TEL-AVIV

1-92

The Sackler Institute of Economic Studies

מכון סאקלר לכלכלה

אוניברסיטת תל-אביב

ON THE UNIQUENESS OF SUBJECTIVE PROBABILITIES

by

Edi Karni* and David Schmeidler**

Working Paper No.1–92

Revised December, 1991

* The Johns Hopkins University
** Tel–Aviv University and The Ohio State University

THE SACKLER INSTITUTE OF ECONOMIC STUDIES
Faculty of Social Sciences
Tel–Aviv University, Ramat Aviv, Israel.

ON THE UNIQUENESS OF SUBJECTIVE PROBABILITIES

Edi Karni

The Johns Hopkins University

and

David Schmeidler[1]

Tel Aviv University and The Ohio State University

Summary

The purpose of this paper is twofold: First, within the framework of Savage (1954), we suggest axiomatic foundations for the representation of event–dependent preference relations over acts. This representation has the form of expectation of event-dependent utility with respect to non-unique subjective probabilities on the set of states. Second, we give an economic-theoretic motivation for selecting a unique probability distribution as an appropriate concept of "subjective probabilities." However, unlike in Savage's theory, this notion of subjective probabilities does not necessarily represent the decision-maker's belief regarding the likelihood of events.

Our approach involves a departure from Savage's postulate P4, which guarantees the completeness of Savage's likelihood relation on the set of all events. Instead, we assume the existence of a finite partition of the set of states, $\{S_1,...,S_n\}$, such that, for events within each element of this partition P4 is satisfied. This weakening of Savage's axioms suffices for the existence of an expected event-dependent utility representation, but not for the uniqueness of the subjective probabilities.

In many economic problems involving decision-making under uncertainty the existence of a unique probability is presumed and, in fact, is essential for the statement of the result. An example is Arrow's (1965) finding that all risk averse decision-makers will invest in a risky asset provided its expected rate of return exceeds that of an alternative risk-free asset. We show that a unique probability distribution can be chosen so as to render such results meaningful. Namely, any risk averse decision-maker will hold a positive position in the risky asset if and only if its expected rate of return with respect to the chosen probability exceeds that of the riskless asset.

---

[1] Part of the research described here was carried out at the Santa Fe Institute, Santa Fe, NM, U.S.A.

1. Introduction

The uniqueness of subjective probabilities in decisions under uncertainty is implied by Savage's theory. This theory postulates preference relations over acts that are representable as mathematical expectation of a utility on consequences with respect to a unique, nonatomic, subjective probability measure on states of nature. The interest in this representation stems from several reasons: First, this specification proved to be convenient for the analysis of problems such as portfolio selection, insurance contracts, and other topics in economics of uncertainty. Second, the existence of a unique (prior) subjective probability measure permitted the application of Bayesian statistics methods for incorporating new information.

There are situations of decision-making under uncertainty, in which Savage's postulates, requiring that preferences over consequences are independent of the state in which they are obtained are not satisfied. In such circumstances, i.e., when the decision-maker's preferences are state-dependent, the existence of unique subjective probabilities is not implied by Savage's theorem. (See for example Arrow (1974)). Indeed, consider a set of states, S, a set of consequences C, (both non-empty) and a preference relation, $\succeq$, on the set of acts $A := \{a\colon\ S \to C\}$. To grasp the problem let $\{S_1, \ldots, S_n\}$ be a partition of S and restrict attention to acts that are constant on each event in the partition. Suppose that the preference relation is representated by a functional $a \to \sum_{i=1}^{n} \alpha_i u_i(a_i)$, where $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i > 0$, a_i is the consequence that the act, a, assigns to all states in S_i, $i = 1, \ldots, n$, and for each i, $u_i : C \to \mathbb{R}$ is a state-dependent (or event-dependent) utility function. The representation of preferences by this functional means that for all $a, b \in A$, $a \succeq b \iff \sum_{i=1}^{n} \alpha_i u_i(a(S_i)) \geq \sum_{i=1}^{n} \alpha_i u_i(b(S_i))$. The same preference relation $\succeq$ is also represented by $a \to \sum_{i=1}^{n} \beta_i v_i(a(S_i))$ where for $i = 1, \ldots, n$, $v_i = \gamma_i u_i + \delta_i$, $\gamma_i > 0$, $\beta_i = (\alpha_i/\gamma_i)/\sum_{j=1}^{n}(\alpha_i/\gamma_j)$. By the term *state (or event)–dependent preferences* we mean that for some i and j (for simplicity imagine that for all $i \neq j$) u_i is not a positive linear transformation of u_j, i.e., there are no $\mu > 0$ and τ such that $u_i = \mu u_j + \tau$. Because the utility functions for different states are essentially different and each is unique up to positive linear transformations, there is no obvious way of deciding which utility, say u_i or v_i, is the appropriate one to be used in the representation. Consequently, there is no obvious way of deciding whether the distribution $\{\alpha_1, \ldots, \alpha_n\}$ or $\{\beta_1, \ldots, \beta_n\}$ is the more appropriate concept of subjective probabilities. In other words, it is not clear which of the two distributions, if any, represent the decision-maker's subjective beliefs regarding the likely realization of the alternative states of nature.

