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NEO–BAYESIAN DECISION MAKING WITH SUBJECTIVE MODELS

by

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ABSTRACT

The neobayesian decision models $(X^S, \geq)$ considered, with X set of consequences, S set of states of the world and $\geq$ a binary relation on acts in X^S, are seen as models $(X^S \cup X, \geq)$ with $\geq$ on $X^S \cup X$ satisfying the identification axiom: $x^* \cong x \;\; \forall x \in X$, where $x^*(s) = x \;\; \forall s \in S$ and $\cong$ denotes indifference. It is contended that this identification characterizes the objectivity of the models, i.e. their being models of a 'grand decision', as Savage puts it. This identification axiom is replaced by two axioms, one of agreement: $x^* \geq y^*$ iff $x \geq y$, $x, y \in X$, and one of revealed limited intelligence:

$$x \;\substack{\geq \\ \leq}\; x^* \quad \text{and} \quad y \;\substack{> \\ <}\; x \;\Rightarrow\; y \;\substack{> \\ <}\; y^* ,$$

where $>$ denotes strict preference. The resulting models are then interpreted as subjective.

By coupling these axioms with existing theories (i.e. sets of axioms) on the restriction of $\geq$ to X^S, theories on $(X^S \cup X, \geq)$ are obtained which, besides separating beliefs about the events in S from preferences over cosequences in X, enable to identify an element of pessimism/optimism and an element of 'trust' in one's model.

A stroger form of axioms and results is given for preferences extended to $\Delta(X)$, the set of simple probability measures on X.

1. Introduction

After the classical work of Savage (1952,1954), neo-bayesian analysis of decision making has concentrated more on decision Theories (i.e. sets of axioms of choice) than on the underlying decision Model itself. The present elaborates upon the latter, and it is hopefully a contribution to the analysis of the problems of realism and applicability that the model presents.

To be more specific, a decision model will be a pair $(X^S, \geq)$, where S and X are sets and $\geq$ is a binary relation on X^S. A decision theory is a set of axiomatic restrictions on $\geq$. Elements of S represent states of the world, and elements of X consequences; the elements of X^S, maps from S to X, are interpreted as courses of action - acts; and the d.m. being represented assesses preferences $\geq$ on acts in order to direct choice.

For an introduction to the problems under discussion we may read Savage himself. On the basic elements S and X, he writes:

"A state of the world is a description of the world, leaving no relevant aspect undescribed" (The Foundations of Statistics 1972, FS, p.9). And: "Consequences might involve [...] anything at all about which the person could possibly be concerned. Consequences might appropriately be called states of the person, as opposed to states of the world" (FS, p.14).

With this extra-mathematical specification the resulting model becomes, as we shall say, an Objective model. Savage is well aware of the fact that real-life decision making is not based on objective models. In the section of FS devoted to the problem, 'Small worlds', he writes:

"Making an extreme idealization, which has in principle guided the whole argument of this book thus far, a person has only one decision to make in his whole life. He must, namely, decide how to live, and this he might in principle do once and for all. Though many, like myself, have found the concept of overall decision stimulating, it is certainly highly unrealistic and in many contexts unwieldy." (FS, p.83)

And this is his view:

"Any claim to realism made by this book-or indeed by almost any theory of personal decision of which I know-is predicated on the idea that some of the individual decision situations into which actual people tend to subdivide the single grand decision do recapitulate in microcosm the mechanism of the idealized grand decision.[...] The problem of this section is to say as clearly as possible what constitutes a satisfactory isolated decision situation. The general method of attack I propose to follow, for want of a better one, is to talk in terms of the grand decision-tongue in cheek-and in those terms to analyse and discuss isolated decision situations. I hope you will be able to agree, as the discussion proceeds, that I do not lean too heavily on the concept of the grand decision situation." (FS, p.83)

However, he finds his own solution

"[...] unsatisfactory in that it seems incapable of verification without taking the grand world much too seriously." (FS, p.90)

The point of view of the present work is different from that of Savage. It is argued that individual decision situations do not quite 'recapitulate' the grand one, so Savage's problem of characterizing 'satisfactory' isolated decision situations in terms of the grand decision does not arise. Individual decision situations are simply what they are and, it is contended, they constitute the proper object of analysis. So we shall study what we interpret as Subjective decision models -those used in decisions other than the grand one-, with no formal reference to any objective model.

In order to expand on this, let us first say that the analysis will be confined to what we call Separating models and theories. Separating models are those in which consequences in X are described with no reference to the space S and interpreted as states of the d.m. -as in Savage. In these models, 'umbrella in the rain' is not an appropriate (component of a) consequence. Nor is 'umbrella', for it is not a state of the d.m. An example of Savage is in the FS, p.25. Another one could be 'getting pleasantly wet', which may result from an act of having a cold shower in a state of hot weather or a hot one in cold weather.

Separating models are tied up with separating theories, i.e. those which enable to identify separately subjective beliefs about the events is S and 'state-independent' preferences on the consequences in X (again, like that of Savage). The separating theories on $(X^S,\geq)$ considered here characterize preferences on X^S for which there exist a utility function $u:X \to \mathbb{R}$ and a (possibly non-additive) probability measure μ on S such that each $f \in X^S$ is ranked according to the value of the (Lebesgue or Choquet) integral of $u \circ f$ with respect to μ.

Actually, the extra-mathematical specification of a separating model almost forces state-independence axioms (hence separating theories) on the preference relation $\geq$ (see FS p.25 up to P3); and conversely, separating theories appropriately apply only to such models. As blurred as it is, the distinction between models and theories will continue to be invoked in the sequel.

To proceed we start with Savage again:

"a smaller world is derived from a larger by neglecting some distinctions between states." (FS, p.9)

In our interpretation, the d.m. refines his state space (i.e. enlarges his world) by including 'relevant aspects' he is aware of, and at some point he stops, conscious of ending up in a small world in the above sense (it is instructive to see to what extent this description reflects the way in which Savage and Aumann actually proceeded in the context of concrete examples in a private exchange of letters now available (Drèze 1987, appendix A to ch. 2)). The d.m.'s subjective state space will not be a set of subsets of a larger ('finer') space, as it is in FS (p.84). For, from our point of view, there are no distinctions between states which are 'neglected', but only distinctions of which the d.m. is not aware. The subjective states are those which he can distinguish on the basis of the relevant aspects (facts, propositions) of which he is aware, and are perceived and represented as singletons. However, our d.m. suspects he may have left some 'relevant aspects undescribed'. In fact, the d.m. who assumes that his state space is the objective space is essentially covered by existing theory (if he does so wrongly, he will just be surprised of being wrong when unexpected events materialize). At any rate the point is important for the present approach. To illustrate it, we quote Savage once more. On the consequences he had described to Aumann in the aforementioned letter he writes:

"Of course, they are not ultimate. [...] An ultimate analysis might seem desirable, but probably it does not exist and certainly threatens to be cumbersome."

