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ECONOMIC GROWTH WITH TECHNOLOGICAL UNCERTAINTY: EFFICIENT STATIONARY PLANS

by

Itzhak Zilcha*

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* Department of Economics, Tel–Aviv University

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THE FOERDER INSTITUTE FOR ECONOMIC RESEARCH
Faculty of Social Sciences
Tel–Aviv University, Ramat Aviv, Israel

Introduction

Our economy is a one–sector neoclassical growth model with stochastic production function; thus capital investment and consumption over time are random variables. This type of model has been studied extensively in the literature (see for example Brock and Mirman (1972), Majumdar and Zilcha (1987)). The theoretical investigation of such economies, which have been applied widely, had concentrated on properties of optimal plans, their long–run stability etc.; particularly, it is assumed that utility functions are known at the outset, hence characterizing **Pareto optimality** in such stochastic models has been a central issue (see Foldes (1989) for a comprehensive discussion).

The concept of *efficiency* in economics has not been tied to a particular given utility function. Thus, following Zilcha (1990a), we shall use stochastic ordering to define 'efficient plans.' However, in an intertemporal framework with a sequence of random states of nature there are several possible definitions of efficiency each yielding different economic consequences. In our approach we consider in each period t the information about the states of the environment up till that date and compare the (random) input–output possibilities for that period using either first or second degree stochastic dominance. This way, using conditional probability distributions, we obtain two different definitions of efficiency. We derive a simple characterization of efficient (type I) *stationary plans*: $E[\log r(\omega)] \geq \log(1+n)$, where the function $r(\omega)$ generates the *interest factors* stationary process and n is the rate of population growth.

Another issue studied in this paper considers the case where the utility function of the decision maker is *given*. We prove that the stochastic modified golden rule allocation is efficient (of type II). We also prove explicitly the existence of a stochastic modified golden–rule for any discounting factor $0 < \delta < 1$. This generalizes the result of Marimon

(1989), where the existence is proved for δ close to 1, in the *one–sector* case.

The notions of efficiency and Pareto optimality were studied also in the stochastic overlapping generations models. Pareto optimality has been characterized by Abel, Mankiw, Summers and Zeckhauser (1989), Aiyagari and Peled (1988), and Manuelli (1990b). Characterization of efficient stationary allocation in stochatic overlapping generations model has been attained by Zilcha (1990b).

Dynamic efficiency plays an important role in achieving certain results in infinite horizon economies. Thus the generalization of the well–known criteria of efficiency and Pareto optimality to the stochastic models is significant.

2. Notions and Preliminaries

Let $I = [\alpha,\beta]$ where $0 < \alpha < \beta < \infty$, and let μ be the Lebesgue measure on I. Define $\Omega = X_{k=-\infty}^{\infty} I_k$ where $I_k = I$ for all k. Denote by $\mathcal{F}$ the Borel sigma–field on Ω (i.e., $\mathcal{F}$ is the sigma–field generated by cylinder sets in Ω) and let σ be a given probability measure on $(\Omega, \mathcal{F})$. Let $\mathcal{F}_t$ be the sigma–field generated by all the cylinder sets $X_{k=-\infty}^{\infty} B_k$ where $B_k = I$ for all $k > t$. A particular $\omega \in \Omega$, $\omega = (\ldots,\omega_0, \omega_1, \omega_2,\ldots,\ldots)$, is a *possible state of the environment*; ω_t is referred to as the environment at date t; the sequence $(\ldots,\omega_0,\ldots,\omega_{t-1})$ is a particular history of the environment up to period t. The probability measure σ is referred to as the stochastic law of the environment. The fact that in each period t certain economic decisions must be made on the basis of information about the history of the environment up to that period (and not on the basis of the environment) is made precise formally by requiring that the relevant random variables be measurable with respect to the sigma–field $\mathcal{F}_t$.

$L_t^1(\Omega, \mathcal{F}_t, \sigma)$ is the set of all integrable functions $g(\omega)$ from Ω into R^1 which are $\mathcal{F}_t$–measurable. L_t^+ stands for the nonnegative functions in L_t. A sequence of functions $\{g_k(\omega)\}_{k=0}^{\infty}$ is an adapted integrable process if $g_k \in L_k$ for $k = 0,1,\ldots$.