The purpose of this paper is twofold: First, within the framework of Savage (1954), we suggest axiomatic foundations for the representation of event–dependent preference relations over acts. This representation has the form of expectation of event-dependent utility with respect to non-unique subjective probabilities on the set of states. Second, we give an economic-theoretic motivation for selecting a unique probability distribution as an appropriate concept of "subjective probabilities." However, unlike in Savage's theory, this notion of subjective probabilities does not necessarily represent the decision-maker's belief regarding the likelihood of events.

Our approach involves a departure from Savage's postulate P4, which guarantees the completeness of Savage's likelihood relation on the set of all events. Instead, we assume the existence of a finite partition of the set of states, $\{S_1, \ldots, S_n\}$, such that, for events within each element of this partition P4 is satisfied. This weakening of Savage's axioms suffices for the existence of an expected event-dependent utility representation, but not for the uniqueness of the subjective probabilities.

In many economic problems involving decision-making under uncertainty the existence of a unique probability is presumed and, in fact, is essential for the statement of the result. An example is Arrow's (1965) finding that all risk averse decision-makers will invest in a risky asset provided its expected rate of return exceeds that of an alternative risk-free asset. We show that a unique probability distribution can be chosen so as to render such results meaningful. Namely, any risk averse decision-maker will hold a positive position in the risky asset if and only if its expected rate of return with respect to the chosen probability exceeds that of the riskless asset.

2. Event-Dependent Preferences and a Weakening of Savage's Postulate P4

Let S and C be non-empty sets of states and consequences, respectively, and let $A := \{a : S \to C\}$ be the set of Savage's acts. Consider a binary relation $\succeq \subset A \times A$, satisfying all of Savage's postulates except P4. Since P1, namely, transitivity and completeness of $\succeq$, is satisfied, we refer to $\succeq$ as a preference relation. Recall that P2 is the sure thing principle, P3 is the (ordinal) state-independence, P4 guarantees that the relation "more likely to obtain" on events is complete. P5 requires the nondegeneracy of $\succeq$. P6 is nonatomicity or continuity axiom, and P7 is a technical uniformity assumption.

We weaken postulate P4 by assuming the existence of a finite partition of the set of states such that within each event of this partition Savage's P4 applies, i.e., the comparison of any two events by the relation "more likely to obtain" is possible if both are subsets of the same event of the partition. To illustrate and motivate this axiomatization we consider the example of life insurance in which the state-dependence of the preference relation is natural. For simplicity we consider two events: in one event the insured person is alive and in the other complementary event he is dead. In both cases the individual's utility as a function of his wealth is strictly monotonic increasing, but we assume that the decision-maker displays greater risk aversion in the second event. This may reflect, for instance, the perception that when alive the decision-maker is better able to cope with random financial losses. For concreteness, let $C = [0, M] \subset \mathbb{R}$ and $S = [0, 1]$, and let λ be a nonatomic purely finitely additive probability measure on the set of all subsets of the unit interval such that on intervals $\lambda([\alpha, \beta]) = \beta - \alpha$. Suppose that the decision-maker's preferences are represented by the functional

$$a \mapsto \int_0^{1/2} a(s) d\lambda(s) + \int_{1/2}^1 a(s)^{1/2} d\lambda(s) \quad a \in A.$$

This preference relation displays risk neutrality in the event $[0, 1/2)$ and risk aversion in the event $[1/2,\ 1]$. To show that this preference relation does not satisfy Savage's postulate P4, we recall that:

P4 For all $x, y, x', y' \in C, x \succ y, x' \succ y'$ and, for all events F and G,

$$\begin{bmatrix} x & \text{on} & F \\ y & \text{on} & F^c \end{bmatrix} \succ \begin{bmatrix} x & \text{on} & G \\ y & \text{on} & G^c \end{bmatrix} \text{ iff } \begin{bmatrix} x' & \text{on} & F \\ y' & \text{on} & F^c \end{bmatrix} \succ \begin{bmatrix} x' & \text{on} & G \\ y' & \text{on} & G^c \end{bmatrix}$$

where, for each $E \subset S$, E^c is the complement of E in S.

Suppose next that C, the set of consequences, is a bounded interval of real numbers which includes 0 and 100, representing wealth in dollars. Let $x = 100$, $x' = 25$, $y = y' = 0$, $F = [0, 1/20]$, and $G = [1/2, 1/2 + 1/3]$. Then, the expected utility of the act [x on F, y on F^c] is 100/20 which is larger then the expected utility of the act [x on G, y on G^c], which is 100/30. On the other hand, the expected utility of the act [x' on F, y' on F^c], namely 25/20 is smaller than the expected utility of [x' on G, y' on G^c] which equals 5/3. Thus, our interpretation of P4 is state-independence of the utility functions.