Our shifting attention to consequences was on purpose. Indeed, the above quotation also brings out an essential symmetry between the space S of states of the world and the space X of states of the person (d.m.): as far as 'refinement' is concerned, the d.m. is in the same position with respect to both, and what was said before on subjective state space also applies to subjective consequence space.

Now we turn to the domain of the d.m.'s preference relation. Although existing separating theories are intended to separate beliefs about S from intrinsic preferences on X, in the model $(X^S,\geq)$ the domain of $\geq$ is originally just X^S, a set which formally does not include X. X^S contains the subset X^* of constant acts, with elements x^* assigning to each $s\in S$ the same consequence $x\in X$ (i.e. $x^*(s)=x$, $s\in S$); and preferences on X are usually superimposed on the model by a definitory condition of agreement with preferences on X^*: $x\geq y$ iff $x^*\geq y^*$, $x,y\in X$ (we have kept the same notation $\geq$ as customary). This does not specify a model $(X^S\cup X, \geq)$ with $\geq$ defined on $X^S\cup X$, for there are many transitive extensions to $X^S\cup X$ of the original preference relation on X^S which satisfy the above agreement condition. And on the other hand, the formal conclusions of the theories are about the model $(X^S, \geq)$ which they study, as they should be (in particular, they do not contain assertions about preferences on X). However, interpreting these as theories which enable to single out an element of preference over consequences in X is tantamount to imposing the stronger condition:

1.1 Identification. $x^* \cong x$, $x\in X$,

where $\cong$ denotes indifference. And this does determine a unique transitive extension of $\geq$ to $X^S\cup X$. Therefore, existing separating theories on $(X^S, \geq)$ are effectively theories on $(X^S\cup X, \geq)$ which include the above identification axiom. We stress in passing that explicit assessment of preferences on X (objective or subjective) is part of the definition of the model $(X^S\cup X, \geq)$. Indeed, X representing the set of states of the d.m. in the decision situation represented by the model, we might say that preferences on X transform the machine with state space X -an object- into a subject of decision.

After this lenghty preamble, we can now state the idea which generates the analysis of the following sections, namely that it is precisely the identification axiom 1.1 above that confers to the space S its objective character and reflects it. In other words, we contend that interpretation of the model as objective is appropriate if and only if that axiom is imposed. We shall then introduce separating theories on $(X^S \cup X, \geq)$ which do not include the identification axiom, and interpret the resulting models as subjective (regardless of the objective or subjective character of X).

To motivate this, we recall that in the present interpretation of an isolated 'small world' decision situation the d.m. refines his spaces S and X, and stops being conscious of their subjective character. He then assesses his preferences on what he has, namely $X^S \cup X$. Maps from S to X represent different courses of action as the d.m. perceives them, and he knows this. Therefore, we argue, the d.m. will not regard an act which 'looks like' x^* and x itself as the same object. In particular, then, $x^* \gneq x$ may hold for some $x \in X$.

How we replace the identification axiom (while maintaining the agreement condition $x^* \geq y^*$ iff $x \geq y$, $x, y \in X$) is described and discussed in the next section and stated in the abstract; here we complete the description of the model $(X^S \cup X, \geq)$. Under the interpretation of X as set of states of the d.m., the actual objects of choice in a decision situation are different courses of action in X^S, as in $(X^S, \geq)$. And as far as comparisons between elements of X^S are concerned, existing theories apply equally to subjective and objective models. So we embed existing theories on the restriction of

$\geq$ to X^S into the model $(X^S \cup X, \geq)$, the bridge between X^S and X being given by the axioms replacing identification of X^* and X. The result of this is summarised in the abstract.

In the subsequent section 3 model, axioms and results are strenghtened. We leave the presentation of the latter two for that section, and pass to the former. The extra requirement on the d.m. is that his preferences be defined on all $\Delta(X)$, the set of simple probability measures on X, the latter being identified with the set of degenerate measures δ_x, $x \in X$ in $\Delta(X)$ by the axiom: $\delta_x \cong x$, $x \in X$. So we have a model $(X^S \cup X \cup \Delta(X), \geq)$ which is effectively a model $(X^S \cup \Delta(X), \geq)$ in which the elements of $\Delta(X)$ may be interpreted as being generated by some objective device. In the latter form, the identification axiom characterizing objectivity of the model is: $x^* \cong \delta_x$, $x \in X$. We again replace this, with strenghtened counterparts of the axioms on $(X^S \cup X, \geq)$ which lead to a stronger type of results.

2. Theories on $(X^S \cup X, \geq)$

In the model of this section the d.m.'s preferences are defined on $X^S \cup X$. hr two axioms replacing identification 1.1 of X^* and X will reflect subjectivity of the model. By adjoining to these axioms a theory on the restriction of $\geq$ to X^S one then obtains representation theorems for $\geq$ on $X^S \cup X$. Here this is done with a theory of Wakker (1989a), which includes as a special case expected (continuous) utility theory.

The continuity aspect should be emphasized. The axioms bridging X^S and X below are designed for a model in which X is 'rich' enough, as will be clear. In the topological approach of Wakker X is a connected space.

First, a basic representability condition on $\geq$ must be imposed.

2.1 Representable Weak Order (RWO). There exists a function $V: X^S \cup X \to \mathbb{R}$ such that (i) $V(\xi) \geq V(\zeta)$ iff $\xi \geq \zeta$, $\xi, \zeta \in X^S \cup X$, i.e. V represents $\geq$; and (ii) counterimages of $\{\xi \mid \xi \geq \zeta\}$ and $\{\xi \mid \zeta \geq \xi\}$ under V are closed in $\mathbb{R}$.

The technical condition equivalent to RWO, that $X^S \cup X$ be separable in the order topology generated by the simple order on indifference classes associated with $\geq$, may be found in Debreu (1954) or Krantz, Luce, Suppes and Tversky (1971, p.40).

Our two 'small world' axioms, as anticipated, are:

2.2 Agreement. $x^* \geq y^*$ iff $x \geq y$, $x, y \in X$.

2.3 Revealed Limited Intelligence (RLI).

$$x \begin{smallmatrix} \geq \\ \leq \end{smallmatrix} x^* \text{ and } y \begin{smallmatrix} > \\ < \end{smallmatrix} x \Rightarrow y \begin{smallmatrix} > \\ < \end{smallmatrix} y^*, \quad x, y \in X.$$

Under the present interpretation, x^* is constant only up to subjective approximation; one might think of it as x plus unperceived 'suspected' variability. The former axiom says that at least the corresponding rankings on X and X^* agree. The latter says that if x is good enough to be better than x^*, and y is even better than x, then y will be surely better than y^*; and that if x is so bad that x^* is better, and y is even worse, then y will surely be worse than y^*.

