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A NEO²BAYESIAN FOUNDATION OF THE MAXMIN VALUE FOR TWO–PERSON ZERO–SUM GAMES

by

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A NEO^{2}BAYESIAN FOUNDATION OF THE MAXMIN VALUE FOR TWO-PERSON ZERO-SUM GAMES*

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Abstract. A joint derivation of utility and value for two-person zero-sum games is obtained using a decision theoretic approach. Acts map states to consequences. The latter are lotteries over prizes, and the set of states is a product of two finite sets (m rows and n columns). Preferences over acts are complete, transitive, continuous, monotonic and certainty-independent (Gilboa and Schmeidler (1989)), and satisfy a new axiom of strategic flexibility which we introduce. These axioms are shown to characterize preferences such that (i) the induced preferences on consequences are represented by a von Neumann-Morgenstern utility function, and (ii) each act is ranked according to the maxmin value of the corresponding $m \times n$ utility matrix (a two-person zero-sum game). An alternative statement of the result deals simultaneously with all finite two-person zero-sum games in the framework of conditional acts and preferences.

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1. Introduction.

In their "Theory of Games and Economic Behavior", von Neumann and Morgenstern (1944) present the theory of two-person zero-sum games as an extension of the axiomatic theory of decision under risk, which from their point of view is a theory of rational behavior in one-person games.

Although von Neumann and Morgenstern (1944) do not define preferences on games, they 'suggest' a ranking of two-person zero-sum games by their (maxmin) value by asserting (in section 17.8) that the 'good' way of playing such games is to choose, from among the alternative feasible strategies, the ones which ensure for each game the attainment of its value. However, as already noted by McClennen (1976), validity of this assertion is not implied by the von Neumann-Morgenstern axioms. Thus there is a gap between the axioms characterizing expected utility maximization in individual decision under risk and the presumption that expected utility maximizers evaluate two-person zero-sum games by their value.

The purpose of this paper is to fill this gap by means of a unified decision theoretic analysis resulting in a simultaneous derivation of utility and value.

Ellsberg (1956) and Aumann and Maschler (1972) also criticize the completeness of the von Neumann-Morgenstern argument justifying the use of maxmin strategies, but they do not discuss the relation - or lack of it - between utility theory and behaviors in two-person games. Roth (1982) refers to the above mentioned gap, but his work is more in the direction of Vilkas (1963) and Tijs (1981) who characterize the 'value' as a functional on matrices.

The basic decision model we use is Anscombe and Aumann's (1963) simplified version of Savage (1954) model, consisting of a set of acts and a preference relation $\succsim$ over it, where acts are mappings from a space S of states into a space C of consequences, and the latter are 'roulette lotteries', i.e. probability distributions with finite supports over a fixed set of outcomes. A *state* is here interpreted as a *state of the world* and not as a *state of nature*. The distinction was first introduced by Mertens and Zamir (1985): a state of nature is chosen by a neutral nature according to some (additive) probability distribution which may be unknown to the decision-maker(s), and nature is thought of as beyond the decision maker(s)' control. The *world* may include in addition to neutral nature several decision makers each having his own goal, and a state of the world is a consequence of a joint selection *by all the world*, so that in this case occurrence of events may be partially under the control of the decision maker(s). Moral hazard is an example of such a situation, special in that, in addition to nature, only the single decision maker under study has influence on events. In the general case different decision makers, possibly with conflicting interests, may partially influence events.

In the next section, after describing the model, we posit a set of basic axioms of (individual) choice, borrowed from Gilboa and Schmeidler (1989). These axioms imply in particular (i) (Lemma 2.6 below) existence of a von Neumann-Morgenstern utility $u : C \to \mathbb{R}$ on consequences, and (ii) (Lemma 2.7) existence of a real valued mapping $I : \mathbb{R}^S \to \mathbb{R}$ such that for any acts $f, g : S \to C$ one has $f \succsim g$ iff $I(u \circ f) \geq I(u \circ g)$.

