Physical Description of the Empire of China, Chinese Tartary and Tibet., Volume 1. 3rd edition. J. Watts, London., Page 8

[9] Duhalde, P (1772) "The Chinese Traveler Containing a Geographical, Commercial, and Political History of China" Printed for E. and C. Dilly, London, Page 211

[10] Malthus, T. R. (1888). An essay on the principle of population: or, A view of its past and present effects on human happiness. Reeves & Turner, Page 104

robbers…and that in times of famine which are here but too frequent, millions of people should perish with hunger, without having recourse to those dreadful extremities… "[11].

But Malthus also took care not to equate China's woes with those of Europe: "*It cannot be said in China, as in Europe, that the poor are idle, and might gain a subsistence if they could work. The labours and efforts of these poor people are beyond conception. A Chinese will pass whole days digging the earth, sometimes up to his knees in water, and in the evening is happy to eat a little spoonful of rice, and to drink the insipid water in which it was boiled.*" [12] It is unsurprising with Duhalde's and Malthus' observations that China's agricultural economy would find itself entrapped in a cycle of poverty. Persistent poverty in agriculture ultimately deprives the general economy from much needed savings and capital, while an abundance of labor suppresses the need to innovate.

1.3 Low-Level Equilibrium Traps

Several demand-failure models have been developed to explain the lapse in agricultural innovation in China. Prominent among these is the model put forth by Elvin, but before this were Nurkse and Nelson. Nurkse (1952) makes the argument that balanced growth rests ultimately on the need for a balanced diet[13]. The imbalance results, at least in part, to the inelasticity of demand for consumables at low real income levels, so that almost all goods are seen as necessities. Thus begins the circular relationship in low income economies that the inelasticity of demand leaves little capacity to save, and thus the capital to invest, and thus to low productivity. The lack of buying power impedes any incentives to invest in a diversified industrial base that would ordinarily provide complementary goods and services so that the new entrepreneurs become each other's customers and slowly extract themselves from the deadlock of a low-level equilibrium trap.

Drawing on this Nelson (1956) defined what he refers to as a low-level equilibrium trap[14]. A low-level equilibrium trap arises when the population growth rate equals the rate at which capital stock is accumulating. Under such an economy the amount of capital per worker is not increasing and the economy cannot grow. Capital formation, changes in population, changes in output and

[11] Malthus, T.R. (1888) *Op Cit*. Page 105
[12] Malthus, T.R. (1888) Op Cit. Page 105
[13] Nurkse, R. (1952). Some international aspects of the problem of economic development. The American economic review, 42(2), 571-583.
[14] Nelson, R. R. (1956). A theory of the low-level equilibrium trap in underdeveloped economies. The American Economic Review, 46(5), 894-908.

the social , political, and economic organization of the economy ultimately determine various equilibria where changes in population equal changes in output. Changes in population are bounded by the maximum biological rate and it is assumed that this arises only after a period of time at which per capital output was substantially higher than a subsistence rate. The boundary to the low-level equilibrium poverty trap is distinguished by the boundary of subsistence; that is (typically underdeveloped) economies in which output per capita are at or below subsistence, versus those (typically developing or developed) in which output per capita is above subsistence. The low level equilibrium trap relates to low income-low technology states and is a stable equilibrium – an equilibrium that persists – when the population growth rate intersects the output growth curve from above. The crucial insight from Nelson's Malthusian Trap model is representation of the existence of persistent poverty traps that arise from the relationship between changes in population and changes in output or income. John Lossing Buck[15] notes that "*But because of the dense population, the Chinese farmer is doomed and all that can be done is to make the most out of an unfortunate situation*"[16], and later "*The remedies for this too small size of farm business are difficult to find…As China become modernized, it is inevitable that industries will develop and a certain number of the country people be absorbed into them. Yet it can scarcely be hoped that sufficient numbers of them be absorbed as to relive the present agricultural situation very much. The best future solution of the problem seems to be in some method of population control, and the best immediate solution, more intensive methods of raising crops and the growing of crops that produce more food per unit of land. Such productivity, however, will also be useless if population continues to grow*"[17].

1.4 High-Level Equilibrium Traps

Nelson's model is essentially a static short run model. As the technological, economic, and political environment changes the so too would conditions leading into new equilibrium traps, or exiting existing equilibrium traps. An alternative model is Elvin's proposition of a high-level equilibrium trap[18]. In Elvin's model he uses the term "high-level", rather than "low-level" to

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[15] Buck, J.L. (1930) Chinese Farm Economy: A Study of 2866 Farms in Seventeen Localities and Seven Provinces in China. University of Chicago Press, Chicago Illinois, Pages 147-148,
[16] Buck (1930) *Op Cit.* page 314
[17] Buck (1930) *Op Cit.* Page 424
[18] Elvin (1973*) Op Cit.*.
Sinha, R. P. (1973). Competing ideology and agricultural strategy: current agricultural development in India and China compared with Meiji strategy. *World Development*, 1(6), 11-30.
See also
Elvin, M. (1984). Why China failed to create an endogenous industrial capitalism. Theory and Society, 13(3), 379-391.

describe an equilibrium in which all innovations towards a maximal level of agricultural productivity have been exhausted at both the intensive an extensive margin. As populations increased the cost of labor fell relative to investments in capital, removing any economic incentives to develop labor savings technology. In addition as poorer quality land came to be cultivated the rental value of that land – its marginal value – also fell so that the demand for technology also fell. As lower productivity land was being brought into production for an increasing rural population, the retentions held back for household consumption as a proportion of total output would also increase. Elvin's pathway to a (high level) equilibrium trap is much consistent with the anthropologic approach of Boserup (1975; see also Darity 1980)[19]. Where Elvin and Boserup depart is that Boserup's intensification path always stays ahead of population pressure and thus avoids the equilibrium trap. In fact, Darity's dynamic interpretation of Boserup's anthropology reveals an unstable equilibrium that is sensitive to initial conditions. In one world, Boserup's evolution guarantees perpetual poverty, while in another world wages rise sufficiently high to absorb excess output.