Notice that under RLI 2.3: $x \cong x^*$ and $y \cong y^* \Rightarrow x \cong y$. There can be at most one indifference class of consequences indifferent to their corresponding constant acts. Therefore no non-trivial relation $\geq$ on $X^S \cup X$ satisfying 2.3 can satisfy the 'grand world' objectivity axiom 1.1, and vice versa. It is because of this incompatibility that from the present point of view individual decision situations characterized by agreement 2.2 and RLI 2.3 do 'not quite' recapitulate the grand one (cfr. section 1): in our subjective models the identification axiom 1.1 of the objective model is never satisfied.

As for the name 'Revealed Limited Intelligence', intelligence is in the sense of Modica and Schmeidler (1991), who briefly discuss its aspects of ability to reason, computational power, information, and understanding of interactions. In the present interpretation, construction of an objective model requires unlimited intelligence. By conforming to RLI 2.3 the d.m. reveals not to have an objective model, hence limited intelligence.

We turn to the continuity aspect mentioned before. RLI 2.3 implies that there is at most one indifference class of consequences indifferent to their constant-act counterparts, but it does not imply that there is one at all. First, RLI 2.3 (together with RWO 2.1 and agreement 2.2) may be satisfied by preference relations such that $x > x^* \; \forall x \in X$, or $x < x^* \; \forall x \in X$, in which in a sense all the consequences the d.m. thinks of are good, or all are bad. These cases exist formally, but seem void of interest. More interesting is when there exist good and bad enough consequences $x_g \geq x_g^*$ and $x_b \leq x_b^*$ (by non-triviality of $\geq$ -which we always assume- and RLI 2.3, one must be strict). It is in this case that the topological assumption of connectedness of X plays its role: 'richness' of X is needed to ensure existence of an

$\bar{x}\in X$ such that $\bar{x} \cong \bar{x}^*$.

It is convenient to have some notation at this point. We let $F:=X^S$, the set of acts. With subscript 'o' for 'ordinary',

$$X_o := \{ x\in X \mid x \cong f \text{ for some } f\in F \}$$

$$F_o := \{ f\in F \mid f(s)\in X_o \text{ for all } s\in S \} .$$

Notice that under identification 1.1, $X_o=X$ (and $F_o=F$). If $\psi:A\to B$ and $A'\subseteq A$, the restriction of ψ to A' is denoted by $\psi\big|_{A'}$ as usual.

2.4 Lemma. Let $V: F\cup X \to \mathbb{R}$ represent $\geq$, $v:= V\big|_X$, $\mathcal{U}:= V\big|_F$ and $u: X\to\mathbb{R}$ be defined by $u(x)=\mathcal{U}(x^*)$. Let V' be another representation of $\geq$ and v', $\mathcal{U}'$, u' analogously defined. If $\mathcal{U}' = \phi\circ\mathcal{U}$ for some $\phi:\mathbb{R}\to\mathbb{R}$, then $v'\big|_{X_o} = (\phi\circ v)\big|_{X_o}$.

Proof. Letting $x\in X_o$ and $f_x\in F$ such that $f_x\cong x$, one has

$$v'(x) = \mathcal{U}'(f_x) = \phi(\mathcal{U}(f_x)) = \phi(v(x)) .$$

A simple consequence of axioms 2.1-2.3 is this:

2.5 Lemma. Assume RWO 2.1, Agreement 2.2 and RLI 2.3. Let V, v, $\mathcal{U}$, u be as in Lemma 2.4, and suppose that there exists $\bar{x}\in X$ such that $\bar{x} \cong \bar{x}^*$. Then there exists a function $\alpha_{(\mathcal{U},v)} : X \to (0,1)$ such that

$$u(x) - u(\bar{x}) = \alpha_{(\mathcal{U},v)}(x) \ (v(x) - v(\bar{x})) .$$

If, furthermore, $\mathcal{U}$ is cardinal -that is, unique up to positive linear transformations-, then $\alpha_{(\mathcal{U},v)}\big|_{X_o}$ is independent of $(\mathcal{U},v)$.

Proof. For the first assertion: by RLI 2.3,

$$x \underset{<}{\overset{>}{\cong}} \bar{x} \quad \text{iff} \quad x \underset{<}{\overset{>}{\cong}} x^*$$

so

$$V(x) \underset{<}{\overset{>}{=}} V(\bar{x}) \quad \text{iff} \quad V(x) \underset{<}{\overset{>}{=}} V(x^*)$$

i.e.

$$v(x) \underset{<}{\overset{>}{=}} v(\bar{x}) \quad \text{iff} \quad v(x) \underset{<}{\overset{>}{=}} u(x)$$

Also, $\bar{x}^* \cong \bar{x}$ means $u(\bar{x}) = v(\bar{x})$ so the last relation holds iff

$$v(x) - v(\bar{x}) \underset{<}{\overset{>}{=}} u(x) - u(\bar{x}).$$

The assertion now follows from the fact that, by agreement 2.2, one has $(v(x)-v(\bar{x}))(u(x)-u(\bar{x}))\geqq 0$, with equality iff both factors are zero.

The second assertion follows from the first and Lemma 2.4 (with ϕ affine). ∎

2.6 Remarks. (i) the function α depends only on indifference classes, i.e. $\alpha_{(\mathcal{U},v)}(x) = \alpha_{(\mathcal{U},v)}(y)$ iff $x \cong y$, by agreement 2.2.

(ii) $\alpha_{(\mathcal{U},v)}(\bar{x})$ may be defined arbitrarily.

Given $\bar{x} \cong \bar{x}^*$, the function α of Lemma 2.5 -or rather $(1-\alpha)$- is a measure of the difference the d.m. perceives between x (better or worse than $\bar{x}$) and x^*: the larger this difference, the smaller the value of $\alpha(x)$. In this sense, the function α includes an element of 'trust' in one's model, for the more accurate the d.m. thinks his model is, the smaller is the perceived difference between x and x^*, hence the closer to 1 is the function α.

With α close to 1, subjective models approach objectivity. Indeed, with the identification 1.1 of objective models in place of agreement 2.1 and RLI 2.2, $V|_X = u$, so α would be identically 1.