Note that in this example Savage's postulate P3 as well as all the other axioms are satisfied. In general, however, state-dependent preferences do not have to satisfy P3. Thus, for instance, when taking a stroll in the park, it is conceivable that one may prefer carrying an umbrella to not carrying it if it rains and he may prefer not carrying an umbrella to carrying it if it is sunny. In this paper we deal with the case of state or event-dependence when P3 is satisfied. This serves to simplify the exposition and highlight the role of P4. (Alternatively, one can weaken P3 in the same way as P4 with respect to the same partition. In this case P7 must also be similarly modified.) In some sense, the relaxation of P4 is a minimal form of state-dependence.

Next we introduce the weakened version of postulate P4.

P4* There exists a finite partition, say $S_1, \ldots, S_n$, such that for all x, x', y, y' in C with $x \succ y$ and $x' \succ y'$ and for all events G, F, where for some i, $i = 1, \ldots, n$, $G \subset S_i$ and $F \subset S_i$ the following implication holds:

$$\begin{bmatrix} x & \text{on} & F \\ y & \text{on} & F^c \end{bmatrix} \succ \begin{bmatrix} x & \text{on} & G \\ y & \text{on} & G^c \end{bmatrix} \text{ iff } \begin{bmatrix} x' & \text{on} & F \\ y' & \text{on} & F^c \end{bmatrix} \succ \begin{bmatrix} x' & \text{on} & G \\ y' & \text{on} & G^c \end{bmatrix}$$

3. Expected Utility Theory with State-Dependent Preferences

In this section we assume that $\succeq$ is a binary relation on $A := \{a : S \rightarrow C\}$ and that this binary relation satisfies Savage's axioms P1-P3, P5-P7, and P4* instead of P4. Given an event $E \subset S$ recall Savage's definition of conditional (on E) preferences on A: For all a, b, and c in A, $a \succeq_E b$ if and only if $[a \text{ on } E; c \text{ on } E^c] \succeq [b \text{ on } E; c \text{ on } E^c]$. In view of the sure thing principle (P2) this definition makes sense, (i.e, the relation above is independent of c.) An event $E \subset S$ is null if $\succeq_E = A \times A$. For $i = 1, \ldots, n$, we denote by $\succeq_i$ the preferences conditional on S_i. To simplify the presentation we introduce a strengthening of the nondegeneracy axiom P5 and assume that it is satisfied:

P5* For all $i = 1, \ldots, n$, $\succ_i$ non-empty. (Equivalently, we may state that for all i, S_i is non-null.)

Clearly, for each i the preference relation $\succeq_i$ satisfies Savage's postulates, $P1 - P7$. In particular, P4* implies P4 for each $\succeq_i$. By Savage's theorem, for each i, $i = 1, \ldots, n$, there exists a unique nonatomic, finitely additive probability measure π_i on events in S and a bounded utility function $w_i : C \to \mathbb{R}$ such that

$$\text{(3.1)} \quad \text{For all } a, b \in A: \; a \succeq_i b \Leftrightarrow \int_{S_i} w_i(a(s))d\pi_i(s) \geq \int_{S_i} w_i(b(s))d\pi_i(s).$$

Note that since S_i^c is π_i-null, integrating over S_i with respect to π_i is equivalent to integrating over S with respect to π_i.

Next we state our main representation theorem for state-dependent preferences.

Theorem.

(i) Given a binary relation $\succeq$ on A the following two conditions (i.i) and (i.ii) are equivalent:

(i.i) The relation $\succeq$ satisfies P1, P2, P3, P4, P5*, P6, and P7.*

(i.ii) For $i = 1, \ldots, n$ there exist nonatomic (finitely) additive probability measures π_i on S_i and utility functions $u_i : C \to \mathbb{R}$, such that,

(3.2)

$$\text{For all } a, b \in A : a \succeq b \Leftrightarrow \sum_{i=1}^{n} \int_{S_i} \left[u_i(a(s)) - u_i(b(s))\right] d\pi_i(s) \geq 0.$$

(ii) If condition (i.ii) of part (i) is satisfied then for any (finitely) additive probability measures π_i' on S_i and any utility functions $u_i' : C \to \mathbb{R}$, $i = 1, \ldots, n$, the following two conditions (ii.i) and (ii.ii) are equivalent:

(ii.i) For all $a,\, b \in A,\, a \succeq b \Leftrightarrow \sum_{i=1}^{n} \int_{S_i} \left[u_i'(a(s)) - u_i'(b(s))\right] d\pi_i'(s) \succeq 0.$

(ii.ii) For $i = 1, \ldots, n,\;\; \pi_i = \pi_i'$*, and there are* $\beta > 0$ *and* α_i *such that* $u_i' = \beta u_i + \alpha_i$.