1.5 Equilibrium and Poverty Traps

A resurgent view of the Malthusian Trap in more recent years has opted for the more broadly defined 'Poverty Trap'. In fact to our sensibilities the use of the term poverty trap is preferable to the Malthusian trap or its neo-Malthusian counterparts because it does not exclude output to population dynamics at the macro level, while allowing for a micro, short-run focus simultaneously. Perhaps more important is that it opens the economics to random influences in the small and in the large and thus widens the range of policy option to include credit, insurance, and other forms of agricultural stabilization and social welfare.

In the poverty trap literature a dynamic equilibrium exists when a unit of well-being (income, assets) neither increases nor decreases in real terms between one period and the next. An equilibrium exists as an attractor of sorts in which economic forces, good or bad, will move a household away from that initial equilibrium into an alternative state. How long the household remains in that state depends on degrees of resilience and asset dynamics. The particular problem

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Elvin, M. (1972) '*The high-level equilibrium trap: the causes of the decline of inventions in the traditional Chinese textile industries*', in W. E. Willmott (ed.), Economic Organization in Chinese Society .Stanford University Press, Stanford CA.
Elvin M. (1996) "Another History: Essays on China from a European Perspective" The University of Sydney East Asian Series #10. Wild Peony, Broadway, NSW, Australia

[19] Boserup, E. (1975). The impact of population growth on agricultural output. *The Quarterly Journal of Economics*, 257-270.
Darity, W. A. (1980). The Boserup theory of agricultural growth: a model for anthropological economics. *Journal of Development Economics*, 7(2), 137-157.

addressed in Carter and Barret was the differentiating control of income/expenditure measures which by pure randomness can rise and fall as markets and ecology dictate[20]. In the fractal sense of Barrett and Swallow a poverty trap is one in which multiple simultaneous equilibria exist in much the same way as the equilibrium trap literature suggests[21]. In the Barrett and Swallow context fractal poverty traps can exist simultaneously at multiple scales (micro, meso and/or macro) and are self-reinforcing through feedback effects. The essential element of a fractal poverty trap is that the pattern repeats at all scales of aggregation; that is the forces which drive farm households into poverty by a particular dynamic, are the same forces that drive a county into poverty, and are the same forces that drive a country into poverty. The forces are endogenous to each other and are self-reinforcing.

1.6 A Stochastic-Dynamic Model for Low-level/ High-Level Equilibrium and Poverty Traps

As hinted above, the concept of equilibrium in a stochastic world is difficult to rationalize since nothing ever is stable. Instead from any particular set of initial conditions the stochastic path sets a trajectory towards some distant loci to which it oscillates in some random fashion. In the interim there are a multitude of paths, in fact an infinite of paths upon which random and seemingly independent shocks can change economic trajectories in a permanent way. Where China ended up in 1921 is not due to any single factor but the accumulation of seemingly random effects which accumulate in China's history. Each event has the effect of shifting the population-land-economic dynamic in setting a new trajectory which may be exacerbated or reversed by subsequent random events.

In the equilibrium trap literature the boundary between economic surplus (gains in income, savings and consumption) and poverty states (low income, depleted savings, low consumption) is determined by the boundary of the output to population ratio that describes bare subsistence. Below this line affected households fall in to a state of below-subsistence poverty which has numerous impacts on population and productivity.

These states of nature can be viewed in a random world as excursions, and how long they persist depends upon the nature of the shocks that occurred, the intensity of the shocks and the

[20] Carter, M. R., and C.B. Barrett, (2006). The economics of poverty traps and persistent poverty: An asset-based approach. *The Journal of Development Studies*, 42(2), 178-199.
[21] Barrett, C. B., and B.M. Swallow, (2006). Fractal poverty traps. *World development*, 34(1), 1-15.

duration. In some years, a drought may cause distress but abundance in the following year can reverse conditions. The number of people who die in the interim depends upon resilience. Consequently, the dynamic evolution of agriculture forms a stochastic differential equation of the Ito type which appears to be fractional in typology. By fractional we mean that certain measures such as output or population or ratios of the two do not follow a random walk in the classical Brownian sense, of say a stock market. Instead, the nature of things interact in ways that are self-reinforcing. Thus events of politics and nature are not simply independent random draws but are characteristically correlated across time in a systematic manner. For example, if population growth is dependent on the capacity of the land to feed its population and the resilience of the population to calamities and conflicts,

1.6.1 Population Dynamics

Under the Malthusian argument population growth is based on the natural growth rate comprised of birth, death and migration rates, $g = (\mu_P - d - m)$. We assume that population growth is based on this natural growth rate, but also mitigated by the capacity of agricultural output to feed the population, P^* and random effects arising from natural and man-made calamities and conflicts. Here, $P_t^* = \frac{Y_t}{c}$ is the ratio of aggregate output divided by per capita output requirements. We state this as a geometric Brownian motion in continuous time with ϕ being a measure of resilience that can exacerbate or moderate the population to capacity ratio[22].

$$(1)\quad \frac{dP}{P} = g\left(1-(1-\phi)\frac{P}{P^*}\right)dt + \sigma_P dZ_P .$$

The ratio $(1-\phi)\frac{P_t}{P^*}$ captures the adjustment due to food supply with $0 \le \phi <\approx 1$ as a resilience adjustment. As P^* increases relative to population $\frac{P_t}{P^*}$ declines, which increases the population

[22] The dynamic is close to Boulding's and other specifications. He measures the population above subsistence but also includes what he refers to as an improvement coefficient to capture technological change, and a scarcity coefficient to capture resilience. Boulding (1955) *Op Cit*. Pages 199-201

growth rate. This moderates population growth as Malthus describes. Whether this is due to decreasing birth rates versus increasing death rates is difficult to discern.

1.7 Agricultural Output

We assume that agricultural output per unit of land (Mou, acre, hectare) is determined by labor and capital employed. We start off with the standard output model of labor and capital, $Y = AP^{\alpha}L^{\beta}$. We assume that population is proportional to labor supply and land is proportional to capital. However, as discussed this conventional function is unsustainable in the long run because of the diminishing capacity of land productivity as the amount of land increases, and the necessity of human capital to adapt to these conditions. We thus depreciate the elasticity of land by defining

(2) $\hat{\beta} = \beta - \lambda_L L$

and

(3) $\hat{\alpha} = \alpha + \lambda_P L$

Where λ_L and λ_P are the rates of productivity depletion and human capital appreciation respectfully. When $\lambda_P \geq \lambda_L$ the model avoids the problem of involution as described by Huang.

