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Abstract

A decision maker is characterized by two binary relations. The first reflects decisions that are rational in an "objective" sense: the decision maker can convince others that she is right in making them. The second relation models decisions that are rational in a "subjective" sense: the decision maker cannot be convinced that she is wrong in making them. We impose axioms on these relations that allow a joint representation by a single set of prior probabilities. It is "objectively rational" to choose f

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in the presence of g if and only if the expected utility of f is at least as high as that of g given each and every prior in the set. It is "subjectively rational" to choose f rather than g if and only if the minimal expected utility of f (relative to all priors in the set) is at least as high as that of g.

1 Introduction

1.1 Reasoned choice

Consider a policymaker who has to make a decision such as the determination of environmental, economic, or foreign policy. The decision maker wishes to know what her policy should be. That is, she constructs her preferences in as rational a way as she can.

Economic theory typically assumes the existence of a binary relation $\succsim$, reflecting preferences between pairs of alternatives, acts, or courses of action. When consumer theory is discussed, this relation is most commonly interpreted descriptively, assumed to reflect the consumer's preferences, as revealed by her choices. It is almost a truism that this relation is complete, namely, that between any two courses of action, f and g, we will observe $f \succsim g$ or $g \succsim f$.[1] Moreover, the leading interpretation of the relation $\succsim$ is of a preference that exists without a complicated reasoning process.

By contrast, a decision maker such as a government official who seeks to determine environmental policy does not necessarily have pre-defined preferences $\succsim$. Rather, she is in the process of *determining* these preferences. In the terms of Gilboa, Postlewaite, and Schmeidler (2004, 2007) the relation $\succsim$ in this problem reflects "reasoned choice" rather than "raw preferences". Correspondingly, it is not obvious that such a relation may be assumed complete.

[1] The completeness axiom is not vacuous, as it implicitly requires that the same choices will be made under the same conditions. However, it cannot be refuted by a single choice between two alternatives, whereas other axioms typically can be refuted by a single observation of the preference between any pair of alternatives involved.

At the end of the reasoning process completeness should better be satisfied, or else the decision maker will be caught in indecision. But at the outset completeness typically does not hold. Well-defined preferences are the goal, not the data.

1.2 Two notions of rationality

We submit that standard models in decision theory, using a single binary relation $\succsim$, are too austere to describe the process by which a decision maker generates her preferences. Such models are also not rich enough to distinguish between choices that the decision maker feels strongly about, and choices that are made out of necessity.

The purpose of this paper is to extend the standard model in a modest way, upgrading it to have *two* binary relations as primitives, rather than one. These two relations would distinguish between preferences that are based on sound reasoning and those that are not necessarily so. Clearly, a pair of binary relations is also too limited to describe the dynamic process of generating preferences, or an entire dialog between policymakers and their consultants. But a model with two relations will allow us to capture more of the subtleties of decision under uncertainty, without losing too much in terms of parsimony.

1.2.1 Objective rationality

Let one binary relation, $\succsim^*$, denote preferences that are *rational in the objective sense*: when we write $f \succsim^* g$, we mean to say that the decision maker *can be convinced* that act f is at least as desirable as act g. That is, this preference can be justified and defended on more or less objective grounds, given the decision maker's goals, values, and desires.[2] If the decision maker seeks expert advice, the relation $\succsim^*$ would reflect the preferences that the advisor could derive, using logical, statistical, and decision-theoretic reasoning, from the decision maker's utility, data, and his own expertise.

[2] We use the term "objective" in a highly qualified way. See subsection 3.4 below.

The informal definition of "objective rationality" revolves around the ability to convince others. We wish to focus on the ability to convince based on sound arguments, rather than on rhetorical ruses or personal style. It is therefore useful to think of "being convinced" as saying "being convinced and being able to convince others in turn". That is, the relation $f \succsim^* g$ can be read as saying "the decision maker finds f at least as desirable as g, and she also feels quite confident that she can convince any reasonable person that, according to her utility, f is indeed at least as desirable as g." For example, if the decision maker would hire an assistant, she believes that the latter would see the logic behind the decisions described by $\succsim^*$.

Unfortunately, in many decision problems under uncertainty, a relation $\succsim^*$ that can be interpreted as "objectively rational" would fail to be complete. There will typically be many pairs of acts f and g between which no well-reasoned preferences exist. Even if the decision maker's utility function is clearly defined, absence of information is likely to leave the decision maker unable to logically justify preferences that depend on plausibility judgments. Indeed, the scientific method allows us to settle many questions of belief, but it has to remain silent on others.

How should the theory of decision under uncertainty cope with the challenge posed by incompleteness? One approach is to make do with an incomplete relation. According to this approach, if there is no compelling reason to prefer f to g nor g to f, we might be better off explicitly modeling this absence. Models of incomplete preferences date back to Aumann (1962), Kannai (1963), and Peleg (1970). Walley (1981) and Bewley (2002) focused on incompleteness that is due to uncertainty, namely to the absence of an agreed-upon probability. Such models have recently received renewed attention (cf. Ok, 2002, Dubra, Maccheroni, and Ok, 2004, Mandler, 2005, Evren and Ok, 2007, Nehring, 2008, Ok, Ortoleva, and Riella, 2008). Many of these authors have also argued that there is nothing irrational about incompleteness of preferences. In the absence of information, it appears more rational to be silent than to pretend to have knowledge that one does not have.

1.2.2 Subjective rationality

Despite the arguments for allowing incompleteness, the standard justification of the completeness axiom for rational choice still remains: eventually, a decision will be made. If we do not describe this decision in the model, we might be left with a very well-reasoned relation $\succsim^*$ that has little to do with actual decisions. The relation $\succsim^*$ might be the epitome of rationality, while the decisions that will be taken in practice fail to satisfy basic consistency requirements such as transitivity. An expert who derives the relation $\succsim^*$ for a decision maker might be appalled to learn what follies were allowed by his cautious analysis.

We are therefore led to introduce a second binary relation, $\hat{\succsim}$, which we expect to be complete. The relation $\hat{\succsim}$ will reflect preferences that are *rational in the subjective sense*: when we write $f\hat{\succsim}g$, we mean to say that the decision maker *cannot be convinced* that choosing f in the presence of g is *wrong*. Intuitively, such a choice does not lead to any contradiction with other choices of the decision maker, and does not seem illogical given the decision maker's goals and the data available to her.

Thus subjective rationality is also defined by the ability to convince others. But it does not require that the decision maker be able to convince others that she is right, only that others will not be able to convince her that she is wrong. Should the decision maker hire an assistant, she may not be certain that he would come up with the choices reflected in $\hat{\succsim}$; but she feels confident that these choices would not appear silly to him.

1.2.3 Analogy: statistics

The relations $\left(\succsim^*, \hat{\succsim}\right)$ are analogous to the classical and the Bayesian approaches to statistics, respectively. Classical statistics aspires to objectivity, at the price of completeness. When a hypothesis H_0 is rejected by a scientific study, it is expected that any reasonable person would find H_0 incompatible with evidence. This high standard of objective rationality has the obvious implication that in many cases neither a hypothesis H_0 nor its negation H_1 can be rejected. Seeking objectivity, science has to remain silent on many issues.[3]

Bayesian statistics, by contrast, has a well-defined probability for any event of interest. In this sense, it obeys the completeness axiom: it can state, for any hypothesis and given any data base, whether, given the evidence, the hypothesis is more or less likely than its negation, and, indeed, precisely how likely it is. Such likelihood judgments cannot be derived based on evidence and logical reasoning alone, and therefore they cannot be expected to be shared by all. Hence, Bayesian statistics depends on a subjective prior.

1.2.4 Analogy: law

The distinction between the two relations, $\succsim^*$ and $\hat{\succsim}$ is reminiscent of that between criminal and civil law. Criminal law requires that guilt be proven beyond a reasonable doubt. Thus, a verdict of "guilty" can be read as "judging the defendant to be guilty is preferred, in the sense of objective rationality, $\succsim^*$, to acquitting him". It is expected that the court be able to convince others that such a verdict was indeed justified. It is accepted, however, that questions of guilt may remain doubtful. In other words, what can be legally "proven" defines an incomplete relation $\succsim^*$. This incomplete relation is completed by the default of a "not guilty" verdict.[4]By contrast, civil cases are more symmetric in their treatment of the two parties involved. In the absence of an obvious default, an incomplete order is unsatisfactory, as it does not specify the court's ruling. Thus, the decision in a civil case can be thought of as a complete order, which may be less robustly justified than a decision in criminal case. That is,

[3]Rather than the statements made by science, one may consider science's choice among the three alternatives, "state H_0", "state H_1", and "remain silent". In this meta-problem we may consider complete preferences, where "remain silent" is the preferred choice unless one of the hypotheses may be rejected. When classical statistics is used for decision making, preferences are often completed by resorting to a default such as the status quo.

[4]This is a little different from the default choice of "remain silent" in science. The law is committed to treat "not proven guilty" and "proven innocent" in the same way, whereas science should better not treat "not proven false" as equivalent to "proven true".

as compared to criminal law, civil law is closer to subjective rationality, $\hat{\succsim}$, than to objective rationality.

1.3 The role of axioms

Decision theory offers sets of axioms that are shown to be equivalent to particular representations of preferences. For example, a complete and transitive relation over a finite set can be represented as maximizing a certain utility function. The literature also offers a variety of axiomatic models for decision making under uncertainty. Most notably, building on ideas of Ramsey (1931) and de Finetti (1937), Savage (1954) and Anscombe and Aumann (1963) provided axiomatic models of subjective expected utility maximization. These models are often interpreted descriptively, as supporting the claim that economic agents can be modeled as expected utility maximizers, relative to their subjective probabilities. In this paper we are mostly interested in the normative interpretation of such models, supposedly helping the decision maker to determine what her preferences should be.

1.3.1 Normative role of universal statements

There are at least two rather different ways in which axioms can be normatively interpreted. The first is as general mathematical conditions, and the second – as specific instances of preferences. Consider, for example, the transitivity axiom. One might consider a hypothetical dialog with a decision maker, in which a decision theorist says, "Consider the claim that, for every three choices, f, g, and h, if you prefer f to g and also g to h, you should prefer f to h.[5] Wouldn't you like to satisfy it? Wouldn't you feel uneasy with ever be found to violate it?" If the decision maker is sophisticated enough to understand this type of general statement, involving a universal quantifier over the variables f, g, and h, she might say, "I adopt this axiom. I would hate to find myself violating it." Then the decision theorist can, as it were, flash a slide with a

[5] For simplicity, we use "prefer" instead of "prefer or find equivalent".

representation theorem, and say, "Well, if you agree with this axiom, and your preference is complete, you have to maximize a utility function. It's a theorem. Now wouldn't it be simpler to try to estimate your utility function for each alternative?" That is, the decision theorist uses a set of axioms, viewed as abstract universal statements, to convince the decision maker that she should adopt a particular model of decision making. The axioms do not necessitate preference between any two particular choices f and g; they only impose a general structure on the totality of the decision maker's choices.

1.3.2 Normative role of concrete instances

By contrast, axioms such as transitivity can also be interpreted in a concrete way, as building blocks in a reasoning process. For example, assume that a consultant tells the decision maker, "If I recall correctly, we have already determined that f is preferred to g. Moreover, last week you have chosen g over h. Now it would seem to me that, if you put these two decisions together, you should also prefer f to h." In this type of reasoning, f, g, and h are particular choices. The decision maker does not engage in an abstract argument with variables and universal quantifiers. Rather, she is shown the logic of the axiom in a particular instance.

In the concrete interpretation, axioms are viewed as "reasoning templates", namely as ways to use arguments for some preferences in order to construct from them arguments for other preferences. If one were to model this process formally, one could consider particular instances of preferences as propositions, and decision theoretic axioms as "inference rules", allowing the concatenation of such propositions to generate the formal object of a "proof".[6]

1.3.3 Comparison

When axioms are interpreted as universal statements, they demand a rather high degree of sophistication on the decision maker's part. Relatedly, when the

[6]Observe that the decision theoretic axioms are the inference rules, not the "axioms" as used in propositional logic.

decision maker is asked to judge the plausibility of axioms in the abstract, she is susceptible to “framing effects”: an axiom may appear more or less compelling depending on its representation. Moreover, an axiom that is logically stronger may be more compelling than an axiom that is weaker. For example, the axiom, “there should be no cycles of strict preference” is probably more compelling than the axiom “there should be no cycles of strict preference of odd length.”