After comparisons between elements of F with elements of X, it is left to consider the restriction of $\geq$ to F -denoted by $\geq_F$-, i.e. preferences among acts. By adding a set of axioms on $\geq_F$ to 2.1-2.3, one then obtains a theory on $\geq$. We shall illustrate this with a theory of Wakker (1989a), as anticipated. The structural assumption is the following:

2.7 Structure. S (is finite and) contains at least two 'essential' states. X is a connected topological space.

Finiteness of S is in parenthesis because it is assumed here for ease of exposition. Wakker (1989b) has extended the theory to infinite S, by relaxing the assumption that the set of acts F is all of X^S and imposing on $\geq_F$, loosely speaking, the restriction that exceptionally good or bad consequences do not matter too much. 'Essential' roughly means not deemed impossible, and the presence of at least two essential states rules out decisior situations in which there is effectively no uncertainty (for details see Wakker 1989b). Technically, existence of two essential states makes it possible to dispense with the assumption of separability of X which is needed otherwise (see Wakker 1989a, sec.10 and 1989b, rem.48). From our point of view, it is important because it ensures cardinality of the representation of $\geq_F$ (Wakker 1989a, (5.1)), which will imply uniqueness of the function α in the theorem of this section. Connectedness of X, as already mentioned, is essential. Under structure 2.7, Wakker's theory on $\geq_F$ is equivalent to the following:

2.8 Continuous Choquet Expected Utility (CCEU). There exist a unique capacity μ on S and a cardinal, continuous function u: $X \rightarrow \mathbb{R}$ such that $\geq_F$ is represented by the function $\mathcal{U}$: $F \rightarrow \mathbb{R}$ given by

$$\mathcal{U}(f) = \int_X u(x)\ d(\mu \circ f^{-1})(x)\ ,\quad f \in F\ .$$

A capacity on S is a real-valued set function on the subsets of S such that $\mu(\emptyset)=0$, $\mu(S)=1$ and $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$. The integral is Choquet integral. In particular, if μ is additive ($A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$), the integral reduces to Lebesgue integral and the theory to (continuous) expected utility theory. Details are of course in Wakker.

As anticipated, under RLI 2.3 the following is a regularity condition:

2.9 Regularity. There exist x_g, $x_b \in X$ such that $x_g \geq x_g^*$ and $x_b \leq x_b^*$.

We will comment on dropping it later. Using RWO 2.1(ii) and the fact that the range of u (as continuous image of a connected set) is an interval, it is routine to show that Regularity 2.9 implies existence of an $\bar{x} \in X$ such that $\bar{x} \cong \bar{x}^*$.

One more piece of notation is that brackets will always enclose indifference classes, so for $x \in X$, $[x] := \{ y \in X \mid y \cong x \}$. Then the result of joining CCEU 2.8 with our subjectivity axioms 2.2 and 2.3 is the following:

2.10 Theorem. Let $\geq$ be a non-trivial binary relation on $F \cup X = X^S \cup X$, with structure 2.7.

Assume that $\geq$ satisfies Representable Weak Order 2.1, Agreement 2.2, Revealed Limited Intelligence 2.3 and Regularity 2.9; and that $\geq_F$ satisfies Continuous Choquet Expected Utility 2.8.

Then there exist, unique, a capacity μ on S, a function $\alpha: X_o \to (0,1)$ and an indifference class $[\bar{x}]$; and a cardinal function $v: X \to \mathbb{R}$, such that any function $V: F \cup X \to \mathbb{R}$ representing $\geq$ is with

$$V = \begin{cases} v(x) , & x \in X \\ v(\bar{x}) + \int_X \alpha(x) \; (v(x) - v(\bar{x})) \; d(\mu \circ f^{-1})(x) , & f \in F_o \end{cases} .$$

This theorem is a direct consequence of CCEU 2.8 and Lemma 2.5: write $u(x) = u(\bar{x}) + (u(x)-u(\bar{x}))$ in 2.8 (with u as in Lemma 2.4) and apply 2.5 to the second term, recalling that by construction $u(\bar{x})=v(\bar{x})$. Details are straightforward, and are omitted. On the other hand, some comments on the result are in order.

First of all, notice that the integral expression for V in the above theorem holds on F_o, not on all of F. The reason is that only on X_o uniqueness of the function α is guaranteed (see Lemma 2.5). Under the continuity assumptions imposed in this section, outside of X_o are only those consequences x such that $x > f$ for all $f \in F$, or $x < f$ for all $f \in F$.

As we have seen before, increased accuracy of the subjective model reduces the perceived difference between any $x \in X$ and its corresponding constant act $x^* \in F$, so that loosely speaking X_o 'approaches' X with α getting close to 1. This is reflected in the representation in the theorem: with the subjective model approaching objectivity, this representation approaches the objective-model CCEU representation, to which it reduces if we set $\alpha\equiv1$ and $F_o=F$ (as it is under identification 1.1).

Something about the function α was said after remarks 2.6, and more will be said in the next section. The other 'new' element in the theorem is the consequence $\bar{x}$ such that $\bar{x} \cong \bar{x}^*$. The utility of $\bar{x}$ may be interpreted as the utility of the unperceived residual variability of the consequences of the various courses of action. For, thinking of x^* as x plus residual variability, it follows from RLI 2.3 that this variability leaves the ranking of x unchanged if and only if $x \cong \bar{x}$. Thus we might say that the higher (resp. lower) the ranking of the $\bar{x} \cong \bar{x}^*$ in the preference relation $\geq$, the more optimistic (resp. pessimistic) is the d.m.'s attitude towards 'the unknown'.

Lastly, we come to regularity 2.9. If it does not hold, (by non-triviality) either $x > x^*\ \forall\ x\in X$ or $x < x^*\ \forall\ x\in X$, so the set F_o may be empty in the first place. On the other hand, there can be no $\bar{x} \cong \bar{x}^*$, so as far as RLI 2.3 is concerned, richness of X is not needed. As we said, it is felt that these cases lack interest. We just mention that taking the case $x > x^*$, $x\in X$ for example, the analogous of Lemma 2.5 and Theorem 2.10 hold with $u(\bar{x})$ and $v(\bar{x})$ replaced by

$$\underline{u} := \inf \{ u(x) \mid x\in X \} ,$$

if the right hand side is finite (but bearing in mind that the function u is cardinal, $\underline{u} = -\infty$ and $x>x^*$ all $x\in X$ together do not seem to make much sense). Analogous statements hold for the case $x< x^*$, $x\in X$.

3. Theories on $(X^S \cup \Delta(X), \geq)$

In this section, as in the original model of Savage, no richness assumptions on X are made. On the other hand, as mentioned in the introduction, the extra requirement is imposed that the d.m. have preferences defined also on the set $\Delta(X)$ of simple lotteries on X, supposed to be generated by some 'objective' device, like an urn. This objectivity induces identification of X with the set of degenerate distributions $\delta_x \in \Delta(X)$, $x \in X$, so it is assumed:

3.1. $\delta_x \cong x$, $x \in X$.

In this way we obtain a model $(F \cup X \cup \Delta(X), \geq)$ with $\geq$ on $F \cup X \cup \Delta(X)$ satisfying 3.1, so effectively a model $(F \cup \Delta(X), \geq)$ with $\geq$ on $F \cup \Delta(X)$. This, in the latter form, is the model of the present section. F, the set of acts, will not always stand for all of X^S, as is specified in the sequel.

In the model $(F \cup \Delta(X), \geq)$, objectivity of the state space S is characterized by identification of X^* and the set of degenerate distributions on X, i.e. by the axiom:

3.2 Identification. $x^* \cong \delta_x$, $x \in X$.

This plays the role of identification 1.1 in the present context. We again will replace this (in another version), and interpret the resulting models as subjective.

First representability, as in the previous section.