(4) $Y = AP^{(\alpha+\lambda_P L)}L^{(\beta-\lambda_L L)} = AP^{\hat{\alpha}}L^{\hat{\beta}}$

Or

(5) $Log(Y) = Log(A) + (\alpha + \lambda_P L)Log(P) + (\beta - \lambda_L L)Log(L)$

Note also that the production elasticities do not adjust linearly in time but with respect to land. Land evolves randomly in time but ultimately has a geographical maximum that places a real boundary on the upper limits of land. An obvious drawback to this construction is that this assumes that growth in human capital diminishes as land approaches its natural boundary. Thus, this model will likely underestimate growth in human capital beyond the years at which land is bounded from above.

1.8 Technological Innovation and Output Uncertainty

To get around this problem we address technological innovations and output risk through the adjustment (intercept) value of A. Perkins too makes allowance for the intercept to grow with innovation in time, but in a deterministic way[23]. We assume that output grows at rate μ_A, but is subject to random events σ_A. This we describe by the following Brownian motion,

$$(6)\quad \frac{dA}{A} = \mu_A dt + \sigma_A dZ_A$$

Where dZ_A is a Wiener process. Ultimately, we follow Justin Y. Lin's lead in correlating the output growth with population so regardless of the land boundary there remains a mechanism for output to adjust in real time to population changes beyond the land constraint.

1.9 Output Dynamics

There are three principal drivers of aggregate output in the above discussion. These are population dynamics and technological innovation. Both of these are described by stochastic differential equations. The third principal driver is the output function itself which feeds of land dynamics and population growth, as well as technological innovation. To extract a dynamic model for output we apply Ito's Lemma and the stochastic calculus:

$$(7)\quad \begin{aligned} dY &= \frac{\partial Y}{\partial A}dA + \frac{\partial Y}{\partial P}dP + \frac{\partial Y}{\partial L}dL \\ &+ \frac{1}{2}\left(\frac{\partial^2 Y}{\partial A^2}dA^2 + \frac{\partial^2 Y}{\partial P^2}dP^2 + \frac{\partial^2 Y}{\partial L^2}dL^2 + 2\left(\frac{\partial^2 Y}{\partial A \partial L}dAdL + \frac{\partial^2 Y}{\partial A \partial P}dAdP + \frac{\partial^2 Y}{\partial L \partial P}dLdP\right)\right) \end{aligned}$$

Using $\frac{dY}{Y} = \frac{dA}{A} + \left(\lambda_P Log\left(P\right) - \lambda_L Log\left(L\right)\right)dL + \hat{\alpha}\frac{dP}{P} + \hat{\beta}\frac{dL}{L}$ we obtain the following first and second order conditions

[23] Perkins (1969) Op Cit. , Mathematical Supplement, Pages 79-84

$$\frac{\partial Y}{\partial A} = \frac{Y}{A}$$
$$\frac{\partial Y}{\partial P} = \hat{\alpha}\frac{Y}{P}$$
$$\frac{\partial Y}{\partial L} = Y\left(\lambda_P Log(P) - \lambda_L Log(L) + \frac{\hat{\beta}}{L}\right)$$
$$\frac{\partial^2 Y}{\partial A^2} = 0$$
$$\frac{\partial^2 Y}{\partial P^2} = \frac{Y}{P^2}\hat{\alpha}(\hat{\alpha}-1)$$
$$\frac{\partial^2 Y}{\partial L^2} = Y\left(\left(\lambda_P Log(P) - \lambda_L Log(L) + \frac{\hat{\beta}}{L}\right)^2 - \frac{(2\lambda_L L - \beta)}{L^2}\right)$$
$$\frac{\partial^2 Y}{\partial A \partial P} = \hat{\alpha}\frac{Y}{AP}$$
$$(8)\quad \frac{\partial^2 Y}{\partial A \partial L} = \frac{Y}{A}\left(\lambda_P Log(P) - \lambda_L Log(L) + \frac{\hat{\beta}}{L}\right)$$
$$\frac{\partial^2 Y}{\partial L \partial P} = \frac{Y}{P}\left(\lambda_P + \hat{\alpha}\left(\lambda_P Log(P) - \lambda_L Log(L) + \frac{\hat{\beta}}{L}\right)\right)$$

Also, by Ito's Lemma we have $\left(g\left(1-(1-\phi)\frac{P}{P^*}\right)dt + \sigma_P dZ_P\right)^2 = \sigma_P^2 dt$, and

$$(9)\quad (\mu_A dt + \sigma_A dZ_A)\left(g\left(1-(1-\phi)\frac{P}{P^*}\right)dt + \sigma_P dZ_P\right) = \rho_{A,P}\sigma_A\sigma_P dt$$

With dA and dP defined by their respected stochastic differential equations we need a term for changes in land. We assume that land is driven by $L = \frac{cP}{P^*}$ where c is the per capita consumption of agricultural goods. This we assume constant so that as the change in Land is

$$(10)\qquad dL = \frac{c}{P^*}dP\,.$$

Substituting for dL , dA, dP and for convenience $L = \frac{c}{P^*}P \rightarrow \frac{L}{P} = \frac{c}{P^*}$, further arranging yields

$$\frac{dY}{Y}=\left(\begin{array}{l}\mu_A+\left(\hat{\alpha}+L\left(\lambda_P Log(P)-\lambda_L Log(L)+\frac{\hat{\beta}}{L}\right)\right)\mu_P \\ +\frac{1}{2}\left(\begin{array}{l}\left(\begin{array}{l}\hat{\alpha}(\hat{\alpha}-1)+L^2\left(\left(\lambda_P Log(P)-\lambda_L Log(L)+\frac{\hat{\beta}}{L}\right)^2-\frac{(2\lambda_L L-\beta)}{L^2}\right) \\ +2L\left(\lambda_P+\hat{\alpha}\left(\lambda_P Log(P)-\lambda_L Log(L)+\frac{\hat{\beta}}{L}\right)\right)\end{array}\right)\sigma_P^2 \\ +2\left(L\left(\lambda_P Log(P)-\lambda_L Log(L)+\frac{\hat{\beta}}{L}\right)+\hat{\alpha}\right)\rho_{AP}\sigma_A\sigma_P\end{array}\right)\end{array}\right)dt+(\sigma_A dZ_A+\sigma_P dZ_P) \quad (11)$$