Then (section 3) we make the structural assumption that S is a product space, $S = S^1 \times S^2$, and also assume that it is finite, so that an act f can be viewed as a game form with outcomes $f(s^1, s^2)$, S^1 and S^2 being interpreted as (pure) strategy sets and the decision maker being identified with player 1 (the row player). In this case, the set $\{(u \circ f)(s^1, s^2) \mid (s^1, s^2) \in S\}$ is an $\#S^1$ by $\#S^2$ real matrix, and act f corresponds to the matrix game with payoffs $u(f(s^1, s^2))$, for $(s^1, s^2) \in S^1 \times S^2$. Within this structure we present an axiom which, as we show, together with the basic axioms of section 2 characterizes preferences ranking game forms (with fixed S) by the value of the corresponding two-person zero-sum games. Formally the result is that the map I above, now defined on the space of $\#S^1$ by $\#S^2$ real matrices, is the 'value' map assigning to each such matrix its maxmin value. Notice however that justifying evaluation of games by value does not automatically imply rationalization of maxmin *strategies*.

In section 4 we recast the model with the purpose of simultaneously considering *all* finite two-person zero-sum games, i.e. S is no longer fixed. We consider all finite rectangular subsets $S = S^1 \times S^2$ of a 'universal' state space, and define conditional acts as pairs (f, S) where f is the map as before and S its domain. We do not assume that all pairs of acts $(f, S), (g, T)$ are comparable (i.e. we do not assume completeness), and show that the basic axioms of section 2 for each S separately plus the appropriate version of the axiom of section 3 characterize, as before, preference relations represented by the 'value' function, defined in this case on the set of all finite real matrices.

A few words on terminology. The term *neobayesian* was used by Savage to describe his and related work which based statistical inference on subjective or *personal* probability. The neo^2 (i.e. neoneo) term is used here to denote the last decade's departure from Savage's sure thing principle and from the independence axiom of von Neumann-Morgenstern utility theory. (We imitate her Stanley Reiter's "New^2 Welfare Economics"). The term *act dependent subjective probability* describes many Neo^{2}Bayesian axiomatizations including non-additive and non-unique priors (surveyed by Karni and Schmeidler (1990)) as well as the present paper. This terminology is consistent with the term *bayesian* used in game theory where the primitive is existence of prior probability as opposed to the primitive being preferences on acts in neobayesian theory.

2. Decision Theoretic Framework.

Let X be a non-empty set and let $\Delta(X)$ be the set of probability distributions over X with finite supports

$$\Delta(X) = \{y : X \to [0,1] \mid y(x) \neq 0 \text{ for only finitely many } x\text{'s in } X \text{ and } \sum_{x \in X} y(x) = 1\} .$$

For notational simplicity we identify X with the subset $\{y \in \Delta(X) \mid y(x) = 1$ for some x in $X\}$ of $\Delta(X)$.

Let S be a finite non-empty set, and denote by $L = \Delta(X)^S$ the set of all functions from S to $\Delta(X)$ and by L_c the constant functions in L. Note that $\Delta(X)$ can be viewed as a subset of a linear space, so $\Delta(X)^S = L$ can also be considered a subset of a linear space. It should be stressed that convex combinations in $\Delta(X)^S$ are performed pointwise, i.e. for f and g in $\Delta(X)^S$ and α in [0,1], $h = \alpha f + (1-\alpha)g$ where $h(s) = \alpha f(s) + (1-\alpha)g(s)$, for $s \in S$.

In the neobayesian nomenclature elements of X are (deterministic) *outcomes*, elements of $\Delta(X)$ are random outcomes or *consequences* and elements of L are *acts*. Elements of S are *states* (of the world) and subsets of S are *events*.

The primitive of a neobayesian decision model is a binary (preference) relation on L to be denoted by $\succsim$. On $\succsim$ we shall impose the following axioms.

2.1 Weak order. *(i) Completeness.* For all f and g in $L : f \succsim g$ or $g \succsim f$. *(ii) Transitivity.* For all f, g and h in L: If $f \succsim g$ and $g \succsim h$ then $f \succsim h$.

As usual, $\succ$ and $\simeq$ denote the asymmetric and symmetric parts, respectively of $\succsim$. The relation $\succsim$ on L induces a relation on $\Delta(X)$ also denoted by

$\succsim$: $y \succsim z$ iff $y^* \succsim z^*$ where $x^*(s) = x$ for all $x \in \Delta(X)$ and $s \in S$. When no confusion is likely to arise, we shall not distinguish between y^* and y.

2.2 Certainty-Independence (C-independence for short). For all f, g in L and h in L_c and for all α in $]0,1[$: $f \succ g$ iff $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

2.3 Continuity. For all f, g and h in L: if $f \succ g$ and $g \succ h$ then there are α and β in $]0,1[$ such that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$.