The concrete interpretation, by contrast, requires less abstract thinking, and leaves less room for different representations of the same statement. Correspondingly, in the concrete interpretation the set of preferences than can be derived from an axiom increases with its logical strength: a more general axiom will allow a larger set of preferences to be deduced from it.

Perhaps the most important distinction between the two normative interpretations of axioms is that the universal one often does not help the decision maker in determining her preferences, only their structure. They deal with form rather than with content. For example, assume that a decision maker wonder whether a certain policy to cope with global warming is to be adopted. She consults an expert who convinces her of the logic of Savage's axioms, viewed as universal statements. Then she is told that, by a mathematical theorem, she should have a utility function and a probability measure, and she should maximize her subjective expected utility. But this general conclusion says nothing about which utility function she should choose, or about which subjective probability she should adopt. In particular, she was just convinced that she should be able to quantify the probability of the globe warming up by at least 2 degrees over the next five years. But nothing tells her what these beliefs of her should be.

By contrast, the concrete interpretation of the very same axioms would take some preferences that the decision maker already has, and build up from them some others. There is no guarantee that this process will end up with a complete relation, but it will typically have more pairs of choices (f, g) in it than the relation that the process started out with.

Yet another distinction between the two interpretations has to do with the scope of the preference relation. Most axiomatic derivations of decision models require a rather rich domain of preferences for the proofs of the theorems to hold true. These may include choices between implausible alternatives. By contrast, the concrete interpretation of axioms requires choices between concrete alternatives that are actually available, and some variations thereof, but typically not between all conceivable pairs of choices.

While the rest of this paper can be read with more than one interpretation in mind, we try to adhere to the concrete interpretation, which we find more conducive to actual decision making processes than the universal one. We imagine the decision maker as starting with some preference propositions and building up to generate new ones. However, which are the initial preference propositions, and which axioms should be used as inference rules would depend on the interpretation of the preference order as reflecting objective or subjective rationality.

1.4 The present model

In this paper we present a model that makes two simplifying assumptions. First, we assume that the decision maker has a well-defined utility function, so that she has a rather clear idea how she would trade-off various goals, what her ethical constraints are, and so forth. Her main difficulty is how to deal with uncertainty. Second, we assume that, should the decision maker consult with experts, her utility function is honestly adopted by them. Thus, we abstract away from the problems discussed in the recent literature on strategic consultants (see, for instance, Scharfstein and Stein, 1990, Prendergast and Stole, 1996, Levy, 2004), and ask a simpler question: how should the decision maker and her consultants work together to obtain the most rational decision (for the decision maker) in the face of uncertainty?

Our focus is on situations where probabilities are neither given, nor can they be easily deduced or estimated. As mentioned above, the works of Ramsey (1931), de Finetti (1937), Savage (1954), and Anscombe and Aumann (1963) famously championed the Bayesian approach, suggesting that any uncertainty can be reduced to risk using the notion of subjective probabilities. The latter are defined behaviorally, as degrees of willingness to bet, embedded in a model of expected utility maximization.

Many statisticians were opposed to this line of reasoning,[7] and Ellsberg's (1961) well-known experiments have shown that people often fail to behave in accordance with the Bayesian approach. In the 1980's two models were proposed, relaxing the axioms underlying subjective expected utility theory and generalizing it by allowing a representation of beliefs by a set of probabilities, rather than by a single probability. These approaches are often referred to as *multiple prior* models, and they tend to be closer to the classical statistics mindset, in which a set of distributions defines the inference problem, but no prior belief over the set can be assumed. One approach (Bewley, 2002, see also Walley, 1981) uses the set of priors to define a partial order by *unanimity*: f is at least as desirable as g if and only if the expected utility of f is at least as high as that of g for *each and every* prior in the set.[8] The other (Gilboa and Schmeidler, 1989) retains the completeness axiom, and derives a representation by the *maxmin* rule: f is preferred to g if and only if the minimal expected utility of f, over all possible priors in the set, is higher than the minimal expected utility of g.

We start with two binary relations, $\left(\succsim^*, \hat{\succsim}\right)$, interpreted as objective and subjective rationality relation, as suggested above. Formally, we assume that the first satisfies the axioms of Bewley (2002),[9] and the second – of Gilboa

[7] See Cifarelli and Regazzini (1996), who describe Cantelli's reactions to de Finetti's ideas as "... speaking to Cantelli about subjective probability ... was tantamount to pulling a tiger by its tail." See also Knight (1921) and Keynes (1921).

[8] Bewley's model dealt with a strict preference, represented by a strict inequality for each prior. Mathematically, it relied on Aumann (1962).

Seidenfeld, Schervish, and Kadane (1995) offer a model in which preferences are described by sets of probability-utility pairs. A derivation of Bewley's result in a purely subjective probability set-up is provided in Ghirardato, Maccheroni, Marinacci, and Siniscalchi (GMMS, 2003).

[9] As explained below, our formulation differs from Bewley's on several minor points: it is closer to those of Shapley and Baucells (1998), GMMS (2003), and Girotto and Holzer

and Schmeidler (1989). This means that each relation can be represented by a set of priors: $\succsim^*$ by unanimity, and $\hat{\succsim}$ by the maxmin rule. However, the two sets of priors are unrelated. They may be different or even disjoint. We therefore introduce two additional axioms, explicitly relating the two relations, and show that these axioms hold if and only if the two sets of priors are indeed identical. Taken together, the axioms imply the existence of a set of priors that represents both $\succsim^*$ and $\hat{\succsim}$ simultaneously: the former via unanimity, and the latter – via the maxmin rule.

We describe the axioms and results in the next section. As a by-product, we offer a version of Bewley (2002) that deals with a general state space. This facilitates the comparison with the Gilboa and Schmeidler (1989) model, but may also be of interest in its own right. Section 3 is devoted to a discussion. In particular, it argues that the present treatment highlights the extremity of the maxmin rule, and suggests alternative notions of subjective rationality. Specifically, we also mention a variation in which the subjectively rational relation is Bayesian, that is, a model in which objective rationality is defined by unanimity with respect to a set of probabilities, but subjective rationality is defined by a Bayesian approach relative to a single probability in this set. We conclude with general discussions of rationality and the related literature.

2 Model and Results

2.1 Preliminaries

We use a version of the Anscombe and Aumann (AA, 1963) model as re-stated by Fishburn (1970).

Let X be a set of *outcomes*. The set of von Neumann-Morgenstern (vNM, 1944) *lotteries* is

$$L = \left\{ P : X \to [0,1] \;\middle|\; \begin{array}{c} \#\{x|P(x)>0\} < \infty, \\ \sum_{x\in X} P(x) = 1 \end{array} \right\}.$$

(2005).

and it is endowed with a mixing operation: for every $P, Q \in L$ and every $\alpha \in [0,1]$, $\alpha P + (1-\alpha)Q \in L$ is given by

$$(\alpha P + (1-\alpha)Q)(x) = \alpha P(x) + (1-\alpha)Q(x) \quad \forall x \in X.$$

The set of *states of the world* is S endowed with an algebra Σ of *events*. The set $\Delta(\Sigma)$ of (finitely additive) *probabilities* on Σ is endowed with the event-wise convergence topology.[10] The set of *(simple) acts* F consists of all simple measurable functions $f : S \to L$. It is endowed with a mixture operation as well, performed pointwise. That is, for every $f, g \in F$ and every $\alpha \in [0,1]$, $\alpha f + (1-\alpha)g \in F$ is given by

$$(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \quad \forall s \in S.$$

The decision maker is characterized by two binary relations $\succsim^*$ and $\hat{\succsim}$ on F, denoting objective and subjective rational preferences, respectively. The relations $\succ^*$, $\sim^*$, $\hat{\succ}$, $\hat{\sim}$ are defined as usual, namely, as the asymmetric and symmetric parts of $\succsim^*$ and $\hat{\succsim}$, respectively.

We extend $\succsim^*$ and $\hat{\succsim}$ to L as usual. Thus, for $P, Q \in L$, $P \succsim Q$ means $f_P \succsim f_Q$ where, for every $R \in L$, $f_R \in F$ is the constant act given by $f_R(s) = R$ for all $s \in S$ and $\succsim$ is either $\succsim^*$ or $\hat{\succsim}$. The set of all constant acts is denoted by F_c.[11]

For a function $u : X \to \mathbb{R}$ we will use the notation

$$\mathrm{E}_P u = \sum_{x \in X} P(x)u(x)$$

for all $P \in L$.[12] Thus, if the decision maker chooses $f \in F$ and Nature chooses $s \in S$, the decision maker gets a lottery $f(s)$, which has the expected u-value of

$$\mathrm{E}_{f(s)} u = \sum_{x \in X} f(s)(x)u(x).$$

[10] A net $\{p_k\}$ converges to p if and only if $p_k(A) \to p(A)$ for all $A \in \Sigma$.

[11] We sometimes abuse the notation writing R instead of f_R and L instead of F_c.

[12] One may replace L by any convex subset of a vector space, or even any mixture space, and $\mathrm{E}_P u$ with the evaluation at P of an affine function u on L. All our results remain valid.

2.2 Several basic conditions

We now turn to discuss the axioms. It will be convenient to start with axioms and conditions that both relations are assumed to satisfy. As discussed in the introduction, completeness will not be among these conditions, as it is not a natural requirement when objective rationality is concerned.

The following conditions are stated for a generic relation $\succsim$. They will be imposed on both relations $\succsim^*$ and $\hat{\succsim}$.

Basic Conditions:

<u>Preorder:</u> *$\succsim$ is reflexive and transitive.*

<u>Monotonicity:</u> *For every $f, g \in F$, $f(s) \succsim g(s)$ for all $s \in S$ implies $f \succsim g$.*

<u>Archimedean Continuity:</u> *For all $f, g, h \in F$, the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succsim h\}$ and $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda)g\}$ are closed in $[0,1]$.*

<u>Non-triviality:</u> *There exist $f, g \in F$ such that $f \succ g$.*

2.2.1 Reflexivity

In general, reflexivity is a matter of notation more than a substantive axiom: it does not say much about the decision maker's preferences. Rather, it reflects the modeler's choice to use the language of weak rather than strong preferences. However, it is important to observe that the language of preference, in which the dialog between the decision maker and her consultants is assumed to take place, does not have a term for strict preference. For example, we may find that $f \succsim^* g$ but not $g \succsim^* f$. In our (standard) notation, this implies that $f \succ^* g$. Yet, it will be inappropriate to read this relation as "the decision maker can be convinced that f is strictly preferred to g". All we can say is that "the decision maker can be convinced that f is at least as good as g. She cannot be convinced of the fact that g is at least as good as f." The latter statement differs from the former. In particular, the proposition "f is strictly preferred to g" cannot be stated in the language of the discussion.

To see a concrete example, assume that there are two states of the world. The payoffs guaranteed by f are $(1, 0)$ and by g – $(0, 0)$. There is no information about the probability of the states. It should be relatively easy to convince the decision maker that $f \succsim^* g$. (In fact, this will also follow from the monotonicity axiom.) Clearly, a reasonable decision maker will not be convinced of the converse. Hence $f \succ^* g$. But the decision maker cannot be convinced that f is strictly better than g. Should she think about it, she might say that it is possible that the probability of state 1 is zero, and then the two acts are equivalent. But the logical reasoning we have in mind does not have strict preference as a primitive of the language.[13]

2.2.2 Transitivity

Transitivity of objective rationality is rather compelling. If a consultant has a compelling argument that f should be at least as desirable as g, and another – that g should be at least as desirable as h, transitivity suggests that these two arguments can be concatenated to generate a compelling argument for concluding that f is at least as desirable as h. In a more formal model, one could model each such argument as a proof, namely, an ordered list of propositions, each one following from its predecessors, and the transitivity axiom would be an inference rule generating a longer proof (that $f \succsim^* h$) from two shorter ones (of $f \succsim^* g$ and $g \succsim^* h$).