There are two components to this Ito Process. The first bracketed term is the expected rate of growth in output which is determined by the contemporaneous amounts of land and population, the variance in population and the correlation between population and technological growth. We can see that the variance of this process is given by the joint relationship between output and population, including the covariance between the two. This correlation is driven by Lin's argument that as the population increases there will likely be more geniuses born and thus more innovation[24]. .Also included are the rates of appreciation and depreciation in human capital and land quality, and importantly the initial elasticities.

The second component of the equation is the variance term which captures the variation in output, via the intercept A and population, P and the covariance between the two. This Wiener process implies that the variance of the change in output per unit of time (in this case yearly) is determined by

(12) $\sigma_A^2+\sigma_P^2+2\rho_{AP}\sigma_A\sigma_P$

More specifically, the dynamics are driven by the drift term which has several components

a) $\mu_A+\left(\hat{\alpha}+L\left(\lambda_P Log(P)-\lambda_L Log(L)+\frac{\hat{\beta}}{L}\right)\right)g\left(1-(1-\phi)\frac{P}{P^*}\right)$ is the natural drift rate where $\mu_P=g\left(1-(1-\phi)\frac{P}{P^*}\right)$ is the mean reverting rate for population

[24] Lin, J.Y. (1995) Op Cit., page 271

b) $\hat{\alpha}+L\left(\lambda_P Log(P)-\lambda_L Log(L)+\frac{\hat{\beta}}{L}\right)$ scales the population growth rate to changes in land and capital. If $\lambda_P=\lambda_L=0$ then the term boils down to $\alpha+\beta$ which =1 is constant returns to scale so that the natural drift would collapse simply to $\left(\mu_A+g\left(1-(1-\phi)\frac{P}{P^*}\right)\right)t$

c) We need to keep in mind precisely the role that λ_P,λ_L plays in our model. The first appreciates human capital and learning whereas the second captures the Malthusian trap by reducing the output elasticity from land as new and poorer quality land is brought into production. If we assume these to be static, then growth in output is dependent only on the natural rates of innovation and population and the production elasticities. For example, if $\lambda_P=\lambda_L=0$ then the equation collapses to the native form

$$E\left[\frac{dY}{Y}\right]=\mu_A+(\alpha+\beta)\mu_P+\frac{1}{2}\left(\left(\beta-\alpha+(\alpha+\beta)^2\right)\sigma_P^2+2(\alpha+\beta)\rho_{AP}\sigma_A\sigma_P\right). \tag{13}$$

If it is further assumed that the initial condition shows constant returns to scale then this reduces even further to

$$E\left[\frac{dY}{Y}\right]=\left(\mu_A+\mu_P+\beta\sigma_P^2+\rho_{AP}\sigma_A\sigma_P\right). \tag{14}$$

Note that this includes the covariance effects presumed by Justin Lin so that as population grows so does innovation because a higher population will provide a larger pool of innovators and entrepreneurs.

As for the relationships themselves, if the economy is to maintain scale neutrality in terms of the production elasticities then $\lambda_P=\frac{k-\alpha-\beta+\lambda_L L}{L}$ which suggests that at any scale $\lambda_P\geq\lambda_L$. In other words if the production coefficient for the initial agriculture economy was $k=1$ then $\lambda_P=\lambda_L$. If the economic goal was to achieve better than constant returns to scale then for $\frac{\Delta k}{\Delta L}>0\rightarrow\lambda_P>\lambda_L$.

If we assume that $\frac{\hat{\beta}}{L}$ is arithmetically negligible then the crucial term in the drift of the stochastic differential equation for growth is $\lambda_P Log(P) - \lambda_L Log(L)$. This should always be positive so long as $\lambda_P = \lambda_L$ and per capita land is greater than 1. Conflicts, calamities and catastrophes can from time to time violate this presumption, but rearranging terms $\frac{\lambda_P}{\lambda_L} \geq \frac{Log(L)}{Log(P)}$ or $\lambda_P \geq \lambda_L \frac{Log(L)}{Log(P)}$. Thus as λ_P rises above λ_L, not only does the rate at which human capital rise relative to the depletion of land quality, but perhaps more importantly the slack between the two, i.e. $\lambda_P - \lambda_L > 0$, lowers the chance that the growth condition $\frac{\lambda_P}{\lambda_L} \geq \frac{Log(L)}{Log(P)}$ will be violated even when the population growth rate exceeds the rate at which new lands are brought into production.

1.10 Operationalizing the Simulation Model

The model presented above results in an intertemporal stochastic differential equation of an Ito form. As presented it assumes certain contemporaneous adjustments which are impracticable and unrealistic from a modelling point of view. We start with the following initial conditions which are drawn largely from Huang's reporting on Hebei and Shandong. There he reports combined population of 7.183 million persons in 1393 and 68 million in 1913. Over 520 years the exponential growth rate was 0.4323%/year. Citing Perkin's estimates of land in 1502, Huang argues that a number of about 88.721 million Mou would be appropriate for 1400 and this capped to 238 million Mou at the turn of the 20th century[25].

Models based on certain forms of Brownian motion (random walks) are notoriously difficult to converge to real world observations because by design there is an infinite number of possible pathways emanating from the initial conditions. Thus, we calibrate the natural population growth rate to be .71%/year over 500 years from 1400 to 1900 so that given an initial population of 7.183 million the mean simulated population in 1900 is 68 million. This also yields a net

[25] Huang (1985) *Op Cit.*Appendix B and Appendix C
Perkins, D. (1969) *Op Cit.*

population growth rate of 0.4037% over 500 simulated years which compares favorably to the actual 0.4323% actual growth rate between 1393 and 1913.