Let E be the expectation operator. For $f \in L_t$ and $1 < k < t$, $E_k f(\omega) = E[f(\omega) \mid \mathcal{F}_{k-1}]$. We write $f \geq 0$ if $f(\omega) \geq 0$ a.s. and $f \neq 0$; $f > 0$ if $f(\omega) > 0$ almost surely.

Define a shift operator $T:\Omega \to \Omega$ by $(T\omega)_k = \omega_{k+1}$ for all k and $\omega \in \Omega$. Also for $f \in L_t$ let $Tf \in L_{t+1}$ be defined by $Tf(\omega) = f(T\omega)$ for all ω.

3. The Model

We shall use the well–known infinite horizon one–sector growth model with production uncertainty (see for example Brock and Mirman (1972)). Consumption and production take place at each date. The aggregate production function at date t is $F(K_t, L_t; \omega_t)$, where K_t, L_t are the aggregate capital and labor at period t, and ω_t is the environment at this date. It is assumed that $F(K_t, L_t; \omega_t)$ exhibits constant returns–to–scale for all ω_t. As was assumed in the last section, the stochastic process governing the state of the environment is given by the probability measure σ on $(\Omega, \mathcal{F})$. We assume that this probability measure σ satisfies [see Breiman (1968, pp.106–109) for definitions],

(A1) T is measure preserving and ergodic.

This condition holds, for example, for the following processes:

(a) (ω_t) is a sequence of i.i.d. random variables;

(b) The environment process is a Markov chain with a transition probability matrix which is irreducible.

Note also that (A1) implies that T^{-1} is also a measure preserving transformation [see Breiman (1968, proposition 6.18)].

We assume that the labor force growth rate is the constant n and hence $N_t = (1+n)^t N_0$. Capital is a perishable homogenous good which can be either consumed or used as a production factor. The per–capita production function at date t is $f(k,\omega_t)$, where k is the (per capita) capital stock and the environment at date t is ω_t. It satisfies the following assumptions, common in the neoclassical growth literature:

(A2) $f(k,\theta)$ is strictly concave, strictly increasing and twice continuously differentiable in k for all $\theta \in [\alpha,\beta]$. Also $f'(0,\theta) = \infty$, $f'(\infty,\theta) = 0$ for all θ. $f(\cdot,\theta)$ and $f'(\cdot,\theta)$ are uniformly continuous in θ on $[\alpha,\beta]$.

Let k_0^* be the initial (per–capita) capital stock. A *feasible program* (FP) from k_0^* is a pair of adapted integrable processes $(\underset{\sim}{k},\underset{\sim}{c})$ where $\underset{\sim}{k} = (k_0,k_1,\ldots)$, $\underset{\sim}{c} = (c_0, c_1,\ldots)$, $k_{t+1},c_t \in L_t^+$ for all t, $k_0 = k_0^*$ and

$$c_t(\omega) + (1+n)k_{t+1}(\omega) = f(k_t(\omega), \omega_t) \text{ a.s. } t = 0,1,2,\ldots \tag{1}$$

where $k_t(\omega)$ and $c_t(\omega)$ are the capital stock and consumption at date t. A FP is called *interior* if for some $\lambda > 0$ $c_t(\omega) \geq \lambda$ a.s. for all t. We denote by $P(k_0^*)$ the set of all feasible programs from initial capital stock k_0^*.

A feasible program $(\underset{\sim}{k}, \underset{\sim}{c})$ is *stationary* if there exist a pair of functions $k(\omega)$, $c(\omega)$ in L_0^+ such that

$$k_t(\omega) = k(T^{t-1}\omega) \ \text{a.s.} \ t = 0,1,2,\ldots$$
$$c_t(\omega) = c(T^t\omega) \ \text{a.s.} \ t = 0,1,2,\ldots \ .$$

Thus the consumption and investment over time are stationary stochastic processes generated by some k and c, and this plan is feasible from the initial capital $k(T^{-1}\omega)$. We shall denote a *stationary* FP by (k,c).

Given a stationary FP (k,c) define its corresponding *interest factors* (stationary) process by:

$$r(\omega) = f'(k(T^{-1}\omega), \omega_0) \ \text{a.s.} \tag{2}$$

The *interest rates* stationary process is generated by $\rho(\omega) = r(\omega) - 1$.