3.3 Representable Weak Order (RWO). There exists a function $V: F \cup \Delta(X) \to \mathbb{R}$ such that (i) $V(\xi) \geqq V(\zeta)$ iff $\xi \geq \zeta$, $\xi, \zeta \in F \cup \Delta(X)$, i.e. V represents $\geq$; and

(ii) counterimages of $\{\xi \mid \xi \ge \zeta\}$ and $\{\xi \mid \zeta \ge \xi\}$ under V are closed in $\mathbb{R}$.

In the context of this section, agreement 2.2 and RLI 2.3 would be written with $\delta_x, \delta_y \in \Delta(X)$ in place of $x,y \in X$, respectively as:

3.4 $x^* \ge y^*$ iff $\delta_x \ge \delta_y$, $x,y \in X$

and

3.5 For all $x,y \in X$

$$\delta_x \underset{\le}{\ge} x^* \quad \text{and} \quad \delta_y \underset{<}{>} \delta_x \quad \Rightarrow \quad \delta_y \underset{<}{>} y^* .$$

But stronger versions will be used in this setting. With no richness conditions on X, there need not exist an $x \in X$ such that $\delta_x \cong x^*$. So 3.5 must be strenghtened to ensure existence of something analogous to the consequence $\bar{x} \cong \bar{x}^*$ of section 2 (it will be a lottery $\bar{p} \in \Delta(X)$). As for the agreement condition 3.4, it could be left as it is. By extending it in a 'natural' way, we will present a result which is much sharper than theorem 2.10 of the previous section (no restriction to F_o and the function α equal to a constant).

As in section 2, we shall embed in $(F \cup \Delta(X), \ge)$ a theory on $\ge_F$ (Sarin and Wakker 1990) which includes the theory of Savage as a special case. However, doing this directly would require some notation and definitions which are not needed in order to present the central features of the result and of the subjectivity axioms which lead to it. We choose to do the latter first by dealing, in the main body of the section, with the special case where the theory on $\ge_F$ is exactly the theory of Savage. Axioms and result for the more general case, with the necessary measure-theoretic preliminaries, will be stated at the end of the section. The extension will be completely straightforward.

Some notation is still needed. First, $F=X^S$ if X is finite. If not, F denotes the subset of X^S consisting of all finite-outcome acts. This restriction, to which a measurability condition is added in the more general case, is maintained throughout. Let A be a set and ψ: A $\to \mathbb{R}$ a bounded function on A; let $\mathcal{P}(A)$ be the set of capacities on the (measurable where appropriate) subsets of A. Then $\hat{\psi} : \mathcal{P}(A) \to \mathbb{R}$ will denote the Choquet-integral of ψ with respect to $\nu \in \mathcal{P}(A)$, the latter taken as variable:

$$\hat{\psi}(\nu) := \int_A \psi \, d\nu \ , \quad \nu \in \mathcal{P}(A) \ .$$

We recall that the additive probability measures are special capacities. If ν is additive, $\hat{\psi}(\nu)$ is simply the expectation of ψ with respect to ν.

To begin theory, we start this time with $\geq_F$, on which we assume, neglecting redundancies with RWO 3.3, Savage P1-P6 (dealing with step acts P7 is not needed). In the notation just introduced, this is equivalent to:

3.6 Savage Theory. There exist a unique additive, convex-ranged probability measure μ on S and a cardinal, bounded function $u : X \to \mathbb{R}$ such that $\geq_F$ is represented by the function $\mathcal{U} : F \to \mathbb{R}$ given by

$$\mathcal{U}(f) = \hat{u} \, (\mu \circ f^{-1}) \ , \quad f \in F.$$

Notice that with our definition of F, $\mu \circ f^{-1} \in \Delta(X)$ for all $f \in F$. We mention in passing that in Wakker (1989b) this is extended to unbounded -but integrable- u.

We turn to comparisons between elements of F and elements of $\Delta(X)$. As we have seen, according to the present interpretation in subjective models $(F\cup\Delta(X), \geq)$ the d.m. does not regard x^* and δ_x as the same object. With μ as in 3.6, he will more generally not regard an act $f\in F$ and the lottery $\mu\circ f^{-1}\in\Delta(X)$ which it generates as the same object -in analogy to section 2, we might think of f as $\mu\circ f^{-1}$ plus residual variability. So under 3.6 the objectivity of the model is characterized by the identification:

3.7 With μ from 3.6, $f \cong \mu\circ f^{-1}$, $f\in F$,

which extends 3.2 above. This we have to replace. Since μ is convex-ranged, one has

$$(3.8) \quad \{ \mu\circ f^{-1} \in\Delta(X) \mid f \in F \} = \Delta(X) ,$$

so Savage theory reduces $\geq_F$ to a preference relation on $\Delta(X)$, which will be denoted by Π. Therefore to replace identification 3.7 and obtain what we interpret as subjective models, we shall have to specify the relationship between the Π induced on $\Delta(X)$ by $\geq_F$ and the restriction of $\geq$ to $\Delta(X)$, $\geq_{\Delta(X)}$. Paralleling section 2, this will be done by means of two axioms, one of agreement and one of revealed limited intelligence, which will be stated in turn.

So, with u and μ as in 3.6, define Π to be the von Neumann-Morgensten preference relation on $\Delta(X)$ induced by u (more precisely by the family $a + bu$, $a\in\mathbb{R}$, $b>0$), that is,

$$p \,\Pi\, q \quad \text{iff} \quad \hat{u}(p) \geqq \hat{u}(q) , \quad p,q\in\Delta(X).$$

From 3.6 and this definition one then has

(3.9) $f \geq g$ iff $\mu \circ f^{-1} \sqcap \mu \circ g^{-1}$, $f,g \in F$.

Coming to our first small world axiom, suppose that agreement condition 3.4 above were to hold. Then (3.9) would imply

$$\delta_x \sqcap \delta_y \quad \text{iff} \quad x^* \geq y^* \quad \text{iff} \quad \delta_x \geq \delta_y, \qquad x,y \in X,$$

so that $\sqcap$ and $\geq_{\Delta(X)}$ would agree on degenerate distributions. We shall assume the following extended version of 3.4:

3.10 Agreement. With μ from Savage Theory 3.6 and $f,g \in F$:

$$f \geq g \quad \text{iff} \quad \mu \circ f^{-1} \geq \mu \circ g^{-1}.$$

This imples that $\sqcap$ and $\geq_{\Delta(X)}$ agree on all of $\Delta(X)$ (from (3.8) and (3.9)). Therefore, by cardinality, they are represented by the same family $a + b\,\hat{u}$.

We remark that this implication of agreement 3.10 is what yields a constant function α in the theorems of this section. By sticking to 3.4 one would obtain a non-constant α, as in section 2.