We model population in year 1 of the Monte Carlo simulation using the mean reverting Brownian motion

$$(15)\ P_t = P_{t-1} e^{\left(g\left(1-(1-\phi)\frac{P_{t-1}}{P^*_{t-1}}\right) - \frac{1}{2}\sigma_P^2 + N(0,1)\sigma_P \right)}$$

Where $g = 0.0071$ is the natural growth rate , $\phi = 0.25$ is a measure of resilience (with 1.0 showing no resilience and 0.0 being fully resilient), $P^*_{t-1} = \frac{Y_{t-1}}{c}$, the capacity of aggregate output divided by per capita consumption ($c = 1,037.53 kg / year$), is the capacity of the land to support the population at subsistence, $\sigma_P = 0.05$ is assumed to be the annual volatility in population meaning that the population might increase or decrease by 5% in approximately 68% of sampled years. Finally, since we are assuming lognormality in population $N(0,1)$ is a randomly drawn standard normal deviate.

The output function is modelled in the following way. We assume as in our model a Cobb-Douglas form,

$$(16)\qquad Y_t = A_t P_{t-1}^{(\alpha+\lambda_P(L_{t-1}-L_0))} L_{t-1}^{(\beta-\lambda_L(L_{t-1}-L_0))}$$

The intercept term is not a constant as is usually the case but allows for technical innovations to increase aggregate output at the rate of $\mu_A = 0.002$ per year. This number is loosely based on reported numbers for growth in wheat yield by Elvin. The volatility in aggregate output of 5% is assumed. It may be significant at the very local level but across two provinces there would likely have been spatial correlation and covariance between good and bad years that would moderate risk. The intercept is modeled as a Brownian motion. The volatility component of this random walk is used as the source of exogenous variation on output. However we add an additional component to this by correlating the randomness in this part with the randomness in the population equation to capture Lin's conjecture.

(17) $$A_t = A_{t-1} e^{\left(\left(\mu_A - \frac{1}{2}\sigma_A^2\right) + N(0,1)\sigma_A\right)}$$

We set the parameters as follows; $\alpha = 0.1668, \beta = 0.8332, \lambda_P = 0.00000186, \lambda_L = 0.00000186$. These assume constant returns to scale. The parameters for human capital growth and land depreciation were calibrated to initial conditions. This ensures that over time the elasticity for labour increases at the same rate that the elasticity for land declines to ensure constant returns to scale at each time step. Finally, we use $(L_{t-1} - L_0)$ to scale the human capital and land depreciation properties. We do this because the initial values were calibrated to initial conditions which presumed the initial land base of 88.721 mou.

With output determined, the population capacity is computed using $P_t^* = \frac{Y_t}{c}$. Land dynamics evolve as follows

(18) $$L_t = L_{t-1} + \frac{c}{\hat{Y}_{t-1}}(P_{t-1} - P_{t-2}) - \frac{cP_{t-1}}{\hat{Y}_{t-1}^2}\left(\hat{Y}_{t-1} - \hat{Y}_{t-2}\right),$$

where

(19) $$\hat{Y}_t = 0.5\frac{Y_{t-1}}{L_{t-1}} + 0.5\frac{Y_{t-2}}{L_{t-2}}$$

is output per mou. This we smooth over two previous years on the assumption that farmers have some sense of rational expectations. Smoothing land in the same way makes the land dynamic less choppy while allowing for some lag between the time of an event or shock and a decision to increase or decrease the land base.

1.11 Some Results from Monte Carlo Simulation

We operationalize the above model using Monte Carlo methods for 5,000 iterations over a time span of 500 years initialized to 1400 and concluding in 1900. Under the stochastic model presented above it is assumed that the actual history and evolution of population-land-output dynamics is driven largely by chance. The reality is that the historical path could have been altered at any point by a random shock to output, or population that was good, bad, or neutral. The driving force are the stochastic differential equations which follow Brownian motion with time-independent shocks. Essentially this means that in the course of China's agricultural history, anything can happen at any time; history is a random walk.

Casting the problem as a random walk also alters our view of equilibrium. In the sense of Nelson and others an equilibrium is a steady state- a point of attraction of largely deterministic economic forces that once reached, it remains in place. In a stochastic world the point of equilibrium, measured in various ways by land to population, or output per capita is fleeting. It is a point of breakeven that is breached from above or below, but is transitory.

In Figure 0-1, Figure 0-2, Figure 0-3 we present three simulations. The upper left quadrant plots the stochastic and dynamic paths of population and land. The output to land ratio is provided in the upper right quadrant. Land per capita is provided in the lower left quadrant and the Capacity to Population Ratio in the lower right. Capacity is measure by the population that can be sustained by agricultural output and is a traditional measure of equilibrium trap. When this ratio is greater than 1 the economy will generally thrive. However when this value falls below 1.0 outcomes are dire – there is not enough food to support the population. This is the entry point of a poverty trap. How long a poverty trap persists is determined not only by population and land adjustments but also random events. A poverty trap is thus defined as an excursion below the equilibrium capacity line.

Figure 0-1 and Figure 0-2 are interesting because the final population after 500 simulated years approximates the observed population in the 1900, and as well the land assumed to be under cultivation at that time. While both start and end up at the same place, the pathways are remarkably different. Figure 0-3 is provided to illustrate an outcome that was possible, but comes nowhere close to the observed conditions in 1900.

The output to land ratio is the aggregated output per Mou. In our model we made some allowance for innovation, but depreciated the elasticity of land while increasing human capital, i.e. the elasticity of labor, to compensate. But in a random growth model from time to time disaster strikes, Whether this be from a natural calamity or war the model does not differentiate, but it does capture the periodic booms and busts in China's agricultural history . Land per capita adjusts. This can be through a population increase relative to land, or the abandonment of land. In Figure 0-1, for example land hits the natural limit after 300 years, or about the year 1700 as simulated. But once this maximum is reached there are periods of abandonment which could result from transitory increases in output relative to population, or declines in population. The capacity per capita in Figure 0-1 shows a period of relative prosperity up until year 200. There is then a 50-year excursion into a poverty trap, a slight recovery and then an extended poverty trap as the capacity to population ratio falls. In essence the population is growing at a rate in excess of the cultivation of new land and aggregate output. This extended poverty trap never recovers.