Observation Condition (i.ii) in the Theorem states that the functional $a \to \sum_{i=1}^{n} \int_{S_i} u_i(a(s)) d\pi_i(s)$ represents the preference relation $\succeq$ on A. For any list of positive numbers $(p_i)_{i=1}^{n}$ such that $\sum_{i=1}^{n} p_i = 1$, $a \to \sum_{i=1}^{n} \int_{S_i} \left[u_i(a(s))/p_i\right] p_i d\pi_i(s)$ is the same representation of $\succeq$ on A as the preceding one. For each $E \subset S$ define $\pi(E) = \sum_{i=1}^{n} p_i \pi_i \big(E \cap S_i\big)$ and for $s \in S_i$ and $x \in C$, define $v(x,s) = u_i(x)/p_i$. Hence, π is a probability measure on the algebra of all subsets of S and

$$\text{For all} \quad a, b \in A : a \succeq b \Leftrightarrow \int_S v(a(s), s) d\pi(s) \quad \geq \int_S v(b(s), s) d\pi(s)$$

The subjective probability π is a function of $(p_i)_{i=1}^{n}$ and so is the state-dependent utility v.

The main step in the proof of the Theorem consists of the application of Savage's theorem, using condition (3.1). However, the conclusion (3.2) is not an immediate implication of Savage's theorem. The proof, which consists of several steps is relegated to the Appendix.

Notice that, by the sure thing principle, P2, and the uniqueness of the probability measures in Savage's theorem and in the Theorem, the probability measures π_i of condition (3.1) equals those of condition (3.2) for $i = 1, \ldots, n$. Moreover, for each i, u_i is a positive linear transformation of w_i.

4. A Definition of Subjective Probabilities

In addition to Bayesian statistics (see Lindley (1990)) the interest in a well defined notion of subjective probabilities stems from the economic analysis of decision-making under uncertainty. In particular, numerous results in portfolio theory and insurance economics may not be meaningfully stated without an appropriate notion of probabilities. For instance, consider the classical result, due to Arrow (1965), on investment in risky assets by risk averse decision-makers. Let there be finitely many states of nature, say n, and suppose that consequences are real numbers representing monetary gains or losses. There is a risk-free asset whose rate of return is zero in each state and a risky asset $X = (x_i)_{i \in S}$, where x_i represents the gain (or loss) in state i. In this model a decision-maker is characterized by a concave differentiable utility function $u : \mathbb{R} \to \mathbb{R}$ and by a prior probability $p = (p_i)_{i \in S}$ on S. In this normalization the asset is free and the decision-maker must choose $\alpha \in [0, 1]$ where α represents his position in the risky asset. Arrow's result is that α^*, the optimal position, is positive if and only if the assets expected value, $\sum_{i \in S} p_i x_i$, is positive. Equivalently,

$$(4.1) \qquad \sum_{i \in S} p_i x_i > 0 \Leftrightarrow \exists\ \alpha > 0, \alpha \leq 1 \text{ such that } \sum_{i \in S} p_i u(\alpha x_i) > 0.$$

The surprising aspect of this result is, of course, that all risk averse decision-makers are diversifiers.

To cast the present discussion in terms of the framework of the preceding section we regard the elements of the partition $\{S_i\}_{i=1}^{n}$ of P4* as elementary events, and we restrict attention to acts that are constant on each S_i. Thus, following the conventions of the section, we refer to the elementary event S_i as state i, $i = 1, \dots, n$. In this notation a decision-maker with state-dependent preferences may be represented by a list of probabilities $(p_i)_{i \in S}$ and state-dependent utilities $(u_i)_{i \in S}$. Here each u_i is a differentiable real-valued function on $\mathbb{R}$. To further simplify the presentation we assume that all the elementary events are nonnull and, hence, $p_i > 0$ for all i. Clearly, the representation of the decision-maker is nonunique. The list of probabilities $(p_i)_{i \in S}$ may be replaced by any other list $(q_i)_{i \in S}$ such that $\sum_{i \in S} q_i = 1$ and $q_i > 0$ for all i, and the utilities rescaled accordingly. Hence, the condition $\sum_{i \in S} p_i x_i > 0$ in (4.1) is meaningless. On the other hand, the right hand side of condition (4.1) is independent of the choice of p. Specifically,

$$\sum_{i \in S} p_i u_i(\alpha x_i) > 0 \Leftrightarrow \sum_{i \in S} \lambda_i p_i \left[u_i(\alpha x_i)/\lambda_i\right] > 0. \tag{4.2}$$

We seek a definition of subjective probabilities (i.e., a specific choice of state-dependent utilities on C and probabilities on S) that would render Arrow's theorem meaningful.