Next consider subjective transitivity. In this case, the preferences $f \hat{\succsim} g$ and $g \hat{\succsim} h$ need not be compelling. They may well be nearly arbitrary decisions that the decision maker made, shrugging her shoulders, simply because a decision was called for. Hence, if the consultant starts an argument with "clearly, f is at least as good as g", the decision maker might stop him and say, "It's not so clear. I made this decision and I know exactly what went into the decision process. It was, in fact, a rather arbitrary decision I made under time pressure. Let's not build theories around it." At this point the consultant might

[13] The model can be further elaborated, allowing strict preferences for objective and subjective rationality to be explicitly part of the preference statements.

say, "OK, so maybe it was arbitrary. But if you make arbitrary decisions to choose f in the presence of g, and g in the presence of h, but refuse to choose f because h is available, you'll be in trouble. Just imagine the headlines. Hence, my best advice to you is to now choose f. Even if the first two preferences were arbitrary, the very fact that they were made indicates a certain commitment."

We therefore assume transitivity both for objective and for subjective rationality. We find that transitivity is compelling enough, as an inference rule, to be valid even if the preference statements on which it relies were not fully justified.

2.2.3 Monotonicity

In general, monotonicity is also a condition of internal coherence: it says that if certain preferences hold, then others should hold as well. In the present case, the antecedent has to do with a preference between the vNM lottery obtained by f to that obtained by g at each and every state, whereas the consequent is the preference between f and g. As in the case of transitivity, this axiom appears powerful enough to be a valid inference rule even if its input, namely the statewise preferences between vNM lotteries, may not have been fully justified.

However, we assume that the utility function is given and agreed-upon. Hence the pointwise preference $f(s) \succsim g(s)$ needs no justification, and it is not a matter of arbitrary choice either. Given a utility function, an expected utility of the lottery $f(s)$ that is no lower than that of the lottery $g(s)$ can be viewed as "hard evidence" that at state s, f is at least as good as g. This evidence calls for no further justifications. And using it, a simple inference suggests that $f \succsim g$ for both notions of rationality.

2.2.4 Continuity

The Archimedean continuity axiom is the standard Herstein and Milnor (1953) continuity axiom. It cannot be directly refuted by finitely many observations, and its scientific stature is therefore dubious. It is therefore common to dismiss such axioms as a matter of mathematical necessity and discuss them no further.

We mention in passing that, in the context of the construction of preferences and the rhetorical arguments involved in it, axioms such as continuity may be elevated to a more conceptual realm. For instance, if a decision maker can express a preference $f \succ g$ only at the cost of a discontinuity *somewhere* in her preferences, she might be convinced that it makes more sense to have $g \succsim f$. However, we assume that the decision maker only conducts explicit reasoning in the language of weak preferences, and such an interpretation would be inappropriate in our model. It is therefore more natural to think of the continuity axiom as part of the discussion among decision theorists rather than the discussion between the decision maker and her consultants.

Be that as it may, this standard continuity axiom is be assumed for both relations.

2.2.5 Non-triviality

The non-triviality axiom is a condition designed to rule out the case in which the decision maker might be ascribed a constant utility function. In this case the representation results hold, but the uniqueness results do not: preferences can be represented by a constant utility function and any beliefs whatsoever (whether beliefs are represented by a single probability measure, a set thereof, etc.). Thus, the non-triviality axiom is part of the theoretical discussion rather than the discussion between the decision maker and her consultants. In fact, rather than stating an explicit axiom, one could add a caveat at the end of the representation theorems, qualifying the uniqueness statement.[14] Since the two relations will be assumed to agree on constant acts, both will satisfy this axiom as soon as the utility function is not constant.

[14] The tradition, following Savage's axiom P5, is to state an explicit axiom to rule out the special uninteresting case of trivial preferences. This practice reminds us that the project of elicitation of beliefs from observed choices is predicated on the existence of non-trivial preferences.

2.3 Axioms for objective and for subjective rationality

We now turn to discuss the axioms that are specific to objective or to subjective rationality.[15]

2.3.1 Completeness

C-Completeness: *For every* $f, g \in F_c$, $f \succsim^* g$ *or* $g \succsim^* f$.

Completeness: *For every* $f, g \in F$, $f \hat{\succsim} g$ *or* $g \hat{\succsim} f$.

As discussed above, subjective rationality is required to be complete, because eventually some decision will be taken. Objective rationality, by contrast, is not necessarily complete, because one may not have compelling reasons to determine preferences between certain pairs of alternatives. However, we do require that objective rationality be complete when restricted to the subset of constant acts. C-completeness verifies that the incompleteness of the objectively rational relation $\succsim^*$ is not due to any difficulties that the decision maker might have about determining her preferences under certainty. That is, we are not faced with a decision maker who can't decide whether she prefers chocolate to vanilla ice cream in terms of their immediate hedonic value. Any incompleteness of preferences will therefore be attributed to uncertainty about future outcomes of the options involved. (See Subsection 3.5 below.)

2.3.2 Independence

Independence: *For every* $f, g, h \in F$, *and every* $\alpha \in (0, 1)$,

$$f \succsim^* g \quad \textit{iff} \quad \alpha f + (1-\alpha)h \succsim^* \alpha g + (1-\alpha)h.$$

C-Independence: *For every* $f, g \in F$, *every* $h \in F_c$, *and every* $\alpha \in (0, 1)$,

$$f \hat{\succsim} g \quad \textit{iff} \quad \alpha f + (1-\alpha)h \hat{\succsim} \alpha g + (1-\alpha)h.$$

[15] Since each of the following axioms will be assumed for one relation only, we state them directly in terms of this relation, rather than in terms of an abstract relation $\succsim$ as above. In the sequel, we allow ourselves to use phrases such as "C-Completeness" and "$\succsim^*$ satisfies C-Completeness" interchangeably.

We thus require that objective rationality satisfies the original AA independence axiom, whereas subjective rationality – only the weaker version referred to as C-Independence. The reason is the following: if the preference between f and g is based on objective, perhaps even scientific reasoning, i.e., $f \succsim^* g$, this very preference may be used as a reason to prefer $\alpha f + (1-\alpha)h$ to $\alpha g + (1-\alpha)h$. That is, if preference propositions only refer to preferences that can be "proven", then they are sound enough to build upon, and the independence axiom used by Anscombe and Aumann is a reasonable inference rule. If all reasonable decision makers would accept that f is at least as good as g, they should also accept that $\alpha f + (1-\alpha)h$ is at least as good as $\alpha g + (1-\alpha)h$. This reasoning may also be reversed: if there are good, sound reasons to prefer $\alpha f + (1-\alpha)h$ to $\alpha g + (1-\alpha)h$, one may argue that there are even better reasons to prefer f to g: if a small step from h "towards" f is better than taking the same step "towards" g, continuing in the respective directions presumably only strengthens this preference. In short, the basic intuition of the classical independence axiom is assumed to be compelling when one restricts attention to justified preferences.

This is not the case when subjective rationality is concerned. In this case, the relation $f \hat{\succsim} g$ may follow from more arbitrary considerations, or from lack of information. For example, assume that there are two states of the world, and that $f = (1, 0)$ and $g = (0, 1)$. The decision maker has no information about the probability of the two states, and therefore the objective rationality relation does not rank them. Having to make a decision, the decision maker might shrug her shoulders and decide that they are equivalent, namely, that $f \hat{\sim} g$, due to symmetry.

Next consider the mixing of f and g with $h = f$. For $\alpha = 0.5$, the mixture $\alpha g + (1-\alpha)h$ completely hedges against uncertainty, leaving the decision maker with a risky act. The mixture of f with $h = f$ clearly leaves the decision maker with f, without any reduction of uncertainty. The decision maker might plausibly argue that $\alpha f + (1-\alpha)h$ is not equivalent to $\alpha g + (1-\alpha)h$. Indeed, the former is uncertain whereas the latter – only risky. A consultant might try to construct a "proof" that the two are equivalent, starting with "Don't you recall that you said that f and g were equivalent on your eyes? All we're doing now is to mix both of them with h!" But the decision maker might counter, "Wait a minute, when I said that f and g were equivalent, I didn't *know* they were, I only used a default decision. This is not the kind of decision you can now construct a new theory upon." "Aren't you concerned that you will be perceived as irrational?" the consultant might ask. "Leave this to me" would be the response; "I barely understand this mixture operation of yours and if my worst sin is that $f \hat{\sim} g$ but $\alpha g + (1-\alpha) f \hat{\succ} f$, I can live with that."

By contrast, we maintain that C-Independence is a reasonable inference rule even if the preference propositions are not fully justified. The reason is that mixing f and g with a constant act h can be viewed as a change of scale on the expected utility axis, namely, adding a constant and multiplying by a positive constant. Hence a decision maker might be embarrassed to simultaneously express preferences such as $f \hat{\succsim} g$ and $\alpha g + (1-\alpha) h \hat{\succ} \alpha f + (1-\alpha) h$. Each of these may be a possible decision on its own, but if h is a constant, the conjunction of the two appears inconsistent.

Clearly, certain decision makers will find Independence a reasonable condition for both $\succsim^*$ and $\hat{\succsim}$, while others may find that even C-Independence is too strong for both. How many decision makers actually accept Independence for $\succsim^*$ and (only) C-Independence for $\hat{\succsim}$ is an empirical question. For that reason, the following results are only an example of the way the two notions of rationality can be modeled.[16]

We will resort to an additional axiom:

Uncertainty Aversion: *For every $f, g \in F$, if $f \hat{\sim} g$, then* $(1/2) f + (1/2) g \hat{\succsim} g$.

The uncertainty aversion axiom has been introduced in Schmeidler (1986, 1989) for the subjective preference $\hat{\succsim}$, and it says that the decision maker

[16] For example, C-Independence can be weakened as in the variational preferences of Maccheroni, Marinacci, and Rustichini (2006). In this case we expect that in Theorem 3 a variational representation would hold for $\hat{\succsim}$.

prefers "smoothing out" acts, replacing potential uncertainty about the states of the world by objective risk about the outcomes to be obtained in each and every state. To be more concrete, imagine that a decision maker expressed the preference $f \hat{\sim} g$. This choice might have been due to symmetry considerations, and it might have been completely arbitrary as well. However, the consultant may now approach the decision maker and say, "If you express preference $f \hat{\succ} (1/2)\, f + (1/2)\, g$, it would appear as if you like the uncertain situation. That is, you could have reduced the dependence on unknown probabilities, but you preferred not to. It's fine for a gambler, but it doesn't look very good for a public figure like yourself."

As in the case of C-independence, this reasoning may or may not convince the decision maker. Our focus in this paper is on decision makers who do accept this reasoning, namely, decision makers who find Uncertainty Aversion a reasonable inference rule for subjective rationality propositions. Decision makers who do not accept it might be modeled by more general decision rules, as in Ghirardato, Maccheroni, and Marinacci (GMM, 2004).

The Uncertainty Aversion axiom has no counterpart for objective rationality, because it is implied by the standard Independence axiom of AA, which is assumed to be satisfied by objective rationality.

To conclude, objective rationality, $\succsim^*$, satisfies versions of axioms that appeared in Aumann (1962), Bewley (2002), GMMS (2003), and Girotto and Holzer (2005). Subjective rationality, $\hat{\succsim}$, satisfies the axioms of Gilboa and Schmeidler (1989).

2.4 Representation of objective and of subjective rationality

2.4.1 Representing partial orders

We remind the reader that objective rationality is assumed to be reflexive. As observed above, in the presence of incompleteness, results stated in terms of reflexive relations may not have immediate counterparts in terms of irreflexive relations and vice versa. A brief explanation may be in order.

Aumann (1962) assumed a reflexive relation, corresponding to our $\succsim^*$. He defined a "utility" for $\succsim^*$ to be a function that respects both strict preference $\succ^*$ and indifference $\sim^*$. That is, if $U : F \to \mathbb{R}$ is an Aumann-utility for $\succsim^*$, we have

$$f \succ^* g \Leftrightarrow [f \succsim^* g, \neg g \succsim^* f] \Rightarrow U(f) > U(g)$$

and

$$f \sim^* g \Leftrightarrow [f \succsim^* g, g \succsim^* f] \Rightarrow U(f) = U(g).$$

Aumann proved that such utilities exist, but he did not provide a characterization of $\succsim^*$. Clearly, one does not expect a single utility function to fully characterize incomplete preferences. But a *set* of utilities might provide a joint characterization. In particular, one may consider a "multiple utility" representation by a set of functions $\mathcal{U}$ such that

$$f \succ^* g \Leftrightarrow [\forall U \in \mathcal{U} \quad U(f) > U(g)] \tag{1}$$

or

$$f \succsim^* g \Leftrightarrow [\forall U \in \mathcal{U} \quad U(f) \geq U(g)] \,. \tag{2}$$

Bewley (2002) considers as primitive a strict preference relation, that is, an irreflexive one, and provides a characterization as in (1), where each U is an expected utility functional relative to a certain prior. Ghirardato, Maccheroni, Marinacci, and Siniscalchi (GMMS, 2003) provide a representation as in (2), and this is also the approach we adopt here. Thus, we begin with a reflexive order as does Aumann (1962), but seek a complete characterization as provided in Bewley (2002).