3.11 Note. The technical problem in dealing with non-simple probability measures on infinite X is that the analogous of (3.8) may not hold with $\Delta(X)$ replaced by a larger set. This is due to a problem of finite versus countable additivity of the measures involved (cfr. Sarin and Wakker 1990 p.25 for an example). Therefore (3.9) and agreement 3.10 would not necessarily imply that the extensions of $\sqcap$ and $\geq_{\Delta(X)}$ to a superset of $\Delta(X)$ agree on the latter. So our conclusions would not hold without further restrictions on S. Obviously the previous remark on 3.4 and non-constant α

still applies.

To sum up, what we have so far on $(F \cup \Delta(X), \geq)$ is the following:

3.12 Lemma. Assume Savage Theory 3.6 on $\geq_F$, and RWO 3.3 and Agreement 3.10 on $\geq$. Then, with u and μ from 3.6, $\geq_{\Delta(X)}$ is represented by the family $a + b\,\hat{u}$. So let $V: F \cup \Delta(X) \rightarrow \mathbb{R}$ represent $\geq$, with $V(p) = \hat{v}(p)$, $p \in \Delta(X)$ representing $\geq_{\Delta(X)}$. Then for some $a \in \mathbb{R}$, $b>0$ one has

$$V(f) = a + b\,\hat{v}(\mu \circ f^{-1}) \;, \quad f \in F.$$

Next we state our axiom of revealed limited intelligence, which extends condition 3.5. The extension is in the same spirit as that of 3.4 to agreement 3.10:

3.13 Revealed Limited Intelligence (RLI). With μ from Savage Theory 3.6 and $f,g \in F$:

$$\mu \circ g^{-1} \begin{matrix} \geq \\ \\ \leq \end{matrix} g \quad \text{and} \quad f \begin{matrix} > \\ \\ < \end{matrix} g \;\rightarrow\; \mu \circ f^{-1} \begin{matrix} > \\ \\ < \end{matrix} f \;.$$

This has the same interpretation as RLI 2.3. In the model $(F \cup \Delta(X), \geq)$, $g \in F$ is perceived as $\mu \circ g^{-1} \in \Delta(X)$ with subjective approximation. The actual lottery $\mu \circ g^{-1}$ is something different which, being free from residual variability, if good enough is better than g. So if $\mu \circ f^{-1} > \mu \circ g^{-1}$ -which by agreement 3.10 is equivalent to $f > g$-, then 'a fortiori' $\mu \circ f^{-1} > f$.

At this point, as in section 2, we want to isolate pathological cases. In the present context, these take the form: $\delta_x > x^* \; \forall x \in X$, or $\delta_x < x^* \; \forall x \in X$. By non-triviality again, neither of them occurs if and only if the following analogue of regularity 2.9 holds:

3.14 Regularity. There exist $x_g, x_b \in X$ such that $\delta_{x_g} \geq x_g^*$ and $\delta_{x_b} \leq x_b^*$.

With X being 'enriched' to $\Delta(X)$, the above axiom will imply existence of a $\bar{p} \in \Delta(X)$ such that $f \cong \mu \circ f^{-1}$ if and only if $\mu \circ f^{-1} \cong \bar{p}$, $f \in F$, which is the counterpart of the consequence $\bar{x} \cong \bar{x}^*$ of the previous section. Again, something can be said about pathological cases. We discuss them in the Appendix for the sake of completeness. Under regularity, the result of embedding Savage Theory 3.6 in the model $(F \cup \Delta(X), \geq)$ with $\geq$ satisfying agreement 3.10 and RLI 3.13 is the following:

3.15 Theorem. The following statements are equivalent for the non-trivial relation $\geq$ on $F \cup \Delta(X)$:

(i) $\geq_F$ satisfies Savage Theory 3.6, and $\geq$ satisfies Representable Weak Order 3.3, Agreement 3.10, Revealed Limited Intelligence 3.13, and Regularity 3.14.

(ii) there exist, unique, an additive, convex-ranged probability measure μ on S, a number $\alpha \in (0,1)$ and an indifference class $[\bar{p}]$, $\bar{p} \in \Delta(X)$; and a cardinal, bounded function $v: X \to \mathbb{R}$, such that $\geq$ is represented by the function $V: F \cup \Delta(X) \to \mathbb{R}$ given by:

$$V = \begin{cases} \hat{v}(p) & , \quad p \in \Delta(X) \\ \alpha\, \hat{v}(\mu \circ f^{-1}) + (1-\alpha)\, \hat{v}(\bar{p}) & , \quad f \in F . \end{cases}$$

Proof. Given that the representation in 3.15(ii) holds on all of

$F\cup\Delta(X)$, 'suffficiency' (ii)$\Rightarrow$(i) is routine verification. We prove (i)$\Rightarrow$(ii). From Lemma 3.12 we already have the unique μ, additive and convex-ranged, and the cardinal, bounded $v\colon X\to \mathbb{R}$ such that any $V\colon F\cup\Delta(X)\to\mathbb{R}$ representing $\succeq$ is of the form

$$V = \begin{cases} \hat{v}(p) , & p\in\Delta(X) \\ a_v + b_v\, \hat{v}(\mu\circ f^{-1}) , & f\in F \end{cases}$$

for some $a_v\in\mathbb{R}$, $b_v>0$. We shall show that $b_v\in(0,1)$ and is independent of v; and that $a_v=(1-b_v)\hat{v}(\bar{p})$ for the $\bar{p}$ defined below. Then the assertion follows by setting $\alpha=b_v$.

First we show existence of an $\bar{f}\in F$ such that $\bar{f}\cong\mu\circ\bar{f}^{-1}$. By RLI 3.13,

$$\inf\{\, V(f) \mid \mu\circ f^{-1}\succeq f \,\} \geqq \sup\{\, V(f) \mid f\succeq\mu\circ f^{-1} \,\}.$$

The range of $V|_F$ being an interval, if strict inequality were true there would be an $f\in F$ not comparable to $\mu\circ f^{-1}\in\Delta(X)$, violating completeness of $\succeq$. So equality holds. Let $\bar{f}\in F$ be such that $V(\bar{f})$ is equal to this number. Then by order continuity RWO 3.3(ii), $V(\mu\circ\bar{f}^{-1}) = V(\bar{f})$, i.e. $\bar{f}\cong\mu\circ\bar{f}^{-1}$.

Define $\bar{p} := \mu\circ\bar{f}^{-1}\in\Delta(X)$.

We now show that $a_v = (1-b_v)\hat{v}(\bar{p})$ with $b_v\in(0,1)$. From $V(\bar{f})=V(\bar{p})$, which by defin'tion means $a_v + b_v\hat{v}(\bar{p}) = \hat{v}(\bar{p})$, we get $a_v = (1-b_v)\hat{v}(\bar{p})$. Also, for any $f \ncong \bar{f}$,

$$\frac{V(f) - V(\bar{f})}{V(\mu\circ f^{-1}) - V(\bar{f})} = \frac{b_v(\ \hat{v}(\mu\circ f^{-1})-\hat{v}(\bar{p})\ )}{\hat{v}(\mu\circ f^{-1}) - \hat{v}(\bar{p})} = b_v ,$$

and the leftmost term above is in (0,1) by RLI 3.13.