The proposed definition of subjective probabilities involves a normalization of the utility functions such that the normalized utilities $(v_i)_{i \in S}$ satisfy $\frac{dv_i}{dx_i}(0) = 1$ and $v_i(0) = 0$ for all $i \in S$.

Proposition. *Suppose that state-dependent preferences on acts are represented by probabilities $(p_i)_{i \in S}$ and concave differentiable utilities $(u_i)_{i \in S}$ such that for $i \in S$ $\frac{du_i}{dx_i}(0) = 1$ and $u_i(0) = 0$. Let $(x_i)_{i \in S}$ be a risky asset.*

Then

$$(4.3)\qquad \sum_{i\in S} p_i x_i > 0 \Leftrightarrow \exists\ \alpha > 0, \alpha \leq 1 \ \textit{such that} \ \sum_{i\in S} p_i u_i(\alpha x_i) > 0.$$

The proof is as simple as that of Arrow's theorem. A function U : $[0,\ 1] \rightarrow \mathbb{R}$ is constructed: $U(\alpha) = \sum_{i\in S} p_i u_i(\alpha x_i)$. U is concave differentiable and its deriative at zero is $\sum_{i\in S} p_i x_i$. Further details are omitted.

Remark: The definition of subjective probabilities in the proposition hinges on the definition of the utility function on gains and losses. Implicit in this definition is the decision-maker's initial wealth, w, and state-dependent utility functions $(v_i)_{i\in S}$, where $v_i(w + x_i) = u_i(x_i)$ for all $i \in S$. A change in w, say to w', (stochastically independent of the risky asset) involves a change in the utilities of gains and losses, $u'_i(x_i) = v_i(w' + x_i), i \in S$. In general, following such a change in the initial wealth, the position held in the risky asset $X = (x_i)_{i\in S}$ may change, even to become zero. However, the proposition still holds with $(u'_i)_{i\in S}$ instead of $(u_i)_{i\in S}$ and $(p'_i)_{i\in S}$ instead of $(p_i)_{i\in S}$ is obtained by the joint normalization of $(u_i)_{i\in S}$ so that $u'_i(0) = 0$ and $\frac{d}{x_i}u'_i(0) = 1, i \in S$. In other words, unlike the case of state-independent preferences, the proposed definition of subjective probabilities is not invariant with respect to changes in the decision-maker's wealth.

5. Concluding Remarks

A qualitative probability is a binary relation on events which satisfies certain standard properties, including transitivity and completeness. Under additional assumptions this relation has a unique representation by a probability measure. Interpreting this relation as a decision-maker's beliefs regarding the likely realization of the events, the representation has the interpretation of subjective probability beliefs.

In Savage's theory the qualitative probability relation is derived from the primitive notion of preference relation on acts. Savage's axioms imply that the derived qualitative probabilities satisfy all the requirements guaranteeing the existence of unique subjective probability representation. Furthermore, these axioms imply the existence of an expected utility representation of the preference between acts with respect to this probability measure.

In this paper we have shown that there exists expected utility representation even when such qualitative probabilities cannot be derived. Although the probability that appears in the Observation (Section 3) is nonunique it suffices for the application of Bayesian updating. In other words, if we start from two distinct representations involving two different choices of probability distributions, i.e., two different choices of $(p_i)_{i=1}^n$ and $(p'_i)_{i=1}^n$, and a nonnull event F is observed then, updating using Bayesian formula, we get two distinct posteriors. Nevertheless, they induce the same preference relation over acts conditioned on this event. For a more detailed discussion see Karni, Schmeidler, and Vind (1983).

As was made clear in the Observation, every choice of the numbers $(p_i)_{i=1}^n$ implies a distinct prior, π, and at the same time it implies a distinct state-dependent utility function $v_i(\cdot) \equiv v(\cdot; s) = u_i(\cdot)/p_i$ for all $s \in S_i$. For each i the uniqueness of the representation of the utilities for the given $(p_i)_{i=1}^n$ is "up to" additive constants. One approach to the selection of a unique prior, followed up in Section 4, is to choose an appropriate normalization of each of the utilities $(v_i)_{i=1}^n$. This, in turn, defines a unique vector $(p_i)_{i=1}^n$. An alternative normalization of the utilities u_i exploits the fact that these are bounded functions. In particular, fix $\text{Inf}_{x \in C}\, v(x;\ s) = 0$ and $\text{Sup}_{x \in C}\, v(x;\ s) = 1$ for all $s \in S$ to define unique subjective probabilities. This normalization is implied by recent works of Karni (1991a, 1991b), where an additional assumption on preferences is imposed.