Using a reflexive relation as primitive makes some of the results simpler to state. As opposed to Bewley's model, we do not assume a finite state space, and our results are not restricted to sets of probabilities that are all strictly positive. However, it is important to observe that in our case strict preference would not imply strict inequality for each and every representing functional

$U \in \mathcal{U}$. Explicitly, the representation (2) implies

$$f \succ^* g \Leftrightarrow \begin{bmatrix} \forall U \in \mathcal{U} & U(f) \geq U(g) \\ \exists U \in \mathcal{U} & U(f) > U(g) \end{bmatrix}. \tag{3}$$

If we were to consider a decision matrix in which rows correspond to elements of F and columns – to functionals U in $\mathcal{U}$, the representation we obtain, (3) corresponds to weak dominance, whereas Bewley's, (1) – to strict dominance.

2.4.2 Unanimity representation of objective rationality

The axioms we imposed on $\succsim^*$ deliver a unanimity representation. Our first result extends Bewley (2002) to an infinite state space (see discussion in Section 3.6).

Theorem 1 *The following are equivalent:*

(i) $\succsim^*$ *satisfies the Basic Conditions, C-Completeness, and Independence;*

(ii) there exist a non-empty closed and convex set C^* *of probabilities on* Σ *and a non-constant function* $u^* : X \to \mathbb{R}$ *such that, for every* $f, g \in F$

$$f \succsim^* g \quad \textit{iff} \quad \int_S \mathrm{E}_{f(s)} u^* \; dp\,(s) \geq \int_S \mathrm{E}_{g(s)} u^* \; dp\,(s) \quad \forall p \in C^*. \tag{4}$$

Furthermore, in this case C^* *is unique and* u^* *is cardinally unique.*[17]

Remark 1 *There is a natural trade-off between Archimedean Continuity and Independence. Theorem 1 holds unchanged if we replace Archimedean Continuity with the stronger:*

(a) For all $e, f, g, h \in F$, *the set* $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim^* \lambda h + (1-\lambda) e\}$ *is closed in* $[0,1]$.

and Independence with the weaker:

(b) For every $f, g, h \in F$, *and every* $\alpha \in (0,1)$, $f \succsim^* g$ *implies* $\alpha f + (1-\alpha) h \succsim^* \alpha g + (1-\alpha) h$.

[17] We say that u^* is cardinally unique if it is unique up to a positive linear transformation.

2.4.3 Maxmin representation of subjective rationality

The axioms we imposed on $\hat{\succsim}$ deliver a maxmin rule.

Theorem 2 (Gilboa and Schmeidler, 1989, Theorem 1) *The following are equivalent:*

(i) $\hat{\succsim}$ *satisfies the Basic Conditions, Completeness, C-Independence, and Uncertainty Aversion;*

(ii) there exist a non-empty closed and convex set C *of probabilities on* Σ *and a non-constant function* $u : X \to \mathbb{R}$ *such that, for every* $f, g \in F$

$$f \hat{\succsim} g \quad \text{iff} \quad \min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp\,(s) \geq \min_{p \in C} \int_S \mathrm{E}_{g(s)} u \; dp\,(s)\,. \tag{5}$$

Furthermore, in this case C *is unique and* u *is cardinally unique.*

2.5 Relating objective and subjective rationality

We now come to discuss the relationship between the two orders.

2.5.1 Consistency

Consistency: $f \succsim^* g$ *implies* $f \hat{\succsim} g$.

Consistency seems to be rather compelling given our interpretation of the two relations: if there are sound, objective reasons to weakly prefer f to g, we will not allow the decision maker to exhibit the preference $g \hat{\succ} f$. The choices of the decision maker cannot contradict evidence or logical reasoning. If an expert can prove that f is at least as good as g, given the decision maker's goals, the decision maker should obey this conclusion.[18]

This axiom can also be viewed as part of the definition of subjective rationality: intuitively, we argued that it is subjectively rational to prefer f to g if the decision maker cannot be convinced that she is wrong in exhibiting such

[18] See Nehring (2000,2008) for similar reasoning.

a preference. One way in which the decision maker can be proven wrong is by pointing out internal inconsistencies to her. Indeed, the axioms imposed on $\hat{\succsim}$ rule out such potential embarrassments. However, the decision maker can be proven wrong also directly, namely, if there are compelling, objective reasons to exhibit the opposite preference. Viewed thus, the consistency axiom complements the definition of subjective rationality, making sure that the decision maker will be proven wrong neither by internal inconsistency nor by external inconsistency.

Consistency can also be viewed as a reasoning template, or as an inference rule, provided the language allows preference propositions of both types (objective and subjective).

Observe that we do not require here the strict counterpart of the consistency axiom, namely that $f \succ^* g$ would imply $f \hat{\succ} g$. Given the representation that we have in mind, this condition is somewhat less compelling: $f \succ^* g$ means that it is established that f is as good as g, and that the converse is not established. But it does not mean that f was proven to be better than g – the possibility of equivalence cannot be ruled out. Hence, a thoughtful decision maker may admit that $f \succ^* g$ but still hesitate to strictly prefer f to g.

2.5.2 Caution

Caution: *For $g \in F$ and $f \in F_c$, $g \not\succsim^* f$ implies $f \hat{\succsim} g$.*

This axiom implies that the decision maker in question is rather averse to ambiguity. Comparing a potentially uncertain act g and a constant (risky) act f, the decision maker first checks whether there are compelling reasons to prefer g to f. If there are, namely, $g \succsim^* f$, the axiom is vacuous (and $g \hat{\succsim} f$ would follow from Consistency). If, however, no such reasons can be found, the decision maker would opt for the risky act over the uncertain one.

This ambiguity aversion content of the Caution axiom clearly emerges in Theorem 3, which shows that in our derivation Caution implies that $\hat{\succsim}$ satisfies the Uncertainty Aversion axiom.[19]

[19] In fact, in Theorem 3 the maxmin representation is derived without assuming the Un-

Observe that the decision maker may find that there are compelling reasons to strictly prefer the risky act, that is, it may be the case that $f \succ^* g$. In this case Caution would imply $f \hat{\succsim} g$, as would Consistency. However, the import of the Caution axiom is in completing preferences when pure reason cannot do the job. That is, if objective reasoning can neither suggest that f is preferred to g nor vice versa, then Caution can be invoked to settle the matter by opting for the sure thing.

This axiom is quite extreme in its aversion to uncertainty. See the discussion in Subsection 3.2.

Observe also that Caution differs from the other axioms in that it does not lend itself to a natural description in first order logic. Its antecedent, $g \not\succsim^* f$, is interpreted as "there does not exist a proof that g is at least as good as f". Such a statement is beyond the scope of the simple preference propositions we were referring to in the discussion of the other axioms. However, the practical meaning of Caution is quite intuitive. We can imagine a process by which the consultant works with the decision maker and builds the relation $\succsim^*$ as best they can. At some point, they find that they ran out of preferences that can be inferred from already existing ones and the AA axioms. At this point it is meaningful to compare any g to any risky f and complete the relation between them according to Caution.

2.5.3 Result

Theorem 3 *The following are equivalent:*

(i) $\succsim^*$ *satisfies the Basic Conditions, C-Completeness, and Independence,* $\hat{\succsim}$ *satisfies the Basic Conditions, Completeness, C-Independence, and jointly* $\left(\succsim^*, \hat{\succsim}\right)$ *satisfy Consistency and Caution;*

(ii) There exist a non-empty closed and convex set C *of probabilities on* Σ

certainty Aversion axiom. Since the representation implies the Uncertainty Aversion axiom, the latter is then implied by the other axioms in part (i) of Theorem 3. The only one among these axioms that relates to uncertainty aversion is indeed the Caution axiom.

and a non-constant function $u : X \to \mathbb{R}$ such that, for every $f, g \in F$,

$$f \succsim^* g \quad \text{iff} \quad \int_S \mathrm{E}_{f(s)} u \; dp(s) \geq \int_S \mathrm{E}_{g(s)} u \; dp(s) \quad \forall p \in C \tag{6}$$

and

$$f \hat{\succsim} g \quad \text{iff} \quad \min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp(s) \geq \min_{p \in C} \int_S \mathrm{E}_{g(s)} u \; dp(s) \,. \tag{7}$$

Furthermore, in this case C is unique and u is cardinally unique.

Notice that we do not need to assume that $\hat{\succsim}$ satisfies Uncertainty Aversion. In fact, its connection with $\succsim^*$ through Caution already guarantees that $\hat{\succsim}$ satisfies this property. In other words, Caution can be viewed as "fully" capturing uncertainty aversion in this dual setting.

For this reason, Theorem 3 can be also viewed as providing a novel foundation for the maxmin representation (5), based on the interplay of the two preferences $\succsim^*$ and $\hat{\succsim}$.

Remark 2 *Consider the following, stronger version of Caution:*

(a) For $g \in F$ and $f \in F_c$, $g \not\succsim^* f$ implies $f \hat{\succ} g$.

Conditions (i) and (ii) of Theorem 3 are equivalent to the following:

(iii) $\succsim^*$ satisfies the conditions (i) of Theorem 1, $\hat{\succsim}$ satisfies Preorder, Archimedean Continuity, and Completeness. Jointly, they satisfy Consistency and the above condition (a).

3 Discussion

3.1 Observability

One of the goals of characterization theorems as those presented above is to relate theoretical concepts to observable ones. For instance, Theorems 1 and 2 can be viewed as relating an observable relation – $\succsim^*$ and $\hat{\succsim}$, respectively – to a utility function and a set of probability measures such that these mathematical constructs represent the observable relation via an appropriate condition. Adopting this view, one may ask, which is the revealed preference relation, $\succsim^*$ or $\hat{\succsim}$?

It is probably best to interpret our results as suggesting that both $\succsim^*$ and $\hat{\succsim}$ are observable, though not necessarily through choice behavior alone. Consider a decision maker who consults with experts. After a series of discussions, the decision maker writes down the preferences of which she is sure, $\succsim^*$. If this relation is complete, she is done. If not, she seeks to complete her preferences and generate $\hat{\succsim}$. Alternatively, one may consider the advice of several experts, and view $\succsim^*$ as the relation that reflects the unanimity among them, whereas $\hat{\succsim}$ designates the eventual preference, which may be a result of compromise. Viewed thus, both relations $\succsim^*$ and $\hat{\succsim}$ are observable, though "observability" includes the possibility of preferences being stated, not only revealed through action.

Extending the notion of observability beyond pure choice data seems essential for the discussion of incomplete preference, as well as the process by which preferences are generated. Indeed, a pure revealed preference approach would hold that, since choice is eventually made, incompleteness cannot be observed.[20] If we wish to discuss incomplete preferences, and the process by which preferences are formed, that is, a model in which incomplete preferences become complete, we need to formally refer to other entities beyond the final choices that are observed.

Our main goal, however, is not to represent preferences for their use in descriptive models, but to enrich the language in which the dialog between the

[20]Completeness in fact means a little more than that a certain choice has been observed. It also implies that the same choice is expected to be observed in similar choice situations. But if the repeated choice is modeled formally, it is again not obvious how incompleteness can be observed.

Danan and Ziegelmeyer (2006), for example, propose an interesting revelation approach to incompleteness by allowing subjects to postpone their commitment to alternatives at a small cost.

policymaker and her advisors is conducted. That is, our main application is normative in spirit. We do not think of the economist as an outsider observer, analyzing data generated by "black-box" decision makers, but as an expert whose advice is sought in an open discussion. In this interpretation, the various axioms imposed on $\succsim^*$ and on $\hat{\succsim}$ are not viewed as scientific conditions to be tested for their descriptive accuracy, but as reasoning templates or inference rules, to be used in an open discussion between the expert and the decision maker.