It is left to show that for any positive linear transformation v' of v

and representation V' obtained as above from v', it is $b_{V'} = b_V$. Take a $g \in F$ such that $g > \bar{f}$ (if there is no such g, by regularity 3.14 there must be one such that $g < \bar{f}$, and we may work analogously with that). Recalling that both $V'(\Delta(X))$ and $V'(F)$ are non-degenerate intervals, we can take g such that $g \cong \mu \circ f^{-1}$ for some $f \in F$ (by RLI 3.13 and agreement 3.10, this f will be such that $g > f > \bar{f}$). Therefore

$$b_{V'} = \frac{V'(f) - V'(\bar{f})}{V'(\mu \circ f^{-1}) - V'(\bar{f})} = \frac{V'(f) - V'(\bar{f})}{V'(g) - V'(\bar{f})} = \frac{\hat{V}'(\mu \circ f^{-1}) - \hat{V}'(\bar{p})}{\hat{V}'(\mu \circ g^{-1}) - \hat{V}'(\bar{p})}$$

$$= \frac{\hat{V}(\mu \circ f^{-1}) - \hat{V}(\bar{p})}{\hat{V}(\mu \circ g^{-1}) - \hat{V}(\bar{p})} = \frac{V(f) - V(\bar{f})}{V(g) - V(\bar{f})} = \frac{V(f) - V(\bar{f})}{V(\mu \circ f^{-1}) - V(\bar{f})} = b_V \,. \blacksquare$$

The interpretation of this result is analogous to that of theorem 2.10. Here $f \in F$ is seen as $\mu \circ f^{-1} \in \Delta(X)$ plus unperceived variability. Then the indifference class $[\bar{p}]$ plays the role that $[\bar{x}]$ had in the previous section, reflecting pessimism/optimism. The number α, on the other hand, characterizes trust in one's model in a more clear-cut way than the function $\alpha(\cdot)$ of theorem 2.10, being independent of $x \in X$. About approaching objectivity, with α tending to 1, the same comments following 2.10 aplly here. Lastly, unlike in theorem 2.10, the function V representing $\geq$ has here the same form on all F. This is what allows to state also the 'sufficiency' implication (ii)$\Rightarrow$(i) in theorem 3.15.

We now extend this theorem by relaxing Savage theory 3.6 to a more general theory on $\geq_F$ (Sarin and Wakker 1990) including 3.6 as a special case. To do this we have to introduce what will be referred to as 'measurability structure'.

The space S is to be endowed with a σ-field Σ with respect to which all acts in F are measurable. So Σ is a family of subsets of S, closed under complements and countable unions, which includes S itself; and F will denote the set of (finite-outcome) $f \in X^S$ such that $f^{-1}(C) \in \Sigma$ for all $C \subseteq X$. The σ-field Σ is assumed to contain a ('rich enough') sub σ-field $\Sigma^{ua} \subseteq \Sigma$ of the so called unambiguous events. The set of unambiguous acts $F^{ua} \subseteq F$ is the set of acts in F which are measurable with respect to Σ^{ua}.

The theory of Sarin and Wakker imposes the axioms of Savage on the set F^{ua} of unambiguous acts, and an axiom of 'cumulative dominance' on preferences over ambiguous acts which extends Savage P4 to acts with more than two consequences. Savage theory results when $\Sigma^{ua} = \Sigma$. In the more general case of $\Sigma^{ua} \subseteq \Sigma$, the theory (without, again, P7) is equivalent to the following:

3.16 Sarin-Wakker (S-W). There exist a unique capacity μ on Σ, additive and convex-ranged on Σ^{ua}; and a cardinal, bounded function $u: X \to \mathbb{R}$, such that $\geq_F$ is represented by the function $\mathcal{U}: F \to \mathbb{R}$ given by

$$\mathcal{U}(f) = \hat{u}(\mu \circ f^{-1}) \ , \ f \in F.$$

To embed this in the model $(F \cup \Delta(X), \geq)$, the subjectivity axioms 3.10 and 3.13 have to be slightly modified. For the domain of $\geq$ includes the set $\Delta(X)$ of lotteries on X, but not the larger set of (non-necessarily additive) capacities on X. And $\mu \circ f^{-1} \in \Delta(X)$ if $f \in F^{ua}$, but not otherwise. Therefore, under S-W 3.16, the preference comparisons appearing in 3.10 and 3.13 are not well defined on all F, but only on F^{ua}. The necessary and sufficient modification of 3.10 and 3.13 in this context turns out to be just to restrict them to hold on F^{ua}. So the first is:

3.10^{ua} Agreement. With μ from S-W 3.16 and $f,g \in F^{ua}$:

$$f \geq g \quad \text{iff} \quad \mu \circ f^{-1} \geq \mu \circ g^{-1} .$$

Notice that with Π exactly as before, (3.9) restricted to F^{ua} and (3.8) still hold. Hence it is still true that Π and $\geq_{\Delta(X)}$ are represented by the same family $a+b\hat{u}$ on $\Delta(X)$. On the other hand, the same family represents $\geq_F$ on all F, by S-W 3.16. So, mutatis mutandis, the conclusion of Lemma 3.12 still holds as it is. Formally:

3.17 Lemma. The conclusion of Lemma 3.12 holds with S-W 3.16 replacing Savage Theory 3.6 and Agreement 3.10^{ua} replacing Agreement 3.10.

Finally, the RLI axiom 3.16 in the present context becomes:

3.13^{ua} Revealed Limi ed Intelligence (RLI). With μ from S-W 3.16 and $f,g \in F^{ua}$:

$$\mu \circ g^{-1} \begin{matrix} \geq \\ \leq \end{matrix} g \quad \text{and} \quad f \begin{matrix} > \\ < \end{matrix} g \quad \Rightarrow \quad \mu \circ f^{-1} \begin{matrix} > \\ < \end{matrix} f .$$

Then the extended version of theorem 3.15 is with 3.6 and μ from 3.6 replaced by 3.16 and μ from 3.16; and 3.10, 3.13 replaced by 3.10^{ua}, 3.13^{ua}:

3.18 Theorem. Let $\geq$ be a non-trivial relation on $F \cup \Delta(X)$, with measurability structure. Then the following statements are equivalent:

(i) $\geq_F$ satisfies S-W 3.16, and $\geq$ satisfies Representable Weak Order

3.3, Agreement 3.10^{ua}, Revealed Limited Intelligence 3.13^{ua}, and Regularity 3.14.