In the sequel we shall need the following assumption about the elasticities of the production function and the marginal product [see Mitra (1979)].

(A3) There are positive constants m_1, m_2, m_3, m_4 such that for all $k > 0$ and all θ in $[\alpha, \beta]$ the following conditions hold:

$$m_1 \leq \frac{kf'(k,\theta)}{f(k,\theta)} \leq m_2 \ \text{and} \ m_3 \leq \frac{-k\, f''(k,\theta)}{f'(k,\theta)} \leq m_4. \tag{3}$$

This type of assumption has appeared in deterministic growth models in order to obtain a complete characterization of efficiency.

4. Efficient Intertemporal Allocations

The notion of efficiency in stochastic dynamic economic models can be formulated in various ways. Due to the sequential decision making process the comparison of any two feasible paths can be done in two different ways, (a) in a decentralized manner, i.e., at each point of time, given the history, we compare the relevant random variables at hand, to consider the two plans, or (b) at the very outset the comparison is done according to some aggregative criterion (such as discounted sums of expected utilities. See, for example, Foldes (1989), Karatzas, Lehoczky and Shreve (1987) in the case where the utility is predetermined). Traditionally, the definition of efficiency *did not* use a one particular preference ordering (unlike the *Pareto optimality* notion) and we would like to follow this course. Namely, we shall use the two well–known stochastic orderings to compare random variables and hence derive two definitions in our approach. Denote by U^1 the set of all continuous nondecreasing functions from R^1 to R^1. U^2 is the *subset* of U^1 which contains all the *concave* functions.

Given two feasible programs from k_0 $(\underset{\sim}{k}, \underset{\sim}{c})$ and $(\underset{\sim}{k}^*, \underset{\sim}{c}^*)$ we shall compare $c_t(\omega)$ and $c_t^*(\omega)$ as follows: For each possible history $(\ldots\omega_{t-2}, \omega_{t-1})$, $E_t u(c_t(\omega)) \geq E_t u(c_t^*(\omega))$ $\forall u \in U^1$ with strict inequality for some $v \in U^1$. This is denoted by $c_t >_1 c_t^*$ and it is a conditional first degree stochastic dominance (FDSD). We say that $(\underset{\sim}{k}, \underset{\sim}{c})$ *dominates* $(\underset{\sim}{k}^*, \underset{\sim}{c}^*)$ in the *first degree stochastic dominance* if $c_t \geq_1 c_t^*$ for $t = 0,1,2,\ldots$ and for some τ we have strict $>_1$. An FP in $P(k_0)$ is *efficient of type* I if it is not dominated in the FDSD by any other FP in $P(k_0)$. Similarly, using second degree stochastic dominance, i.e., replacing U^1 by U^2 in the above definition, we define "efficient of type II". Since $U^2 \subset U^1$ it is clear that if $(\underset{\sim}{k}, \underset{\sim}{c})$ is *efficient of type II* then it is *efficient of type I.*

Let us note that in the stochastic overlapping generations models a similar approach which compares the random consumption of each given generation, conditioned on the current information, was taken by Peled (1982) in justifying his "conditional pareto optimal" concept. However, the utility function is predetermined in this model as well as in Manuelli (1988) and Aiyagari–Peled (1988).

5. Characterization of Efficient(Type I) Stationary Plans

Our aim in this section is to provide a simple efficiency criterion for stationary plans relating the corresponding random interest rates to the rate of population growth.

THEOREM 1: *Assume that (A1)–(A3) hold. An interior stationary feasible plan* (k,c) *is efficient of type I if and only if its corresponding interest factors, defined in (2), satisfy:*

$$E[\log r(\omega)] \geq \log(1+n) \qquad (4)$$

We bring all the proofs in the last section. Note that due to the strict concavity of the logarithmic function the condition $Er(\omega) > 1+n$ *does not* guarantee efficiency. Also we do not assume risk aversion on the part of the decision makers since this criterion characterizes efficiency of type I. The proof of Theorem 1 applies the characterization of efficiency for nonstationary intertemporal allocations attained in Zilcha (1990a). Theorem 1 generalizes Cass's (1972) result for deterministic growth model, since in the certainty case (4) reduces to $1+r = f'(k) \geq 1+n$.