APPENDIX: Proof of the Theorem

We prove that (i.i) implies (i.ii). The opposite direction as well as part (ii) of the Theorem are almost obvious or very easy to prove (when taking into account Savage's theorem, both directions, and our proof here). The general approach to the proof is first to show the result for the special case where there is a most preferred consequence and a least preferred consequence in C (Lemma 2). For the general case, the set C is extended by adding a most preferred and a least preferred consequence. The set of acts and the preference relation are correspondingly extended. The 'in-between' case where one of the two, a most preferred consequence or a least preferred consequence does exist is not discussed explicitly.

The proof is carried out in several steps, a few of which are stated as lemmas. We freely make use of Savage's theorem and the representations (3.1). Our first step is an implication of P2 and P1.

Lemma 1. *If $a \sim_i b$ for $i = 1, \ldots, n$, then $a \sim b$.*

Proof of Lemma 1 For $i = 0, \ldots, n$ define the acts,

$$a_i = [a \quad \text{on} \quad S_1 \cup \cdots \cup S_i, \quad b \quad \text{on} \quad S_{i+1} \cup \cdots \cup Sn] \,.$$

Clearly, $a_0 = a$ and $a_n = b$. By P2 and the definition of conditional preferences: $a \sim_i b$ implies $a_{i-1} \sim a_i$. Hence, (by transitivity), $a \sim b$. □

Notations For $x, y \in C$, $x \succ y$ let:

$$A\{x, y\} := \{a \in A \mid a(s) = x \quad \text{or} \quad a(s) = y\}$$

$$A[x, y] := \{a \in A \mid x \succeq a(s) \succeq y\}$$

Remark When $x \succ y$, the preferences $\succeq$ restricted to $A\{x, y\}$ satisfy all Savage's axioms. Hence there is a unique nonatomic probability π, (depending on x and y), such that for any two acts a and b in $A\{x, y\}$:

$$a \succeq b \Leftrightarrow \pi(\{s \in S \mid a(s) = x\}) \geq \pi(\{s \in S/b(s) = y\}) \,.$$

On the other hand, for $i = 1, \ldots, n$ and $a, b \in A\{x, y\}$:

$$a \succeq_i b \Leftrightarrow \pi_i(\{s \in S_i \mid a(s) = x\}) \geq \pi_i(\{s \in S, \mid b(s) = x\}) \,.$$

Since the preferences $\succeq$ restricted to $A\{x, y\}$ and then conditioned on S_i coincide with $\succeq_i$ restricted to $A\{x, y\}$, the uniqueness of Savage's probability implies that:

$$\pi(E) = \pi_i(E)\pi(S_i) \quad \text{for all} \quad E \subset S_i, i = 1, \ldots, n \,.$$

Conclusion For all a and b in $A\{x,y\}$: $a \succeq b$ iff

$$\sum_{i=1}^{n} \pi(S_i)\pi_i(\{s \in S/a(s) = x\}) \geq \sum_{i=1}^{n} \pi(S_i)\pi_i(\{s \in S/b(s) = x\}) \ .$$

Our next step is to show that (i.ii) of the Theorem holds for the special case where there is a most preferred consequence, say $\overline{x}$, and a least preferred consequence, say $\overline{y}$, in C. We will utilize the conclusion above where Savage's probability π has been constructed for $\overline{x}$ and $\overline{y}$. Note also that by $P5^*$, $\pi(S_i) > 0$ for $i = 1, \ldots, n$.

To simplify presentation, first, we assume, without loss of generality, that the utility functions w_i in 3.1 satisfy $w_i(\overline{x}) = 1$ and $w_i(\overline{y}) = 0$. Next, we introduce the following notation for all $d \in A$ and $i = 1, \ldots, n$.

$$W_i(d) := \int_{S_i} w_i(d(s))d\pi_i(s)$$

Lemma 2. *For all* $a, b \in A = A[\overline{x}, \overline{y}]$: $a \geq b$ *iff* $\sum_{i=1}^{n} \pi_i(S_i)W_i(a) \geq \sum_{i=1}^{n} \pi_i(S_i)W_i(b)$.

Before proving the lemma, we point out that by defining $u_i(z) := w_i(z)/\pi(S_i)$ for $i = 1, \ldots, n$, Lemma 2 implies (3.2).