The simulation in Figure 0-2 is similar, except that in this model land does not reach its maximum boundary until year 450, or the middle of the 19th century. Again with this particular iteration randomness in output reveals a chaotic path in output per capita. Again, under this set of circumstances the capacity to population ratio falls into poverty trap at around year 365 (1765) and remains there until year 500 (1900).

Figure 0-1 and Figure 0-2 are dire, but in terms of the question posed in Chapter 2 – how did China's agricultural economy get to its condition in 1921? – illustrates how randomness in populations, productivity, and innovation can explain the conditions. But if chance is the driving force then under this model alternative pathways were feasible. Figure 0-3 ends with a population of about 77 million supported by only 67 million Mou. In fact the land base in 1900 is less that it was in 1400. Why is this? The output per capita in Figure 0-3 had a good run, and increased almost exponentially throughout this period. This could be a combination of strong innovation, good luck, or both but as output increased faster than the population, the amount of land required decreased. There was still variance, but the capacity to population ratio shows more positive excursions. Poverty traps still existed now and again, but not to the same extent as the other simulations in Figure 0-1 and Figure 0-2.

Again these are only simulations. Actual records on land cultivated, population, and output are scarce and dubious, so we can only presume what actually happened in China. By observation and anecdote we know that lands in Shandong and elsewhere were contoured many hundreds of years ago for example. If the max was reached in 1600-1700 then it is not unreasonable to conjecture that any drought, plague, or war would lead to abandonment of mountain plots that were labor intensive. The continual parceling of land across generations as the population grew has repeatedly been asserted to explain the poverty conditions at the turn of the Republican era.

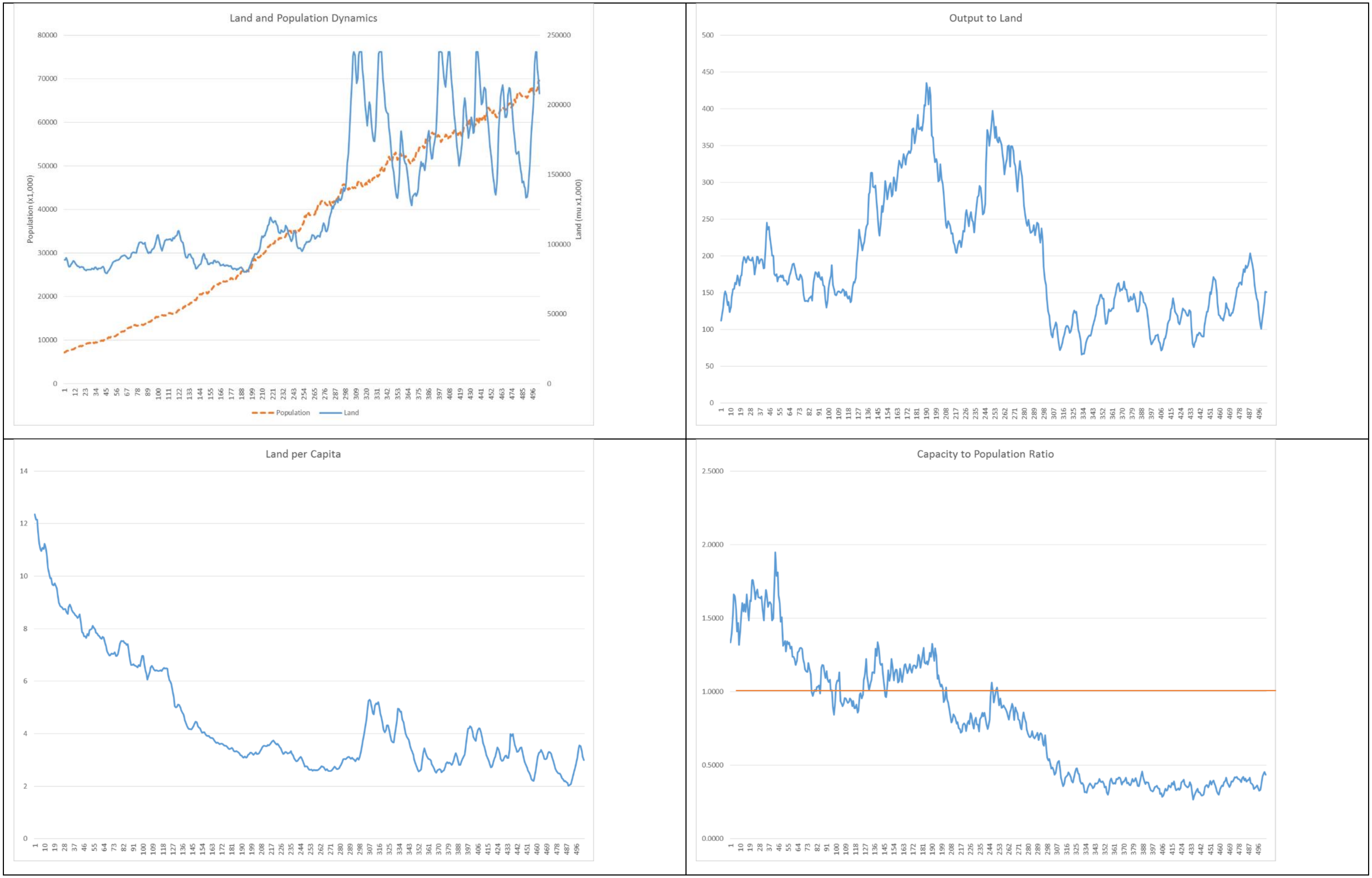

Figure 0-1 :Fractional Poverty Trap Dynamics showing possible path over 500 years with Land and Population, upper left; Output to Land on upper right; Land per Capita, lower left; and Capacity to Population Ratio, Lower right. Start and endpoints approximate China's observed growth. Poverty trap dynamics illustrated in lower right, where excursions below values of 1 indicate below subsistence living standards. In this scenario land reaches its maximum about 300 years in, or around the year 1700. Beyond that he combination of low innovation and high output risk causes periodic rises and falls in land cultivated. Output uncertainty, the continual rise in population, and the land constraint results in a poverty trap that extends for nearly 300 years with this simulation.