6. Efficiency of Stochastic Modified Golden–Rules

Consider a stochastic one–sector optimal growth model as in Brock and Mirman (1972) and Mirman and Zilcha (1975). In this economy there are technological shocks as above, and the comparison of any two feasible plans is done through *some given* concave utility function, $u\colon R^1_+ \to R^1$, defined on one period (per–capita) consumption using the *expected* sum of discounted utilities. An *optimal* plan is a solution to

$$\text{Max } E \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{5}$$

s.t.

$$c_t + (1+n)\, k_{t+1} = f(k_t, \omega_t) \text{ a.s. } \; t = 0,1,2,\ldots$$
$$c_0 + (1+n)k_1 = k_0^*$$
$$c_t \geq 0, \;\; k_t \geq 0$$

where $k_0^* > 0$ is the initial capital stock, and future utilities are discounted by δ, $0 < \delta < 1$. We assume that $u' > 0$, $u'(0) = \infty$ and $u'' < 0$. As was shown in Brock–Mirman (1972) and Mirman–Zilcha (1975) the optimal plan, denoted $(\underset{\sim}{k}^*, \underset{\sim}{c}^*)$, satisfies the following Euler's conditions:

$$(1+n)\, u'(c_t^*(\omega)) = \delta\, E\Big[f'(k_t^*(\omega), \omega_{t+1})u'(c_{t+1}^*(\omega)) \mid \mathscr{F}_t\Big] \text{ a.s.} \tag{6}$$
$$\text{for } t = 0,1,2,\ldots .$$

It was demonstrated in the above two papers that the optimal capital stocks k_t^* converge *in distribution* as $t \to \infty$. Let $(k^*, c^*) \in L_0^+ \times L_0^+$ where $\left\{k^*(T^t\omega), c^*(T^t\omega)\right\}_{t=0}^{\infty}$ is a stationary feasible plan from the initial capital stock $k^*(T^{-1}\omega)$. (k^*, c^*) is called a *stochastic Modified Golden–Rule* (SMGR) if,

$$(1+n)u'(c^*(\omega)) = \delta E\left[f'(k^*(\omega), \omega_1)u'(c^*(T\omega)) \mid \mathcal{F}_0\right] \text{ a.s.} \tag{7}$$

We shall consider the question of existence of stochastic modified golden rule in the next section. The result of Brock and Mirman (1972), that optimal capital stocks converge in distribution does not guarantee the existence of SMGR in the above sense.

Let us consider the issue of efficiency (of type II) of the SMGR stationary allocation. We shall prove now that under our assumptions each SMGR plan is efficient of type II.

<u>THEOREM</u> <u>2</u>: *Assume that (A1))–(A3) hold and that the given utility function u is strictly concave. Then any interior Stochastic Modified Golden–Rule allocation is efficient of type II.*

Note that since a stationary plan which is efficient of type II is also efficient of type I Theorem 2 implies:

<u>COROLLARY</u>: Under the conditions of Theorem 2, given the SMGR (k^*, c^*), its corresponding interst factors stationary process $r(\omega)$, defined in (2), satisfies:

$$E[\log r(\omega)] \geq \log(1+n) \tag{8}$$

7. Existence of a Stochastic Modified Golden–Rule

It was shown by Brock and Mirman (1972) that for any optimal plan $(k_t^*, c_t^*)_{t=0}^{\infty}$, from initial k_0^*, the distribution functions of k_t^*, denoted by F_t, $t = 1,2,...$, converge to some limiting distribution function F^*. However, there is no explicit existence proof of a stochastic modified golden rule for the stochastic one–sector growth models for an arbitrary discounting factor δ, $0 < \delta < 1$. To the best of our knowledge there is no generalization of the Peleg–Ryder (1974) existence theorem of modified golden rule in an n–sector growth model to the stochastic case. It was shown by Marimon (1989) that in a stochastic multisector economy, when δ is close enough to 1, there exists an SMGR. Thus we find it useful to demonstrate existence of SMGR in the one–sector stochastic model with an arbitrary δ, $0 < \delta < 1$. In this case by Theorem 2 the golden–rule allocation is efficient of type II (when the interiority condition is guaranteed). To simplify our existence proof we shall strengthen assumption (A1) as follows:

(A1*) The stationary stochastic process $\left\{\omega_t\right\}_{t=-\infty}^{\infty}$ is i.i.d. .