Proof of Lemma 2 Since $0 \leq W_i(a) \leq 1$ there is $a_i \in A\{x,y\}$ such that $W_i(a) = W_i(a_i)$. (By nonatomicity of π_i there is $E_i \subset S_i$ such that $\pi_i(E_i) = W_i(a)$. Define $a_i := [\overline{x}$ on $E_i, \overline{y}$ on $E_i^c])$. Define $a' := [a_i$ on S_i for $i = 1, \ldots, n]$. By Lemma 1, $a' \sim a$. Similarly, we construct $b_i \in A\{\overline{x}, \overline{y}\}$ with $F_i = \{s \in S_i \mid b_i(s) = \overline{x}\}$, and b' with $b' \sim b$. So $a \succeq b$ iff $a' \succeq b'$ iff $\sum_{i=1}^{n} \pi(S_i)\pi_i(E_i) \geq \sum_{i=1}^{n} \pi(S_i)\pi_i(F_i)$ iff $\sum_{i=1}^{n} \pi(S_i)W_i(a) \geq \sum_{i=1}^{n} \pi(S_i)W_i(b)$ where the middle implication uses the Conclusion. □

We now deal with the case where a most preferred consequence and a least preferred consequence do not exist. We still use the normalizations for w_i such that $\sup\{w_i(z)/z \in C\} = 1$ and $\inf\{w_i(z) \mid z \in C\} = 0$. Hence, there exist two sequences $(x(k))_{k=1}^{\infty}$ and $(y(k))_{k=1}^{\infty}$ in C such that when $k \uparrow \infty$, $w_i(x(k)) \uparrow 1$ and $w_i(y(k)) \downarrow 0$. Our next result is of interest on its own within Savage's framework.

Lemma 3. *For $i = 1, \ldots, n$, there exists a most (least) preferred act in A with respect to the preferences $\succeq i$.*

Proof of Lemma 3 For all i; π_i is additive, atomless, bounded and defined on all subsets of S_i. Hence π_i is purely finitely additive, i.e., there is a countable partition $(H_i(k))_{k=1}^{\infty}$ of S_i such that $\pi_i(H_i(k)) = 0$ for all k. (This classical result assumes the continuum hypothesis.)

Define $\overline{a}_i := [x(k) \text{ on } H_i(k) \text{ for } k = 1, 2, \ldots]$ ($\overline{b}_i := [y(k) \text{ on } H_i(k)$ for $k = 1, 2, \ldots]$). Clearly, $W_i(\overline{a}_i) \geq w_i(x(k))$ for all k. So, $W_i(\overline{a}_i) = 1$ (Similarly, $W_i(b_i) = 0$). □

In the proof of Lemma 3, the acts $\overline{a}_i$ and $\overline{b}_i$ were not defined on $S \setminus S_i$ since $\pi_i(S \setminus S_i) = 0$ and any definition may do. As an immediate implication of Lemmas 1 and 3 we get:

Corollary. *The act $\overline{a} := [\overline{a}_i$ on S_i for $1 = 1, \ldots, n]$ ($\overline{b} := [\overline{b}_i$ on S_i for $i = 1, \ldots, n]$) is a most (least) preferred consequence in the relation $\succeq$ in A.*

We now extend the set of consequences, C, to include a most and a least preferred consequence (in the preferences over consequences, i.e., over constant acts). Formally, let $\widehat{C} := C \cup \{\overline{x}, \overline{y}\}$ where $\overline{x} \notin C$ and $\overline{y} \notin C$. Next let $\widehat{A} := \{a : S \to \widehat{C}\}$. Finally, we extend $\succeq$ to complete preferences on $\widehat{A}$. Given $\widehat{a} \in A$, let $E := \{s \in S \mid \widehat{a}(s) = \overline{x}\}$ and $F := \{s \in S \mid \widehat{a}(s) = \overline{y}\}$. Define now an act $\widetilde{d} \in A$ corresponding to $\widehat{d}$ as follows: $\widetilde{d} :=$ [$\overline{a}$ on $E, \overline{b}$ on F and $\widehat{d}$ otherwise], i.e., for $s \in E \cap H_i(k), \widetilde{d}(s) = x(k), \ldots$ Given $\widehat{a}$ and $\widehat{b}$ in $\widehat{A}$, we define $\widehat{a} \succeq b$ iff $\widetilde{a} \succeq \widetilde{b}$.

To state the next Lemma, we first define (for $i = 1, \ldots, n$) the extention of w_i to $\widehat{C}$ in the obvious way: $\widehat{w}_i(\overline{x}) = 1$ and $\widehat{w}_i(\overline{y}) = 0$. The definitions of $\widehat{W}_i$ and $\widehat{\succeq}_i$ are equally obvious.

Lemma 4. *(i) $\widehat{\succeq}$ satisfies $P1$ and $P2$, (ii) $\widehat{a}\widehat{\succeq}_i b$ iff $\widehat{W}_i(\widehat{a}) \geq \widehat{W}_i(\widehat{b})$, for $i = 1, \ldots, n$, (iii) $\widehat{\succeq}$ satisfies P3, P4*, P5*, P6 and P7.*

Proof of Lemma 4 Part (i) is an immediate implication of the definition (and of the fact that $\succeq$ satisfies P1 and P2).