Having said that, we mention that if we observe only the decision maker's final choice, $\hat{\succsim}$, under the assumptions of Theorem 3, $\succsim^*$ is also indirectly observable. In fact, GMM (2004) showed that, in this case,[21]

$$f \succsim^* g \quad \text{iff} \quad \lambda f + (1-\lambda)h \hat{\succsim} \lambda g + (1-\lambda)h \quad \forall \lambda \in [0,1], h \in F.$$

We discuss the relationship between the two papers in subsection 3.6.

3.2 Extremity of the maxmin rule

The Caution axiom is rather extreme. It says that, when an uncertain act is compared to a risky one, unless we know for sure that the former dominates the latter, we should prefer the latter. If, for example, we have no information whatsoever, so that the entire simplex $\Delta(S)$ is considered possible, we may set $C = \Delta(S)$. In this case the relation $\succsim^*$ corresponds to weak dominance, and $\hat{\succsim}$ – to the maxmin rule (without probabilities). Consider an act g such that $\mathrm{E}_{g(s)}u = 1$ for all $s \neq s_0$, and $\mathrm{E}_{g(s_0)}u = -\varepsilon$ for some state s_0 and a small $\varepsilon > 0$. Let f be a constant act with expected utility of zero. Act g has a higher expected utility than does f for almost all priors in $C = \Delta(S)$. Still, for some priors the expected utility of g is below that of f, and Caution dictates that f be preferred to g.

This extreme nature of Caution is reflected in the extremity of the maxmin rule, when the set of probabilities C is interpreted as representing "hard ev-

[21] A similar identification result was proposed by Klaus Nehring in a talk in 1996.

idence". Indeed, it has often been argued that evaluating an act f by its worst-case expected utility is unreasonable.

However, the set C in Gilboa and Schmeidler (1989) is derived from preferences. It need not coincide with a set of probabilities that are externally given to the decision maker. The set C is defined in behavioral terms, as a representation of a binary relation $\hat{\succsim}$, and it need not coincide with any cognitive notion of a set of probabilities. Gajdos, Hayashi, Tallon, and Vergnaud (GHTV, 2007) study the maxmin model given different sets of objectively provided information, and axiomatize a maxmin rule with respect to a class of probabilities that is a subset of the probabilities provided to the decision maker. That is, their model allows the set of probabilities derived from observed behavior to be a strict subset of the set that is cognitively available.

By contrast, if we think of objective rationality as a cognitive concept, and, specifically, view $\succsim^*$ as the preferences that are justified by all probabilities that are considered possible, then Caution does take a strict interpretation of the set of priors, identifying the set of measures used in the maxmin rule with the set of measures used to define objective rationality.

It follows that one may consider alternatives to the axiom of Caution. Simply dropping the axiom allows a representation of $\succsim^*$ by one set of probabilities, C^*, as in (6), and a representation of $\hat{\succsim}$ by another set of probabilities, C, as in (7), where $C \subseteq C^*$ (see the proof of Theorem 3 in the appendix). One may formulate alternative axioms that will correspond to the way that the decision maker selects a subset of priors C as in GHTV (2007).

Another possible direction would be to impose different axioms on subjective rationality, $\hat{\succsim}$. For example, one may assume that this relation involves some aggregation of expected utilities based on second-order probabilities, as in Klibanoff, Marinacci, and Mukerji (2005) or Seo (2007).

Yet another possibility is to assume that the decision maker's notion of internal consistency is structured enough to make $\hat{\succsim}$ an Anscombe-Aumann relation. That is, subjectively rational decisions can be elaborate enough to allow subjective expected utility representation. One obvious way to do so would be for the decision maker to choose a prior out of the set C^*, and to maximize expected utility with respect to this prior. In fact, any other way of complying with Anscombe-Aumann axioms and Consistency is observationally equivalent to such a selection of a prior.

We believe that Consistency is a fundamental axiom. In fact, it may be viewed as part of the definition of $\succsim^*$ and $\hat{\succsim}$: if the former does not imply the latter, it is not clear that these relations can be thought of as objective and subjective rationality of the same decision maker. By contrast, the other axioms presented here should be viewed as examples. One may consider different axioms on $\succsim^*$ and on $\hat{\succsim}$, and certainly also alternatives to the axiom of Caution.

3.3 Rationality

The term "rationality" has been used in many ways. Economic theory tends to identify it with constrained optimization of a utility function, and of expected utility in face on uncertainty. (See Arrow, 1986.) The tradition in philosophy, by contrast, holds that rationality should mean much more than internal consistency.[22] Psychologists, on the other hand, have challenged the concept as too strong to describe human behavior. Simon (1957) introduced the concept of "bounded rationality", and Kahneman and Tversky (1979, Tversky and Kahneman, 1973, 1974, 1981) famously showed failures of basic axioms of rationality. Whereas descriptive failures of rationality need not imply that the concept should be weakened, many authors feel that rationality should be defined in a way that makes in an attainable goal. In particular, both Aumann (1962) and Bewley (2002) argue that there is nothing irrational in having incomplete preferences. Similarly, Gilboa, Postlewaite, and Schmeidler (2004) challenge Savage's axioms as too demanding.

We suggest to define rationality in a way that may simplify the theoreti-

[22] Some modern philosophical essays are closer to the economic notion of rationality. See, for example, Weirich (2007), who offers a discussion of different notions of rationality in the context of group decisions.

cal discussion of decisions and its interaction with actual decisions. A useful definition of rationality would help us distinguish situations in which an expert, or a decision theorist can change the minds of the decision makers she addresses, from situations in which decision makers find the theory irrelevant. For example, decision makers who are sensitive to framing effects (Tversky and Kahneman, 1981) tend to be embarrassed when their decisions are explained to them, and they wish to change these decisions. Thus, a decision theorist can convince such decision makers in the normative appeal of classical decision theory, which can help avoid the pitfalls of framing effects. By contrast, chess players who fail to play chess optimally are rarely embarrassed by this fact. We may dub them irrational, or boundedly rational, but no matter how badly we insult them, they will not change their behavior, simply because they cannot figure out the optimal strategies in chess.

Following this pragmatic line of thought, Gilboa and Schmeidler (2001) suggested to use the term "rationality" as follows: a mode of behavior is *irrational* for a decision maker if, when exposed to the analysis of her behavior, the decision maker feels embarrassed, or wishes to change her choices, and so forth. Clearly, this definition is subjective and qualitative. A mode of behavior might be rational for some decision makers, and not to others. Moreover, less intelligent decision makers may fail to understand the analysis of their choices, or the abstract reasoning involved in certain axioms, and may therefore not exhibit any regret or embarrassment. As a result, they may appear more rational than intelligent decision makers who make the same decisions, but can understand why these decisions are not coherent.

It may appear unfair that, according to this definition, it is easier to be rational if one is less intelligent. But our point of view is that the term "rationality" should not be used as a medal of honor, bestowed upon smart decision makers. Rather, our definitions should facilitate the discussion between decision makers, experts, and decision theorists. As such, the definition of rationality suggested above helps categorize observed deviations from classical decision theory. If a deviation is irrational, explaining the theory may change behavior, and thus the theory may be useful as a normative one. If, on the other hand, no amount of explanation helps, the theory is not very successful as a normative one, and the theorist should better accept that fact, and devise a more acceptable one. The definition of rationality may thus help us decide what we should do, as theorists, in the face of descriptive failures of the theory, in a way that retains the ultimate sovereignty of the decision maker.

The concept of "rationality" in Gilboa and Schmeidler (2001) corresponds to subjective rationality in the present context. A decision maker who is embarrassed by the analysis of her decisions is not subjectively rational; she can be convinced to change her decisions. How can she be so embarrassed? The present paper suggests two ways: first, her decisions may not be internally coherent, as in the case of cyclical preferences. The axioms on $\hat{\succsim}$ are supposed to rule out these internally incoherent patterns of choice. Second, the decisions may appear ridiculous because they are at odds with evidence and basic reasoning, that is, they do not satisfy external coherence. The consistency axiom guarantees that this will not be the case: if there is strong evidence that f is preferred to g, namely, $f \succsim^* g$, then we also require $f \hat{\succsim} g$.

In this context, the present paper refines the definition of rationality by adding the notion of objective rationality. Imagining a dialogue between a decision maker and her advisor, a mode of behavior is subjectively rational if the advisor cannot convince the decision maker to change it. It is objectively rational if the advisor can convince the decision maker to adopt it. One should expect that there will be a grey area between the two, namely that certain modes of behavior will not be irrational enough to be discarded, yet not rational enough to be adopted.

The two notions of rationality may be applied to other contexts as well. In particular, one may delve into the structure of the relation $\succsim^*$ and ask what does it mean to "prove", based on evidence, scientific reasoning, and so forth, that one act is preferred to another. How should evidence be used in such a "proof"? Are there more or less rational ways to interpret and use data for inference? Should one perhaps have a collection of objectively rational relations, depending on one's degree of certainty in a "proof", as there are degrees of significance in hypotheses testing? Such questions are beyond the scope of the present paper.

3.4 Objectivity

The term "objectivity" should be understood in the context of decision and economic theory in the 20th century. The theory assumes that both utility and probability can only be subjective terms, and no reference to a "truly" objective reality is ever made in it. Anscombe and Aumann (1963) assumed that all probabilities are subjective, and used the term "objective" to refer to a probability measure that is shared by all individuals considered. That is, they used the term "objective" where authors in other disciplines would have used "intersubjective" at most.

Our definition of "objectivity" (as in Gilboa and Schmeidler, 2001) requires more than a potentially coincidental agreement among subjective terms. We assume that a view is objective if it is held by the relevant individuals, and if they believe that other, "reasonable" individuals would also share this view. Thus, "objectivity" means an agreement that is not coincidental, and that is believed to be a view than others would be convinced of.

Clearly, it remains a matter of subjective judgment whether another person is "reasonable" and whether such a reasonable person would indeed be convinced of a particular view. Thus, our notion of "objectivity" remains ultimately subjective. But this subjective assessment is at a higher order of belief, that is, a belief about the beliefs that others would hold.

3.5 Incompleteness of tastes and of beliefs

This paper deals with "incompleteness of beliefs", namely, with incompleteness of preferences that is due only to the absence of information, for which the decision maker does not know what the probabilities of various states of the world are. The completeness axiom has also been challenged under certainty, due to the fact that the decision maker simply may not have well-defined preferences. This type of incompleteness, which may be dubbed "incompleteness of tastes", includes the models by Aumann (1962), Kannai (1963), and, more recently, Ok (2002), Dubra, Maccheroni, and Ok (2004), and Mandler (2005). Ok, Ortoleva, and Riella (2008) suggest a model in which there is both incompleteness of tastes and of beliefs.

Incompleteness of tastes is explicitly excluded by our C-Completeness assumption. Observe that, in principle, one might reduce incompleteness of tastes to incompleteness of beliefs. In some cases, such a reduction is rather intuitive. For example, suppose that a decision maker is about the rent a car, and is offered a choice between two models at the same cost. One is smaller and easier to park, the other is more convenient for long trips. The decision maker may find the choice difficult to make, partly because she is unsure about her travel plans, the amount of time she will spend in the car due to traffic jams, and so forth. In this case it is natural to argue that the "certain" outcome of a car is, in fact, an uncertain act, providing different degrees of well-being at various states of the world.

In principle, such a reduction can always be performed, by introducing a "well-being" function whose maximization is tautologically the objective of the decision maker, and by modeling outcomes that cannot be ranked as acts whose outcomes are not known. But such a reduction is not always very intuitive. For example, suppose that a decision maker is at a restaurant she knows well, and she has to make a choice between a steak or a lobster. She is not concerned with long-term effects of this choice, nor does she have any meaningful uncertainty about the quality of the two dishes. She simply can't decide what she feels like having. In such a case, reduction of incompleteness of tastes to incompleteness of beliefs may not generate the most convenient or most intuitive model.

Our general approach, and, in particular, the two definitions of rationality, may apply to incompleteness of tastes as well. Indeed, the analysis above may benefit from generalizations to deal with incompleteness of preferences that derives both from incompleteness of beliefs and of tastes.

3.6 Related literature

GMM (2004) model a preference relation $\hat{\succsim}$ which may exhibit non-neutrality to ambiguity, and they derive from it a relation that captures "unambiguous preferences". This relation, which they also denote by $\succsim^*$, is incomplete whenever $\hat{\succsim}$ fails to satisfy the independence axiom. Moreover, when $\hat{\succsim}$ is a maxmin expected utility relation, $\succsim^*$ turns out to be a unanimity relation with respect to the same set of priors.