Now we prove:[1]

THEOREM 3: *Assume that (A1*) and (A2) hold and that* u *is strictly concave. There exists a Stochastic Modified Golden–Rule* $(k^*(\omega), c^*(\omega))$.

[1] I have benefitted from discussing the idea of the proof of this theorem with Mukul Majumdar and this is gratefully acknowledged.

8. PROOFS

PROOF OF THEOREM 1: The proof of this theorem will be based on a characterization result of efficiency proved by Zilcha (1990a) for nonstationary stochastic plans: A feasible plan $(k_t^*, c_t^*)_{t=0}^{\infty}$ from $k_0 > 0$ is efficient of type I (assuming that $n \geq 0$, but not necessarily zero) if and only if $\sum_{\tau=0}^{\infty}\left[\prod_{\tau=0}^{t} f'(k_t^*(\omega), \omega_t)/_{1+n}\right] = \infty$ a.s. . Thus, given the feasible stationary plan $<k(\omega), c(\omega)>$, its corresponding interest factors stationary process by $r(\omega)$ is defined by (2). Define the following function on Ω,

$$\gamma(\omega) = \sum_{t=0}^{\infty}\left[\prod_{\tau=0}^{t} \frac{r(T^{\tau}\omega)}{1+n}\right]. \tag{9}$$

Let $A = \{\omega \mid \gamma(\omega) < \infty\}$. We claim now that if $\sigma(A) > 0$ then $\sigma(A) = 1$. To show that, assume that $\omega \in A$. Then

$$\gamma(\omega) = \frac{r(\omega)}{1+n}\left[1 + \sum_{t=0}^{\infty}\left[\prod_{\tau=0}^{t} \frac{r(T^{\tau}(T\omega))}{1+n}\right]\right] = \frac{r(\omega)}{1+n}[1 + \gamma(T\omega)]. \tag{10}$$

From (10) it follows that the event A is invariant under T. Since T is ergodic, the probability of A is 1.

Let $\xi_k = \frac{r(T^k\omega)}{1+n}$, $k = 0,1,2,\ldots$. Rewrite $\gamma(\omega)$ as

$$\gamma(\omega) = \sum_{t=0}^{\infty} \exp\left(\sum_{k=0}^{t} \log \xi_k\right). \tag{11}$$

Since $\{r(T^k(\omega))\}$ is an ergodic process, then the process $\{\log r(T^k\omega)\}$ is ergodic as well (see Breiman [1968, proposition 6.31]). Applying Birkhoff's Ergodic Theorem (see Breiman [1968, p.115]) we derive:

$$\frac{1}{m}\sum_{t=0}^{m-1}\log r(T^t\omega) \underset{a.s.}{\longrightarrow} E[\log r(\omega)] \quad \text{as} \quad m \to \infty \tag{12}$$

Now, from (11) and (12) we conclude that:

$$E[\log r(\omega)] > 1 + n \Longrightarrow \gamma(\omega) = \infty \quad \text{a.s.} \tag{13}$$

$$E[\log r(\omega)] < 1 + n \Longrightarrow \gamma(\omega) < \infty \quad \text{a.s.} \tag{14}$$

The verification of (13) and (14) is easy using (11) and (12) and we omit it. However, let us prove the case where $E[\log r(\omega)] = \log(1+n)$. This implies that $E[\log \xi_k] = 0$ for all k. Therefore,

$$E\left[\sum_{k=0}^{T}\log \xi_k\right] = 0 \quad \text{for} \quad T = 1,2,\ldots$$

which clearly shows that $\sigma\{\omega \mid \gamma(\omega) = \infty\} > 0$. But as we have shown before, under our assumption about the operator T this implies that $\sigma\{\omega \mid \gamma(\omega) = \infty\} = 1$ when $E\left[\log\frac{r(\omega)}{1+n}\right] = 0$.