Part (ii) of the Lemma essentially states (for all i), that the preferences $\widehat{\succeq}_i$ are represented by the expected utility functional $\widehat{W}_i$, (which extends W_i), which, in turn, by Savage's theorem implies that $\widehat{\succeq}_i$ satisfies $P1 - P7$. This, together with part (i), i.e., $P1$ and $P2$ satisfied by $\widehat{\succeq}$, imply part (iii), i.e., $\widehat{\succeq}$ satisfies also $P3$, $P4^*$, $P5^*$, $P6$ and $P7$.

So we are left with the proof of part (ii) of the Lemma. Consider the following list of (two-sided) implications for all $\widehat{a}$ and $\widehat{b}$ in $\widehat{A}$: $\widehat{a}\widehat{\succeq}_i\widehat{b}$ iff $\widetilde{a} \succeq_i \widetilde{b}$ iff $W_i(\widetilde{a}) \geq W_i(\widetilde{b})$ iff $\widehat{W}_i(\widehat{a}) \geq \widehat{W}_i(\widehat{b})$. Only the last implication requires proof, and it follows from:

Claim For $i = 1, \ldots, n$ and $\widehat{d} \in A$: $\widehat{W}_i(\widehat{d}) = W_i(\widetilde{d})$.

Fix an i. We have to prove that $\int_{S_i} \widehat{w}_i(\widehat{d}(s))d\pi_i(s) = \int_{S_i} w_i(\widetilde{d}(s))d\pi_i(s)$. Let $E := \{s \in S_i \mid \widehat{d}(s) = \overline{x}\}$ and $F := \{s \in S_i \mid \widehat{d}(s) = \overline{y}\}$. By definition of $\widetilde{d}$, for $s \in S_i \backslash (E \cup F)$, $\widehat{d}(s) = \widetilde{d}(s) \in C$. So $\widehat{w}_i(\widehat{d}(s)) = w_i(\widetilde{d}(s))$ and $\int_{S_i \backslash (E \cup F)} \widehat{w}_i(\widehat{d}(s)) d\pi_i(s) = \int_{S_i \backslash (E \cup F)} w_i(\widetilde{d}(s)) d\pi_i(s)$.

For $s \in E$, $s \in H_i(k)$ for some k and $\widetilde{d}(s) = x(k)$. So $\widehat{w}_i(\widehat{d}(s)) = 1 > w_i(d(s))$ in this case. Hence, $\pi_i(E) = \int_E \widehat{w}_i(\widehat{d}(s)) d\pi_i(s) \geq \int_E w_i(\widetilde{d}(s)) d\pi_i(s)$.

On the other hand, $w_i(x(k)) \uparrow 1$ with $k \uparrow \infty$. Thus, for any $\xi > 0$, there is an m such that for $k \geq m$, $w_i(x(k)) \geq 1 - \xi$. So $\int_E w_i(\widetilde{d}(s)) d\pi_i(s) = \int_{E \cap (\cup_{k=1}^{m-1} H_i(k))} w_i(\widetilde{d}(s) d\pi_i(s) + \int_{E \cap (\cup_{k=m}^{\infty} H_i(k))} w_i(\widetilde{d}(s)) s\pi_i(s) \geq 0 + (1 - \xi)\pi_i(E)$.

The last inequality follows from the fact that $\pi_i(\cup_{k=1}^{m-1} H_i(k)) = 0$ and $\pi_i(\cup_{k=m}^{\infty} H_i(k)) = \pi_i(S_i)$ (and $E \subset S_i$). Since this inequality holds for any $\xi > 0$, we get that $\int_E w_i(\widetilde{d}(s)) d\pi_i(s) = \pi_i(E)$ as required.

An analogue proof shows that $\int_F w_i(\widetilde{d}(s)) d\pi_i(s) = 0$ $(= \int_F \widehat{w}_i(\widehat{d}(s)) d\pi_i(s))$, which concludes the proof of the Claim and with it that of Lemma 4. □

To conclude the proof of (3.2) note that in view of Lemma 4, we have $\widehat{A} = \widehat{A}[\overline{x}, \overline{y}]$, and so Lemma 2 applies in the general case too.

REFERENCES

Arrow, K.J., **Aspects of the Theory of Risk Bearing**, Helsinki: Yrjo Jahnssonin Saatio, 1965.

Arrow, K.J., "Optimal Insurance and Generalized Deductibles," **Scandinavian Actuarial Journal** (1974), 1-24.

Lindley, D.V., "The 1988 Wald Memorial Lectures: The Present Position in Bayesian Statistics." **Statistical Science 5** (1990), 44-71.

Karni, E., "Subjective Probabilities and Utility with Event-Dependent Preferences," Working Paper, The John Hopkins University, (1991a).

Karni, E., "The Measurement of Uncertainty," mimeo (1991b).

Karni, E., Schmeidler, D., and K. Vind, "On State-Dependent Preferences and Subjective Probabilities," **Econometrica 51** (1983), 1021-32.

Savage, L.J., **The Foundation of Statistics**, Wiley, New York (1954).