The present paper is very close to GMM (2004) in terms of the mathematical structure, and we have indeed relied on GMM's derivation of the unanimity rule (as opposed to the earlier work by Bewley, 2002). However, the emphasis is slightly different. In our case, both $\hat{\succsim}$ and $\succsim^*$ are assumed as primitive relations, and the focus is on the relationships between them, as a step in the direction of modeling the reasoning process behind the completion of $\succsim^*$ to a subjectively rational, but complete order $\hat{\succsim}$. If, for instance, one were to replace Caution by the axiom that $\hat{\succsim}$ satisfies independence, the derived relation $\succsim^*$ in GMM would equal $\hat{\succsim}$. By contrast, our model would still distinguish between subjective and objective rationality, and may be used to discuss the process by which a particular prior (corresponding to $\hat{\succsim}$) is selected out of the set of possible priors (corresponding to $\succsim^*$).

Nehring (2000, 2008) also discusses the tension between the inability to have complete preferences that are rationally derived, and the need to make decisions. His model also deals with a pair of relations and the connection between them. In particular, he suggests that "contexts" can be used to choose a way of completing a relation, and has an axiom similar to our Consistency.

Formally, our unanimity representation result for $\succsim^*$, though independent, is very similar to Girotto and Holzer (2005): the setup is slightly different and the proof is simpler.

Rubinstein (1988) discusses preferences between simple lotteries, each guaranteeing a monetary prize x with a probability p, and 0 with probability $(1-p)$.

He assumes two similarity relations, one on the interval of monetary prizes, and the other – on the interval of probabilities, and imposes a certain coherence between the preferences over lotteries and these similarity relations. Our approach is similar to Rubinstein's (1988) in that we assume more than one preference relation as primitive, in an attempt to gain some insight into the process by which preferences are generated. The two models deal, however, with different problems.

Another model that starts out with more than one relation is proposed by Mandler (2005). He suggests to distinguish between "psychological preferences", which may be incomplete, and "revealed preferences", which are complete but may be intransitive. Our decision maker is closer to standard rationality assumptions in two ways: first, the incomplete preferences we assume are due to absence of information, or the inability to reject hypotheses. Second, the complete preferences in our model are supposed to be "subjectively" rational, and, in particular, transitive.

Danan (2006) also deals with two relations, cognitive and behavioral. Cognitive strict preference results in behavioral preference, but cognitive indifference might still be observed as a choice of a particular alternative, and thus appear as strict preference. In his language, our focus is on incompleteness of cognitive preferences. That is, we do not deal with the gap between the "true" preferences and their revelation in choice behavior, but with the problem of generating preferences in the first place.

4 Appendix: Proofs and related material

$B_0(\Sigma)$ is the vector space generated by the indicator functions of the elements of Σ, endowed with the supnorm. We denote by $ba(\Sigma)$ the set of all bounded, finitely additive set functions on Σ, and by $\Delta(\Sigma)$ the set of all probabilities on Σ. As it is well known, $ba(\Sigma)$, endowed with the total variation norm, is isometrically isomorphic to the norm dual of $B_0(\Sigma)$, in this case the weak* topology, w^*, of $ba(\Sigma)$ coincides with the event-wise convergence topology.

Given a non singleton interval K in the real line (whose interior is denoted K°) we denote by $B_0(\Sigma, K)$ the subset of the functions in $B_0(\Sigma)$ taking values in K. Clearly, $B_0(\Sigma) = B_0(\Sigma, \mathbb{R})$.

We recall that a binary relation $\succsim$ on $B_0(\Sigma, K)$ is:

- a *preorder* if it is reflexive and transitive;
- *continuous* if $\varphi_n \succsim \psi_n$ for all $n \in \mathbb{N}$, $\varphi_n \to \varphi$ and $\psi_n \to \psi$ imply $\varphi \succsim \psi$;
- *Archimedean* if the sets $\{\lambda \in [0,1] : \lambda\varphi + (1-\lambda)\psi \succsim \eta\}$ and $\{\lambda \in [0,1] : \eta \succsim \lambda\varphi + (1-\lambda)\psi\}$ are closed in $[0,1]$ for all $\varphi, \psi, \eta \in B_0(\Sigma, K)$;
- *affine* if for all $\varphi, \psi, \eta \in B_0(\Sigma, K)$ and $\alpha \in (0,1)$, $\varphi \succsim \psi$ iff $\alpha\varphi + (1-\alpha)\eta \succsim \alpha\psi + (1-\alpha)\eta$;
- *monotonic* if $\varphi \geq \psi$ implies $\varphi \succsim \psi$;
- *non-trivial* if there exists $\varphi, \psi \in B_0(\Sigma, K)$ such that $\varphi \succsim \psi$ but not $\psi \succsim \varphi$.

Proposition 1 (GMM, 2004, Proposition A.1) *For $i = 1, 2$, let C_i be nonempty subsets of $\Delta(\Sigma)$ and $\succsim_i$ be the relations defined on $B_0(\Sigma, K)$ by*

$$\varphi \succsim_i \psi \Longleftrightarrow \int_S \varphi dp \geq \int_S \psi dp \quad \forall p \in C_i.$$

Then

$$\varphi \succsim_i \psi \Longleftrightarrow \int_S \varphi dp \geq \int_S \psi dp \quad \forall p \in \overline{co}^{w^*}(C_i),$$

and the following statements are equivalent:

(i) $\varphi \succsim_1 \psi \Longrightarrow \varphi \succsim_2 \psi$ *for all* φ *and* ψ *in* $B_0(\Sigma, K)$.

(ii) $\overline{co}^{w^*}(C_2) \subseteq \overline{co}^{w^*}(C_1)$.

(iii) $\inf_{p\in C_2} \int_S \varphi dp \geq \inf_{P\in C_1} \int_S \varphi dp$ *for all* $\varphi \in B_0(\Sigma, K)$.

Proposition 2 (GMM, 2004, Proposition A.2) $\succsim$ *is a non-trivial, continuous, affine, and monotonic preorder on* $B_0(\Sigma, K)$ *if and only if there exists a non-empty subset* C *of* $\Delta(\Sigma)$ *such that*

$$\varphi \succsim \psi \Longleftrightarrow \int_S \varphi dp \geq \int_S \psi dp \quad \forall p \in C. \tag{8}$$

Moreover, $\overline{co}^{w^*}(C)$ *is the unique weak* closed and convex subset of* $\Delta(\Sigma)$ *representing* $\succsim$ *in the sense of Eq. (8).*

4.1 Lemmas

To prove our results we need some lemmas.

Lemma 1 *If* $K = \mathbb{R}$ *and* $\succsim$ *is a preorder, then* $\succsim$ *is affine iff* $\varphi \succsim \psi$ *implies* $\gamma\varphi + \eta \succsim \gamma\psi + \eta$ *for all* $\eta \in B_0(\Sigma)$ *and all* $\gamma \in \mathbb{R}_+$.

Proof. If $\succsim$ is affine and $\varphi \succsim \psi$, then for all $\eta \in B_0(\Sigma)$ we have

$$\begin{aligned}\varphi + \eta &= \frac{1}{2}\varphi + \frac{1}{2}(\varphi + 2\eta) \succsim \frac{1}{2}\psi + \frac{1}{2}(\varphi + 2\eta) \\ &= \frac{1}{2}\varphi + \frac{1}{2}(\psi + 2\eta) \succsim \frac{1}{2}\psi + \frac{1}{2}(\psi + 2\eta) = \psi + \eta.\end{aligned}$$

While if $\gamma \geq 0$ and *per contra* $\gamma\varphi \not\succsim \gamma\psi$, it cannot be $\gamma = 0, 1$. If $\gamma \in (0, 1)$, we have

$$\gamma\varphi + (1-\gamma)\,0 \not\succsim \gamma\psi + (1-\gamma)\,0,$$

which is absurd since $\varphi \succsim \psi$ and $\succsim$ is affine. Else $\gamma > 1$, and $\gamma\varphi \not\succsim \gamma\psi$ together with affinity delivers

$$\varphi = \frac{1}{\gamma}(\gamma\varphi) + \left(1 - \frac{1}{\gamma}\right)0 \not\succsim \frac{1}{\gamma}(\gamma\psi) + \left(1 - \frac{1}{\gamma}\right)0 = \psi,$$

which is absurd.

Conversely, it is obvious that $\varphi, \psi, \eta \in B_0(\Sigma)$, $\alpha \in (0,1)$, and $\varphi \succsim \psi$ imply $\alpha\varphi + (1-\alpha)\eta \succsim \alpha\psi + (1-\alpha)\eta$. On the other hand $\varphi, \psi, \eta \in B_0(\Sigma)$, $\alpha \in (0,1)$, and $\alpha\varphi + (1-\alpha)\eta \succsim \alpha\psi + (1-\alpha)\eta$ imply

$$\varphi = \frac{1}{\alpha}\left(\alpha\varphi + (1-\alpha)\eta\right) + \frac{\alpha-1}{\alpha}\eta \succsim \frac{1}{\alpha}\left(\alpha\psi + (1-\alpha)\eta\right) + \frac{\alpha-1}{\alpha}\eta = \psi.$$

■

Lemma 2 *If $\succsim$ is an affine preorder on $B_0(\Sigma, K)$, then there exists a unique affine preorder $\succsim^\sharp$ on $B_0(\Sigma)$ that coincides with $\succsim$ on $B_0(\Sigma, K)$. Moreover, if $\succsim$ is monotonic (resp. Archimedean), then $\succsim^\sharp$ is monotonic (resp. Archimedean) too.*

Proof. Suppose first $0 \in K^\circ$. We begin with a Claim:

Claim. Given any $\varphi, \psi \in B_0(\Sigma, K)$, the following facts are equivalent:

(i) $\varphi \succsim \psi$,

(ii) there exists $\alpha > 0$ such that $\alpha\varphi, \alpha\psi \in B_0(\Sigma, K)$ and $\alpha\varphi \succsim \alpha\psi$,

(iii) $\alpha\varphi \succsim \alpha\psi$ for all $\alpha > 0$ such that $\alpha\varphi, \alpha\psi \in B_0(\Sigma, K)$.

Proof of the Claim. (i)⇒(ii) and (iii)⇒(i) are obvious. We show (ii)⇒(iii). By (ii), there exists $\alpha > 0$ such that $\alpha\varphi, \alpha\psi \in B_0(\Sigma, K)$ and $\alpha\varphi \succsim \alpha\psi$. If $0 \leq \beta \leq \alpha$, then by affinity

$$\beta\varphi = \frac{\beta}{\alpha}\alpha\varphi + \left(1 - \frac{\beta}{\alpha}\right)0 \succsim \frac{\beta}{\alpha}\alpha\psi + \left(1 - \frac{\beta}{\alpha}\right)0 = \beta\psi,$$

i.e., $\beta\varphi \succsim \beta\psi$. Therefore, if (iii) does not hold, there exists $\beta > \alpha > 0$ such that $\beta\varphi, \beta\psi \in B_0(\Sigma, K)$ and $\beta\varphi \not\succsim \beta\psi$. Then by affinity

$$\alpha\varphi = \frac{\alpha}{\beta}\beta\varphi + \left(1 - \frac{\alpha}{\beta}\right)0 \not\succsim \frac{\alpha}{\beta}\beta\psi + \left(1 - \frac{\alpha}{\beta}\right)0 = \alpha\psi,$$

a contradiction. □

If $\varphi, \psi \in B_0(\Sigma)$, set $\varphi \succsim^\sharp \psi \iff \alpha\varphi \succsim \alpha\psi$ for some $\alpha > 0$ such that $\alpha\varphi, \alpha\psi \in B_0(\Sigma, K)$. By the Claim, $\succsim^\sharp$ is a well defined binary relation on $B_0(\Sigma)$, which coincides with $\succsim$ on $B_0(\Sigma, K)$. Moreover, $\varphi \succsim^\sharp \psi$ if and only if $\alpha\varphi \succsim \alpha\psi$ for all $\alpha > 0$ such that $\alpha\varphi, \alpha\psi \in B_0(\Sigma, K)$. Next we show that $\succsim^\sharp$ is an affine preorder (monotonic if $\succsim$ is monotonic).