□

PROOF OF THEOREM 2: Given an SMGR (k^*, c^*). Let us show first that its stationary allocation $(k^*(T^t\omega), c^*(T^t\omega))_{t=0}^{\infty}$ is optimal from the initial capital stock $k^*(T^{-1}\omega)$. To that end let $(\underset{\sim}{k}, \underset{\sim}{c})$ be any feasible plan from $k^*(T^{-1}\omega)$ and N any finite integer, then

$$E\sum_{t=0}^{N}\delta^t[u(c_t(\omega)) - u(c^*(T^t\omega))] \leq E\sum_{t=0}^{N}\delta^t u'(c^*(T^t\omega))[c_t(\omega) - c^*(T^t\omega)] =$$

$$E\sum_{t=0}^{N}\delta^t u'(c^*(T^t\omega))\Big[f(k_t^*,\omega_{t+1}) - (1+n)k_{t+1}^* - f(k^*(T^{t-1}o),\omega_t) + (1+n)k^*(T^t\omega)\Big] \leq$$

$$E\sum_{t=0}^{N}\delta^t u'(c^*(T^t\omega))\Big[f'(k^*(T^{t-1}\omega),\omega_t)(k_t^*(\omega)-k^*(T^{t-1}\omega))-(1+n)(k_{t+1}^*(\omega)-k^*(T^t\omega))\Big]=$$

$$E\sum_{t=0}^{N-1}\delta^t\Big[\delta f'(k^*(T^t\omega), \omega_{t+1})u'(c^*(T^{t+1}\omega)) - (1+n)u'(c^*(T^t\omega))\Big](k_t^*(\omega)-k^*(T^t\omega))+$$

$$(1+n)\delta^N E\, u'(c^*(T^N\omega))(k_N^*(\omega) - k^*(T^N\omega)) =$$

$$E_t\sum_{t=0}^{N-1}\delta^t\Big\{E_{t+1}\Big[\delta f'(k^*,\omega_{t+1})u'(c^*(T^{t+1}\omega)) - (1+n)u'(c^*(T^t\omega))\Big]\Big\}(k_t^* - k^*(T^t\omega))+$$

$$+ \quad \delta^N E\Big[u'(c^*(T^N\omega))(k_N^* - k^*(T^N\omega))\Big] \leq \delta^N\, Eu'(c^*(T^N\omega))(k_N^* - k^*(T^N\omega))$$

where the RHS converges to 0 as $N \to \infty$ since $u'(c^*(T^N\omega)) \leq Q < \infty$ a.s. for some Q.

Now we can apply the result of Theorem 3 in Zilcha (1990a) which claims that an optimal plan, where the utility function is strictly concave, is efficient of type II.

□

PROOF OF THEOREM 3: To shorten our existence proof we shall use the Brock–Mirman (1972) result and the Mirman–Zilcha (1975) results and notations. It was shown that for each initial $k_0^* > 0$ the optimal stochastic process $\{k_t^*(\omega)\}$ evolves according to:

$$k_{t+1}^*(\omega) = h[f(k_t^*(\omega), \omega_{t+1})] \qquad t = 1,2,\dots$$

where $h(x)$ is the optimal investment policy function (see Mirman–Zilcha (1975)). By our assumption (A2) it follows that there exist $0 < \underline{m} < \overline{m} \leq \infty$ such that for all θ $f(x,\theta) > x$ when $0 < x \leq \underline{m}$ and $f(x,\theta) < x$ for $x > \overline{m}$. Define $H(x,\theta) = h[f(x,\theta)]$, hence $h(0,\theta) = 0$ for all θ and

$$k_{t+1}^*(\omega) = H(k_t^*(\omega), \omega_{t+1}) \;\; k = 0,1,\dots . \tag{15}$$

$$H(x,\theta) > x \;\text{ for all }\; \theta \;\text{ and }\; 0 < x < \underline{m} \tag{16}$$

$$H(x,\theta) < x \;\text{ for all }\; \theta \;\text{ and }\; x > \overline{m} \tag{17}$$

Let $F_t(\xi)$ be the distribution function of k_t^*, where $(\underset{\sim}{k}^*, \underset{\sim}{c}^*)$ is the optimal plan from some initial $k_0^* > 0$. Under our assumptions (see Brock–Mirman (1972) and Mirman–Zilcha (1975)); $F_t(\xi)$ converges, uniformly on $[\underline{m}, \overline{m}]$, to some unique invariant distribution function $F^*(\xi)$. Moreover, by (16) and (17) F^* cannot have all its mass concentrated at 0 since its support is contained in $[\underline{m}, \overline{m}]$.