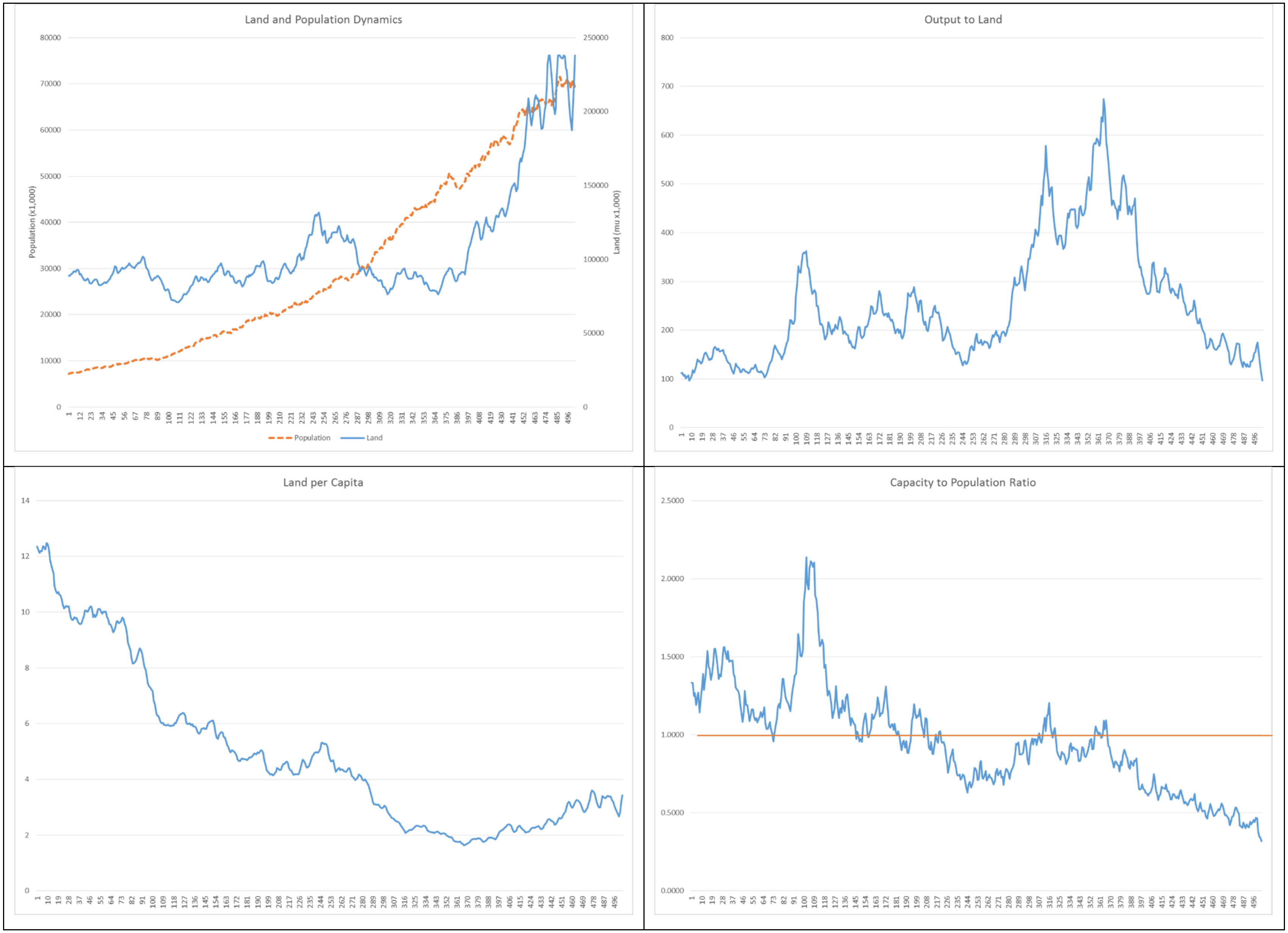

Figure 0-2:Fractional Poverty Trap Dynamics showing possible path over 500 years with Land and Population, upper left; Output to Land on upper right; Land per Capita, lower left; and Capacity to Population Ratio, Lower right. Start and endpoints approximate China's observed growth. Poverty trap dynamics illustrated in lower right, where excursions below values of 1 indicate below subsistence living standards.Note that in this example land reaches its maximum in the mid 1800's which is when some experts conclude the maximum was reached. The combined rise and fall in output, together with the rise in population ensures in this particular path that by the 1900s farmers were locked into a poverty trap.

Figure 0-3: Fractional Poverty Trap Dynamics showing possible path over 500 years with Land and Population, upper left; Output to Land on upper right; Land per Capita, lower left; and Capacity to Population Ratio, Lower right. Start and endpoints differ from observed population and land use in China in 1900. In this scenario large gains in output due to technological innovation reduce the need to cultivate lower quality land, while providing sufficient capacity for population growth. Poverty trap dynamics illustrated in lower right, where excursions below values of 1 indicate below subsistence living standards. Here we observe an excursion pattern that oscillates about the subsistence line.

1.12 Fractional Poverty Traps

A fractional process is a stochastic process which has some form of entrenched memory. A Brownian motion is a stochastic process without memory in the Markovian sense that random shocks from one period to the next are independent of each other. A fractional Brownian motion will exhibit varying degrees of positive or negative correlation across these shocks. If correlation is positive the process is said to be persistent; if it is negative it is said to be mean reverting or ergodic.