Since $0 \in K^\circ$, then for all $\varphi \in B_0(\Sigma)$ there exists $\alpha > 0$ such that $\alpha\varphi \in B_0(\Sigma, K)$, reflexivity of $\succsim$ implies that $\alpha\varphi \succsim \alpha\varphi$ and $\varphi \succsim^\sharp \varphi$. Thus $\succsim^\sharp$ is reflexive.

If $\varphi, \psi, \eta \in B_0(\Sigma)$ are such that $\varphi \succsim^\sharp \psi$ and $\psi \succsim^\sharp \eta$, take $\alpha > 0$ such that $\alpha\varphi, \alpha\psi, \alpha\eta \in B_0(\Sigma, K)$, then

$$\alpha\varphi \succsim \alpha\psi \text{ and } \alpha\psi \succsim \alpha\eta$$

thus $\alpha\varphi \succsim \alpha\eta$ and $\varphi \succsim^\sharp \eta$. Thus $\succsim^\sharp$ is transitive.

If $\varphi, \psi, \eta \in B_0(\Sigma)$ and $\lambda \in (0,1)$, take $\alpha > 0$ such that $\alpha\varphi, \alpha\psi, \alpha\eta \in B_0(\Sigma, K)$, then

$$\begin{aligned}
&\alpha\left(\lambda\varphi + (1-\lambda)\eta\right) = \lambda\left(\alpha\varphi\right) + (1-\lambda)\left(\alpha\eta\right) \in B_0(\Sigma, K), \\
&\alpha\left(\lambda\psi + (1-\lambda)\eta\right) = \lambda\left(\alpha\psi\right) + (1-\lambda)\left(\alpha\eta\right) \in B_0(\Sigma, K).
\end{aligned}$$

Then

$$\begin{aligned}
\varphi \succsim^\sharp \psi &\iff \alpha\varphi \succsim \alpha\psi \iff \lambda\left(\alpha\varphi\right) + (1-\lambda)\left(\alpha\eta\right) \succsim \lambda\left(\alpha\psi\right) + (1-\lambda)\left(\alpha\eta\right) \\
&\iff \alpha\left(\lambda\varphi + (1-\lambda)\eta\right) \succsim \alpha\left(\lambda\psi + (1-\lambda)\eta\right) \\
&\iff \lambda\varphi + (1-\lambda)\eta \succsim^\sharp \lambda\psi + (1-\lambda)\eta.
\end{aligned}$$

Thus $\succsim^\sharp$ is affine.

Assume now that $\succsim$ is monotonic. If $\varphi, \psi \in B_0(\Sigma)$ are such that $\varphi \geq \psi$, take $\alpha > 0$ such that $\alpha\varphi, \alpha\psi \in B_0(\Sigma, K)$, then $\alpha\varphi \geq \alpha\psi$ and monotonicity of $\succsim$ delivers $\alpha\varphi \succsim \alpha\psi$ and $\varphi \succsim^\sharp \psi$. Thus $\succsim^\sharp$ monotonic.

As to uniqueness, let $\succsim^\flat$ be an affine preorder on $B_0(\Sigma)$ that coincides with $\succsim$ on $B_0(\Sigma, K)$. For all $\varphi, \psi \in B_0(\Sigma)$, take $\alpha > 0$ such that $\alpha\varphi, \alpha\psi \in B_0(\Sigma, K)$, then the Claim (applied to $\succsim^\flat$), the fact that $\succsim^\flat$ coincides with $\succsim$ on $B_0(\Sigma, K)$, and the definition of $\succsim^\sharp$ guarantee that

$$\varphi \succsim^\flat \psi \Longleftrightarrow \alpha\varphi \succsim^\flat \alpha\psi \Longleftrightarrow \alpha\varphi \succsim \alpha\psi \Longleftrightarrow \varphi \succsim^\sharp \psi,$$

that is, $\succsim^\flat$ coincides with $\succsim^\sharp$ on $B_0(\Sigma)$.

Suppose $0 \notin K^\circ$. Given any $k \in K^\circ$, for $\varphi, \psi \in B_0(\Sigma, K - k)$ set $\varphi \succsim_k \psi \Longleftrightarrow \varphi + k \succsim \psi + k$. It is easy to verify that $\succsim_k$ is an affine preorder on $B_0(\Sigma, K - k)$ (monotonic if $\succsim$ is monotonic). Since 0 belongs to the interior of $K - k$, by what we just proved there is a unique affine preorder $\succsim_k^\sharp$ on $B_0(\Sigma)$ that coincides with $\succsim_k$ on $B_0(\Sigma, K - k)$ (monotonic if $\succsim$ is monotonic). Such extension coincides with $\succsim$ on $B_0(\Sigma, K)$, and it is the unique affine preorder on $B_0(\Sigma)$ with this property.

Finally, assume $\succsim$ is Archimedean, and denote by $\succsim^\sharp$ the unique affine preorder on $B_0(\Sigma)$ which coincides with $\succsim$ on $B_0(\Sigma, K)$. Notice that, by Lemma 1, if $\varphi, \psi \in B_0(\Sigma)$ and $\alpha > 0$, $\beta \in \mathbb{R}$ are such that $\alpha\varphi + \beta, \alpha\psi + \beta \in B_0(\Sigma, K)$, then

$$\varphi \succsim^\sharp \psi \Longleftrightarrow \alpha\varphi + \beta \succsim^\sharp \alpha\psi + \beta \Longleftrightarrow \alpha\varphi + \beta \succsim \alpha\psi + \beta.$$

Now, for all $\varphi, \psi, \eta \in B_0(\Sigma)$ take $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\alpha\varphi + \beta, \alpha\psi + \beta, \alpha\eta + \beta \in B_0(\Sigma, K)$. For all $\lambda \in [0, 1]$,

$$\alpha (\lambda\varphi + (1 - \lambda)\psi) + \beta = \lambda (\alpha\varphi + \beta) + (1 - \lambda) (\alpha\psi + \beta) \in B_0(\Sigma, K)$$

and

$$\begin{aligned} \lambda\varphi + (1 - \lambda)\psi \succsim^\sharp \eta &\Longleftrightarrow \alpha (\lambda\varphi + (1 - \lambda)\psi) + \beta \succsim \alpha\eta + \beta \\ &\Longleftrightarrow \lambda (\alpha\varphi + \beta) + (1 - \lambda) (\alpha\psi + \beta) \succsim \alpha\eta + \beta. \end{aligned}$$

Then $\{\lambda \in [0, 1] : \lambda\varphi + (1 - \lambda)\psi \succsim^\sharp \eta\}$ coincides with $\{\lambda \in [0, 1] : \lambda (\alpha\varphi + \beta) + (1 - \lambda) (\alpha\psi + \beta) \succsim \alpha\eta + \beta\}$ which is closed since $\succsim$ is Archimedean. A similar argument shows that $\{\lambda \in [0, 1] : \eta \succsim^\sharp \lambda\varphi + (1 - \lambda)\psi\}$ is closed too. Thus $\succsim^\sharp$ is Archimedean. ■

Lemma 3 *An affine and monotonic preorder on $B_0(\Sigma, K)$ is continuous if and only if it is Archimedean.*

Proof. Obviously, continuity implies the Archimedean property.

Conversely, assume $\succsim$ is Archimedean. Since $\succsim$ is monotonic and Archimedean, then the affine preorder $\succsim^\sharp$ on $B_0(\Sigma)$ that coincides with $\succsim$ on $B_0(\Sigma, K)$ is monotonic and Archimedean too (Lemma 2).

If $\varphi_n \succsim^\sharp 0$ for all $n \in \mathbb{N}$ and $\varphi_n \to \varphi$, let $M = \sup_{s\in S} \varphi(s)$, which is indeed a maximum. For all $\varepsilon \in (0,1)$ there is n such that

$$\varphi_n \leq \varphi + \varepsilon\chi_S \leq \varphi + \varepsilon\left((M+1)\chi_S - \varphi\right).$$

In fact, $M\chi_S \geq \varphi$ implies $(M+1)\chi_S - \varphi \geq \chi_S$.[23] Therefore, for all $\varepsilon \in (0,1)$ there is $n \in \mathbb{N}$ such that

$$\varepsilon[(M+1)\chi_S] + (1-\varepsilon)\varphi = \varphi + \varepsilon\left((M+1)\chi_S - \varphi\right) \geq \varphi_n \succsim^\sharp 0.$$

Monotonicity of $\succsim^\sharp$ delivers that, for all $\varepsilon \in (0,1)$,

$$\varepsilon[(M+1)\chi_S] + (1-\varepsilon)\varphi \succsim^\sharp 0. \tag{9}$$

But $\succsim^\sharp$ is Archimedean, hence the set of all ε such that (9) holds is closed, and, containing $(0,1)$, it also contains 0, in particular $\varphi \succsim^\sharp 0$.

Conclude that, if $\varphi_n \to \varphi$, $\psi_n \to \psi$, and $\varphi_n \succsim^\sharp \psi_n$ for all $n \in \mathbb{N}$, then $\varphi_n - \psi_n \succsim^\sharp 0$ for all $n \in \mathbb{N}$ and $\varphi_n - \psi_n \to \varphi - \psi$; therefore $\varphi - \psi \succsim^\sharp 0$, that is $\varphi \succsim^\sharp \psi$. Thus $\succsim^\sharp$ is continuous, which immediately implies that $\succsim$ is continuous too. ■

Now Lemma 3 and Proposition 2 deliver:

Corollary 1 *$\succsim$ is a non-trivial, Archimedean, affine, and monotonic preorder on $B_0(\Sigma, K)$ if and only if there exists a nonempty subset C of $\Delta(\Sigma)$ such that*

$$\varphi \succsim \psi \iff \int_S \varphi dp \geq \int_S \psi dp \quad \forall p \in C. \tag{10}$$

Moreover, $\overline{co}^{w^*}(C)$ is the unique weak closed and convex subset of $\Delta(\Sigma)$ representing $\succsim$ in the sense of Eq. (10).*

All the results we have proved so far hold more generally if $B_0(\Sigma)$ is replaced by any normed Riesz space with unit.

[23] χ_S is the constant function taking value 1 on S.

4.2 Proof of Theorem 1

Assume $\succsim^*$ is a preorder satisfying Monotonicity, Archimedean Continuity, Non-triviality, C-Completeness, and Independence.

Archimedean Continuity, C-Completeness, and Independence, together with the von Neumann-Morgenstern Expected Utility Theorem (see the axiomatics of Herstein and Milnor, 1953), imply that there exists a cardinally unique function $u^* : X \to \mathbb{R}$ such that $P \succsim^* Q$ iff $\mathrm{E}_P u^* \geq \mathrm{E}_Q u^*$, provided $P, Q \in L$. Monotonicity and Non-triviality imply that u^* is not constant. In what follows we write $U(R)$ instead of $\mathrm{E}_R u^*$ if $R \in L$. Clearly $U : L \to \mathbb{R}$ is affine and non-constant.

If $f \in F$ then $U \circ f \in B_0(\Sigma, U(L))$. Conversely, if $\varphi \in B_0(\Sigma, U(L))$, then $\varphi(s) = U(Q_i)$ if $s \in A_i$ for suitable $Q_1, ..., Q_N \in L$ and a partition $\{A_1, A_2, ..., A_N\}$ of S in Σ. Therefore, setting $f(s) = Q_i$ if $s \in A_i$ we have $\varphi = U \circ f$. We can conclude that $B_0(\Sigma, U(L)) = \{U \circ f : f \in F\}$. Moreover, $U \circ f = U \circ g$ iff $U(f(s)) = U(g(s))$ for all $s \in S$ iff $f(s) \sim^* g(s)$ for all $s \in S$, which by Monotonicity implies $f \sim^* g$.

For $\varphi, \psi \in B_0(\Sigma, U(L))$, set

$$\varphi \gtrsim^* \psi \iff f \succsim^* g \text{ for some } f, g \in F \text{ such that } \varphi = U \circ f,\ \psi = U \circ g.$$

By what we have just observed, $\gtrsim^*$ is well defined on $B_0(\Sigma, U(L))$ and it is characterized by

$$\varphi \gtrsim^* \psi \iff f \succsim^* g \text{ for all } f, g \in F \text{ such that } \varphi = U \circ f,\ \psi = U \circ g.$$

For all $\varphi = U \circ f \in B_0(\Sigma, U(L))$, $f \succsim^* f$ implies $\varphi \gtrsim^* \varphi$. Thus $\gtrsim^*$ is reflexive.