Let $k^*(\omega)$ be a random variable in $L^1(\Omega, \mathscr{F}_0, \sigma)$ with a distribution function $F^*(\xi)$. The existence of such R.V. k^* is given by Theorem 3.2 in Billingsley (1971). Note also that $k^*(\omega) \in [\underline{m}, \overline{m}]$ a.s. . Since F^* is an invariant distribution for the process (15) we obtain [see Breiman (1968, proposition 7.11)] that $\left\{k^*(T^t\omega)\right\}_{t=0}^{\infty}$ is a stationary process. In particular it satisfies:

$$k^*(T\omega) = H(k^*(\omega), \omega_1) \quad \text{a.s. .} \tag{18}$$

Let $g(x) = x - h(x)$ be the optimal consumption function, which is strictly increasing and $g(0) = 0$. Define

$$c^*(\omega) = g[f(k^*(T^{-1}\omega), \omega_0)] \quad \text{a.s. .}$$

It can be verified (as in the approach used in Mirman–Zilcha(1975)) that $\left\{k^*(T^t\omega), c^*(T^t\omega)\right\}_{t=0}^{\infty}$ is the optimal consumption plan from the initial capital stock $k^*(T^{-1}\omega)$, hence

$$(1+n)u'(c^*(\omega)) = \delta E_0\left[f'(k^*(\omega), \omega_1)u'(c^*(T\omega))\right] \text{ a.s.}$$

which establishes that $(k^*(\omega), c^*(\omega))$ is a SMGR.

□

REFERENCES

Abel, A., N.G.Mankiw, L.H.Summers and R.J.Zeckhauser, 1989, Assessing dynamic efficiency: theory and evidence, Review of Economic Studies 56, 1–20.

Aiyagari, S.R. and D.Peled, 1988, Dominant root characterization of Pareto Optimality and the existence of monetary equilibria in stochastic overlapping generations models, manuscript.

Breiman, L., 1968, "Probability," Addison Wesley Publishing Company.

Brock, W.A. and L.Mirman, 1972, Optimal economic growth and uncertainty: the discounted case, Journal of Economic Theory, 4, 479–513.

Cass, D., 1972, On capital overaccumulation in the aggregative neoclassical model of economic growth: A complete characterization, Journal of Economic Theory 4, 200–223.

Foldes, L., 1989, Conditions for optimality in the infinite–horizon portfolio–cum–saving problem with semi–Martingale investments, LSE Financial Markets Group Discussion Paper Series, No.53.

Karatzas, I., J.P.Lehoczky and S.E.Shreve, 1987, Optimal portfolio and consumption decisions for a "small investor" on a finite horizon, SIAM Journal on Control and Optimization 25, 1557–1586.

Majumdar, M. and I.Zilcha, 1987, Optimal growth in a stochastic environment: Some sensitivity and turnpike results, Journal of Economic Theory 43, 116–133.

Manuelli, R., 1990, Existence and optimality of currency equilibrium in stochastic OLG models: The pure endowment case, Journal of Economic Theory 51, 268–294.

Marimon, R., 1989, Stochastic turnpike property and stationary equilibrium, Journal of Economic Theory, 47, 282–306.

Mitra, T., 1979, Identifying inefficiency in smooth aggregative models of economic growth: A unifying criterion, Journal of Mathematical Economics 6, 85–111.

Mirman, L., and I.Zilcha, 1975, On optimal growth under uncertainty, Journal of Economic Theory 11, 329–339.

Peled, D., 1982, Informational diversity over time and the optimality of monetary equilibria, Journal of Economic Theory 28, 255–274.

Peleg, B. and H.E.Ryder, 1974, The Modified golden rule of a multi–sector economy, Journal of Mathematical Economics 1, 193–198.

Zilcha, I., 1990a, Efficiency in economic growth models under uncertainty, Journal of Economic Dynamics and Control, forthcoming.

——————, 1990b, Characterizing efficiency in stochastic overlapping generations models, Journal of Economic Theory, forthcoming, The Foerder Institute for Economic Research, Working Paper No.16–90, Tel–Aviv University.