A convenient means of estimating system memory is the Hurst coefficient, H. For $H = \frac{1}{2}$ the system is Brownian motion, for $0 < H < \frac{1}{2}$ the system is mean-reverting, and for $\frac{1}{2} < H \leq 1$. The boundaries at 0 and 1are rarely met in nature with 0 being pure white noise and 1 being almost perpetually reinforcing. Our measure of H is obtained using the scaled variance ratio approach as follows,

(20) $$H = \frac{1}{2}\frac{Log\left(\frac{Var\left[x_{500} - x_1\right]}{Var\left[x_t - x_{t-1}\right]}\right)}{Log\left(T\right)}$$

The variance ratio in the numerator will collapse to T if there is no correlation or covariance. In this case $Var\left[x_{500} - x_1\right] = 500 \times Var\left[x_t - x_{t-1}\right]$ the numerator and denominator cancel out and H=1/2: a geometric Brownian motion. But if $Var\left[x_{500} - x_1\right] > 500 \times Var\left[x_t - x_{t-1}\right]$ then the numerator will have some value $T + n$ so that the variance ratio will be greater than 1 and $H > \frac{1}{2}$. Likewise if $Var\left[x_{500} - x_1\right] < 500 \times Var\left[x_t - x_{t-1}\right]$, then the variance ratio in the numerator will have some value $T - n$, and $H < \frac{1}{2}$.

From the simulations that generated Figure 0-1, Figure 0-2, and Figure 0-3 we computed the Hurst coefficients as follows

Table 0-1: Simulated Hurst Exponents for Population-Land-Output Dynamics

Population	0.701	Persistent
Capacity/Population	0.368	Mean reverting
Output	0.512	Slight persistence
Innovation	0.50	No correlation
Land	0.603	Persistent
Output to Land	0.535	Slight persistence
Output/Population	0.368	Mean reverting
Land/Population	0.654	Persistent

What we find is that the macro forces are, as modeled, fractional. Population, land and land per capita have elements of long memory. This suggests that to some degree events in the present will have some statistical relationship to events in the future, perhaps in an unseen or indescribable way. In general, higher Hurst coefficients will tend to have much longer excursion paths, perhaps indefinite one way or another. In other words if we witness a rise in population this is more likely to persist longer into the future before reversing itself. Likewise with land. But with land we have to take note of the geographic limitations that act as an upper boundary. It is not surprising for those simulated paths such as in Figure 0-1 that reach the capacity sooner than later that this capacity will reverberate in future years, declining and then reversing itself. The interaction between land and population is therefore an interesting one. Our initial specification of the population dynamic as a mean reverting process is indirectly driven by land, and this appears to dominate for population on its own or without the land capacity constraint would naturally rise and fall in response to shocks and the intensity of resilience that the population has available to it.

The land to population ratio is also a common measure in the equilibrium trap literature. We find this to be persistent a 0.654. As a general measure it is not purely random but characterized by longer-than-random excursions. If the ratio is on the rise then it will tend to continue rising for a longer period of time than one would expect with a purely random model; likewise when it is falling it will likely persist and fall for a longer period of time than a purely random model.

The capacity to population ratio (and the output to population ratio) is our key measure for fractional poverty traps, oscillating around a ratio of 1.0 which is the subsistence line. We find this to be highly mean reverting with H=0.368. This is a classic Malthusian results. What it suggests is that the ratio reverses itself much quicker than would be expected under a purely random measure. That is if the capacity to population ratio is rising, this induces population growth, which puts pressure on land, so the ratio soon enough reverses itself.

Likewise if the ratio is falling as a result of some catastrophe then the population adjusts by reducing the growth rate or increased mortality until subsistence is once again reached.

In our interpretation of a fractional poverty trap, we thus consider the number of times that the capacity/population ratio falls below 1.0 in a fixed period of time and once below the line, how long before it reverses itself. This, again will naturally be tied to the land, the time required before all arable land is cultivated, and the innovations and random shocks applies to the output of this land. In Figure 0-1 and Figure 0-2 we can see that as the population rises relative to land (H=0.654) this has the effect of pushing the capacity ratio below 1, and for a much longer period of time. Figure 0-3 on the other hand does not hit the land capacity and in this scenario we can observe a different pattern of reversals in the capacity ratio.

1.13 Summary

In this paper we make a first attempt at merging the economic drivers of various theories of equilibrium and poverty traps with dynamics and stochastics to provide a broader understanding of how China ended up the way it was in the Republican era. The literature provided some good guidance. From the earlier work of Malthus and Ricardo to modernist views of Nurkse, Nelson, and Elvin. While insightful, the prevailing models on low (or high) level equilibrium traps were all short run models. Our history, on the other hand was a long one, and as described in the previous chapter with so many random events from drought and flood and war and dynasties the equilibrium trap models were seemingly incomplete.

It is impossible to replicate China's agricultural history, but enough hints were in these papers to offer guidance to developing a more structured dynamic model that took into account some of the risks and uncertainties that arose, and through this process perhaps gain a better understanding about the forces of poverty traps rather than the absolute causes.

The idea of a fractional poverty trap in our context is that dynamics and structural constraints interact overtime in a persistent manner. Once land hits its maximum capacity and population adjusts accordingly this constraint adds a form of memory to the system. We see that by our measure of the Hurst coefficient that measures tied to land and population have random shocks that are correlated over time – there is memory in the system and this memory drives excursion patterns. Typically the higher the Hurst coefficient the more likely that current observed conditions would persist longer into the future than for lower Hurst coefficients.

Our main measure of poverty trap is the capacity to population ratio, which is closely tied to the land to population ratio used in the literature. We find this to be a mean-reverting fractional process and thus abscond with the terminology introduced by Swallow and Barret of the Fractional Poverty Trap (they used fractal, but with the same meaning). A fractional poverty trap in our context is the consequence of a random process falling below the level of subsistence (capacity ratio=1) and remaining there for some period of time. Our simulated measure of H=0.368 indicates that it is a mean reverting process so that mathematically a poverty trap will ultimately be reversed. The lower the Hurst coefficient the more likely it will be reversed sooner than later, at least in a probabilistic sense. Nonetheless, the model suggests that poverty traps are not necessarily a permanent state, even though a poverty trap might appear as such over decades or even centuries. Eventually the forces of economies and population will cause a reversal a path to escape the poverty trap.

A final note to the reader. We find Figure 0-1 and Figure 0-2 equally plausible. But in a model of this type there are millions of possible paths that could have provided similar initial and final conditions. Should a series be drawn with reliable population, land, and output measures for China (or elsewhere) it would be possible to measure the actual Hurst coefficients. Perhaps of more immediate interest is the modeling and measurement of poverty traps to determine if indeed they are 'fractional poverty traps' as proposed in this paper.