If $\varphi = U \circ f, \psi = U \circ g, \eta = U \circ h \in B_0(\Sigma, U(L))$, $\varphi \gtrsim^* \psi$ and $\psi \gtrsim^* \eta$ amount to $f \succsim^* g$ and $g \succsim^* h$, thus $f \succsim^* h$ and $\varphi \gtrsim^* \eta$. Thus $\gtrsim^*$ is transitive, and a preorder.

Since there are f, g such that $f \succ^* g$ (by Non-triviality of $\succsim^*$), then $U \circ f >^* U \circ g$ and $\gtrsim^*$ is non-trivial.

If $\varphi = U \circ f, \psi = U \circ g, \eta = U \circ h \in B_0(\Sigma, U(L))$ and $\alpha \in (0,1)$, then

$$\begin{aligned}
\varphi \gtrsim^* \psi \iff f \succsim^* g &\iff \alpha f + (1-\alpha)h \succsim^* \alpha g + (1-\alpha)h \\
&\iff U \circ (\alpha f + (1-\alpha)h) \gtrsim^* U \circ (\alpha g + (1-\alpha)h) \\
&\iff \alpha\varphi + (1-\alpha)\eta \gtrsim^* \alpha\psi + (1-\alpha)\eta.
\end{aligned}$$

Therefore $\gtrsim^*$ is affine.

If $\varphi = U \circ f, \psi = U \circ g, \eta = U \circ h \in B_0(\Sigma, U(L))$, then

$$\begin{aligned}
\{\lambda \in [0,1] : \lambda\varphi + (1-\lambda)\psi \gtrsim^* \eta\} &= \{\lambda \in [0,1] : U \circ (\lambda f + (1-\lambda)g) \gtrsim^* U \circ h\} \\
&= \{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succsim^* h\}
\end{aligned}$$

is closed in $[0,1]$ because of Archimedean Continuity of $\succsim^*$, and an analogous argument shows that $\{\lambda \in [0,1] : \eta \gtrsim^* \lambda\varphi + (1-\lambda)\psi\}$ is closed too. Thus $\gtrsim^*$ is Archimedean.

If $\varphi = U \circ f, \psi = U \circ g \in B_0(\Sigma, U(L))$ are such that $\varphi \geq \psi$, then $U(f(s)) \geq U(g(s))$ for all $s \in S$. Therefore $f(s) \succsim^* g(s)$ for all $s \in S$, and by Monotonicity of $\succsim^*$, $f \succsim^* g$, that is $\varphi \gtrsim^* \psi$. Thus $\gtrsim^*$ is monotonic.

By Corollary 1, there exists a unique non-empty weak* closed and convex subset C^* of $\Delta(\Sigma)$ such that, for $\varphi, \psi \in B_0(\Sigma, U(L))$,

$$\varphi \gtrsim^* \psi \iff \int_S \varphi dp \geq \int_S \psi dp \quad \forall p \in C^*,$$

therefore, for $f, g \in F$,

$$\begin{aligned}
f \succsim^* g \iff U \circ f \gtrsim^* U \circ g &\iff \int_S (U \circ f)\, dp \geq \int_S (U \circ g)\, dp \quad \forall p \in C^* \\
&\iff \int_S \mathrm{E}_{f(s)} u^* \; dp(s) \geq \int_S \mathrm{E}_{g(s)} u^* \; dp(s) \quad \forall p \in C^*.
\end{aligned}$$

The rest is trivial.

Alternative Axioms: Next we call *Strong Archimedean Continuity* requirement (a) of Remark 1 and *Weak Independence* requirement (b) of Remark 1. Clearly, Strong Archimedean Continuity implies Archimedean Continuity while Shapley and Baucells (1998, Lemma 1.2) show that Preorder, Strong

Archimedean Continuity and Weak Independence imply Independence. Thus representation (4) holds if Archimedean Continuity and Independence are replaced by Strong Archimedean Continuity and Weak Independence. Conversely, representation (4) implies Strong Archimedean Continuity and (Weak) Independence.

4.3 Proof of Theorem 3

Assume that

- $\succsim^*$ is a preorder satisfying Monotonicity, Archimedean Continuity, Non-triviality, C-Completeness, and Independence;
- $\hat{\succsim}$ is a preorder satisfying Monotonicity, Archimedean Continuity, Non-triviality, Completeness, C-Independence;
- $\left(\succsim^*, \hat{\succsim}\right)$ satisfy Consistency.

By Theorem 1, there exist a non-empty closed and convex set C^* of probabilities on Σ and a non-constant function $u^* : X \to \mathbb{R}$ such that, for every $f, g \in F$

$$f \succsim^* g \quad \text{iff} \quad \int_S \mathrm{E}_{f(s)} u^* \; dp\,(s) \geq \int_S \mathrm{E}_{g(s)} u^* \; dp\,(s) \quad \forall p \in C^*. \tag{11}$$

Set

$$f \succsim' g \in \mathcal{F} \quad \text{iff} \quad \lambda f + (1-\lambda) h \hat{\succsim} \lambda g + (1-\lambda) h \quad \forall \lambda \in [0,1]\,, h \in F.$$

Lemma 1 and Propositions 5 and 7 of GMM (2004), guarantee that there exist a non-empty closed and convex set C of probabilities on Σ, a non-constant function $u : X \to \mathbb{R}$, and a monotonic and constant linear functional $I : B_0\,(\Sigma) \to \mathbb{R}$ such that, for every $f, g \in F$

$$f \hat{\succsim} g \quad \text{iff} \quad I\,(\mathrm{E}_f u) \geq I\,(\mathrm{E}_g u)\,, \tag{12}$$

$$f \succsim' g \quad \text{iff} \quad \int_S \mathrm{E}_{f(s)} u \; dp\,(s) \geq \int_S \mathrm{E}_{g(s)} u \; dp\,(s) \quad \forall p \in C, \tag{13}$$

$$\min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp\,(s) \leq I\,(\mathrm{E}_f u)\,, \tag{14}$$

moreover, equality holds in (14) for all $f \in F$ if (and only if) $\succsim$ satisfies Uncertainty Aversion.

If $Q, R \in L$, then, by (11), Consistency, and (12),

$$\mathrm{E}_Q u^* \geq \mathrm{E}_R u^* \Longleftrightarrow Q \succsim^* R \Longrightarrow Q \hat{\succsim} R \Longleftrightarrow \mathrm{E}_Q u \geq \mathrm{E}_R u.$$

Corollary B.3 of GMM (2004) delivers the existence of $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u^* = \alpha u + \beta$. Wlog, $u^* = u$.

Propositions 4 of GMM (2004) implies that $\succsim'$ is the maximal (relative to the inclusion in $F \times F$) relation on F satisfying Independence and contained in $\hat{\succsim}$. Consistency guarantees that $\succsim^*$ is contained in $\hat{\succsim}$, and $\succsim^*$ satisfies Independence, thus

$$f \succsim^* g \implies f \succsim' g.$$

(11), (13), and Proposition 1 deliver $C \subseteq C^*$.

Assume that also Caution holds. If there is $g \in F$ such that

$$I\left(\mathrm{E}_g u\right) > \min_{p \in C^*} \int_S \mathrm{E}_{g(s)} u \; dp\left(s\right),$$

then, there is $Q \in L$ such that

$$I\left(\mathrm{E}_g u\right) > \mathrm{E}_Q u > \min_{p \in C^*} \int_S \mathrm{E}_{g(s)} u \; dp\left(s\right)$$

that is, $g \not\succsim^* Q$ and $g \hat{\succ} Q$, which violates Caution. Thus, by (14) and $C \subseteq C^*$,

$$\min_{p \in C} \int_S \mathrm{E}_{f(s)} u \, dp\left(s\right) \leq I\left(\mathrm{E}_f u\right) \leq \min_{p \in C^*} \int_S \mathrm{E}_{f(s)} u \, dp\left(s\right) \leq \min_{p \in C} \int_S \mathrm{E}_{g(s)} u \, dp\left(s\right) \quad \forall f \in F$$

and Proposition 1 delivers $C^* \subseteq C$.[24]

The rest is trivial.

Alternative Axioms: Next we call *Default to Certainty* the strong caution requirement (a) of Remark 2.

Assume that

[24]Since, as f ranges in F, $\mathrm{E}_{f(\cdot)} u$ ranges in $B_0\left(\Sigma, K\right)$, where K is the non-trivial interval $\{\mathrm{E}_Q u : Q \in L\}$.

- $\succsim^*$ is a preorder satisfying Monotonicity, Archimedean Continuity, Non-triviality, C-Completeness, and Independence;
- $\hat{\succsim}$ is a preorder satisfying Archimedean Continuity and Completeness;
- $\left(\succsim^*, \hat{\succsim}\right)$ satisfy Consistency and Default to Certainty.

By Theorem 1, there exists a non-empty closed and convex set C of probabilities on Σ and a non-constant function $u : X \to \mathbb{R}$ such that, for every $f, g \in F$

$$f \succsim^* g \quad \text{iff} \quad \int_S \mathrm{E}_{f(s)} u \; dp\,(s) \geq \int_S \mathrm{E}_{g(s)} u \; dp\,(s) \quad \forall p \in C. \tag{15}$$

Let $P, Q \in L$. By Consistency

$$P \succsim^* Q \quad \text{implies} \quad P \hat{\succsim} Q.$$

By Default to Certainty

$$P \succ^* Q \quad \text{implies} \quad P \hat{\succ} Q.$$

Therefore $\hat{\succsim}$ and $\succsim^*$ coincide on L, and $P \mapsto \mathrm{E}_P u$ represents both preorders on L.

In particular, $\hat{\succsim}$ satisfies Monotonicity, in fact, $f\,(s)\,\hat{\succsim} g\,(s)$ for all $s \in S$ implies, by what we have just shown, $f\,(s) \succsim^* g\,(s)$ for all $s \in S$, which, by Monotonicity of $\succsim^*$ implies $f \succsim^* g$, and Consistency delivers $f \hat{\succsim} g$.

For all $f \in F$, let $P, Q \in L$ be such that $P \hat{\succsim} f\,(s)\,\hat{\succsim} Q$ for all $s \in S$, then $P \hat{\succsim} f \hat{\succsim} Q$. By Archimedean Continuity the sets $\{\alpha \in [0,1] : \alpha P + (1-\alpha) Q \hat{\succsim} f\}$ and $\{\alpha \in [0,1] : f \hat{\succsim} \alpha P + (1-\alpha) Q\}$ are closed; they are nonempty since 1 belongs to the first and 0 to the second; their union is the whole $[0,1]$. Since $[0,1]$ is connected, their intersection is not empty, hence there exists $\beta \in [0,1]$ such that $\beta P + (1-\beta) Q \hat{\sim} f$. In particular, for each act f there exists $R_f \in L$ such that $R_f \hat{\sim} f$.

There are two possibilities

- $f \succsim^* R_f$, in this case $E_{R_f} u \leq \int_S \mathrm{E}_{f(s)} u \; dp\,(s)$ for all $p \in C$, that is

$$E_{R_f} u \leq \min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp\,(s)\,.$$

- $f \not\succsim^* R_f$, in this case, by Default to Certainty $R_f \hat{\succ} f$, which is absurd.

Moreover, if $E_{R_f} u < \min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp\,(s)$, take $P \in L$ such that $P \hat{\succsim} f\,(s)$ for all $s \in S$. Then

$$E_{R_f} u < \min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp\,(s) \leq E_P u,$$

and there is $\gamma \in (0,1]$ such that

$$E_{R_f} u < E_{\gamma P + (1-\gamma) R_f} u = \min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp\,(s)$$

thus

$$f \succsim^* \gamma P + (1-\gamma)\, R_f \hat{\succ} R_f$$

and, by Consistency, $f \hat{\succ} R_f$, which is absurd. In conclusion,

$$E_{R_f} u = \min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp\,(s)$$

for all $f \in F$ and all $R_f \in L$ such that $R_f \hat{\sim} f$.

Finally,

$$\begin{aligned} f \hat{\succsim} g &\iff R_f \hat{\succsim} R_g \iff E_{R_f} u \geq E_{R_g} u \\ &\iff \min_{p \in C} \int_S \mathrm{E}_{f(s)} u \; dp\,(s) \geq \min_{p \in C} \int_S \mathrm{E}_{g(s)} u \; dp\,(s)\,. \end{aligned}$$

The rest is trivial.

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