(ii) there exist, unique, a capacity μ on Σ, additive and convex-ranged on Σ^{ua}, a number $\alpha \in (0,1)$ and an indifference class $[\bar{p}]$, $\bar{p} \in \Delta(X)$; and a cardinal, bounded function $v: X \to \mathbb{R}$, such that $\geq$ is represented by the function $V: F \cup \Delta(X) \to \mathbb{R}$ given by:

$$V = \begin{cases} \hat{v}(p) \quad , \quad p \in \Delta(X) \\ \alpha\, \hat{v}(\mu \circ f^{-1}) + (1-\alpha)\, \hat{v}(\bar{p}) \quad , \quad f \in F \, . \end{cases}$$

With the aid of Lemma 3.17, the proof of this theorem follows the same lines of the proof of theorem 3.15 (working on F^{ua} instead of F), and will be omitted. A result for the cases in which regularity 3.14 fails in this context is contained in the Appendix.

APPENDIX (From section 3, when regularity 3.14 fails)

The pathological cases $\delta_x > x^* \; \forall \; x \in X$ and $\delta_x < x^* \; \forall \; x \in X$ in the model $(F \cup \Delta(X), \geq)$ are not quite the same as in the model $(F \cup X, \geq)$ of section 2. We shall add the assumption that the model is a restriction of a non-pathological model with a larger consequence space:

B.1 Extension. Suppose $\delta_x > x^*$ all $x \in X$ (resp. <). Then there exists a set X' with $X \subseteq X'$ and an extension of the model ($\geq$ plus axioms) to one with

X' replacing X, such that there exists $z \in X'$ with $\delta_z \leq z^*$ (resp. $\geq$).

B.2 Remark. By agreement 3.10 and RLI 3.13 one has $z<x$ all $x \in X$ (resp.>).

Under B1 the following holds:

B.3 Theorem. In theorem 3.15, replace regularity 3.14 with Extension B.1. Then the conclusions of the theorem remain valid, except existence of $[\bar{p}]$ and with $\hat{v}(\bar{p})$ replaced by some number $\bar{v}$ such that

$$\bar{v} < \inf \{ V(f) \mid f \in F \} \text{ if } \delta_x > x^* \ \forall \ x \in X,$$
$$\bar{v} > \sup \{ V(f) \mid f \in F \} \text{ if } \delta_x < x^* \ \forall \ x \in X.$$

Proof sketch. I consider the case of $\delta_x > x^* \ \forall \ x \in X$. Start as in the proof of theorem 3.15 to get V as it is there, fix $\hat{v}$ and drop subscripts. Since $\delta_x > x^*$ for some (in fact for all) $x \in X$, it cannot be b=1, a=0. Moreover, by B.1 it cannot be b=1 a≠0 either, for $\hat{v}$ and $a+b\hat{v}$ have to cross each other for some $\bar{v}$; so either fig.1 or 2 below must apply.

fig. 1

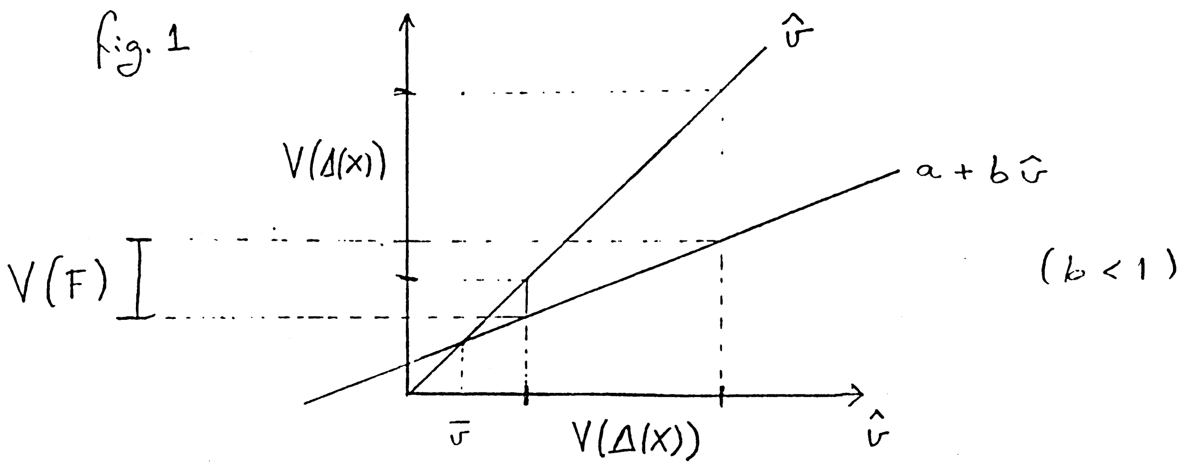
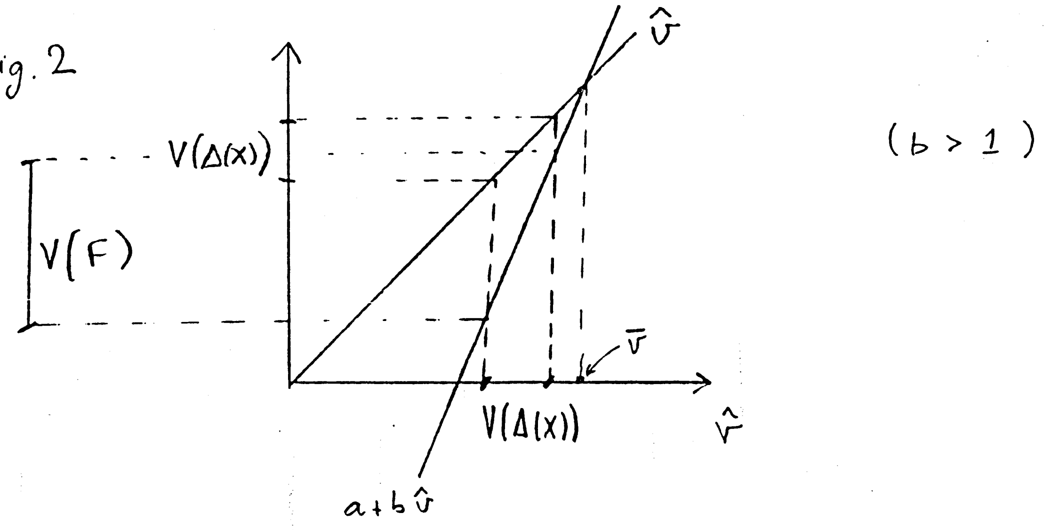


fig. 2



But the latter is excluded by RLI 3.13 (on X') and remark B.2. Conclusion: b<1, and $\bar{v} < \inf \{ V(f) \mid f \in F \}$. Since $\bar{v}=a+b\bar{v}$, $a=(1-b)\bar{v}$, so -reinstating subscripts- if this b_v is independent of v we can set $b_v=\alpha$ and have, as wanted,

$$V(f) = \alpha\, \hat{v}(\mu \circ f^{-1}) + (1-\alpha)\, \bar{v}\ , \quad f \in F.$$

It is left to show that b_v is independent of v. This follows as before, because $\bar{v} = \hat{v}(\bar{p}')$ for some $\bar{p}' \in \Delta(X')$ in the extended model.

We mention, without stating it, that the parallel analogous of theorem 3.18 also holds.

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