2.4 Monotonicity. For all f and g in L: if $f(s) \succsim g(s)$ for all $s \in S$ then $f \succsim g$.

2.5 Non-degeneracy. Not for all f and g in L, $f \succsim g$.

All these assumptions except C-independence, introduced and discussed in Gilboa-Schmeidler (1989) (but see also Drèze (1987) who in effect used it in a slightly different context), are common and essentially define the setup. We have included non-degeneracy for ease of exposition. C-independence is a (quite) weak version of the standard independence axiom which allows h to be any act in L rather than restricting it to be a constant act in L_C.

We shall now state some implications of the above assumptions which will be useful in the presentation of the main result as well as in its proof.

2.6 LEMMA. *Assumptions 2.1, 2.2 and 2.3 imply that there exists an affine $u : \Delta(X) \rightarrow R$ such that for all $y, z \in \Delta(X)$: $y \succsim z$ iff $u(y) \geq u(z)$. Furthermore, u is unique up to positive linear transformations.*

This is (an immediate consequence of) the von Neumann-Morgenstern theorem, since the independence assumption for L_c is implied by C-independence.

We shall henceforth choose a specific $u : \Delta(X) \to \mathbb{R}$. We denote by B the space of all real valued functions on S, i.e. $B = \mathbb{R}^S$. For $\gamma \in \mathbb{R}$, $\gamma^* \in B$ denotes the constant function on S the value of which is γ.

2.7 Lemma. *Under assumptions 2.1, 2.2, 2.3, 2.4 and 2.5, there exists a function $I : B \to \mathbb{R}$ such that:*

(i) For all $f, g \in L$, $f \succsim g$ iff $I(u \circ f) \geq I(u \circ g)$.

(ii) For all $\gamma \in \mathbb{R}$, $I(\gamma^*) = \gamma$.

(iii) I is monotonic (i.e. for $a, b \in B : a \geq b \Rightarrow I(a) \geq I(b)$).

This follows easily from Gilboa-Schmeidler (1989), section 3.

3. Game Theoretical Setting.

In section 2 the state space S was arbitrary. We now introduce the structural assumption that S is a product space:

3.1
$$S = S^1 \times S^2 \ .$$

For any state of the world $s = (s^1, s^2) \in S$, $s^1 \in S^1$ will be the component influenced – in fact determined – by the decision maker, and $s^2 \in S^2$ the component beyond his control.

Notice that under assumption 3.1, act $f \in L$ may be viewed as an $\#S^1$ by $\#S^2$ rectangular array of outcomes (consequences) $f(s^1, s^2)$, $(s^1, s^2) \in S$, and $u \circ f \in B$ as an $\#S^1$ by $\#S^2$ real matrix.

We are going to characterize the decision maker who perceives act f as a game form and $u \circ f$ as a two-person zero-sum game in which he is player 1 (the row player), and evaluates this game according to its maxmin value. In other words, we will characterize preferences on L whose representing map I

(of Lemma 2.7) is the 'value function' $V : B \to \mathbb{R}$ defined as

$$3.1.1 \qquad V(b) = \max_{p\in\Delta(S^1)} \min_{q\in\Delta(S^2)} \sum_{(s^1,s^2)\in S} p(s^1)b(s^1,s^2)q(s^2) \ ,$$

where $b \in B$ and Δ's are simplexes. For this purpose, we shall need the axioms of section 2 plus the following:

3.2 Attitude Toward Substitution of Risk for Uncertainty – Special Case (SR-uncertainty for short*).*[1].

(i) For all $r^1, t^1 \in S^1$, $\alpha \in [0,1]$, $f \in L$:

if $g \in L$ is defined by

$$g(s^1,s^2) = \begin{cases} f(s^1,s^2) & \text{if } s^1 \neq r^1, t^1 \\ \alpha f(r^1,s^2) + (1-\alpha) f(t^1,s^2) & \text{if } s^1 = r^1, t^1, \end{cases}$$

then $f \succsim g$.

(ii) For all $r^2, t^2 \in S^2$, $\alpha \in [0,1]$, $f \in L$: if $g \in L$ is defined by

$$g(s^1,s^2) = \begin{cases} f(s^1,s^2) & \text{if } s^2 \neq r^2, t^2 \\ \alpha f(s^1,r^2) + (1-\alpha) f(s^1,t^2) & \text{if } s^2 = r^2, t^2, \end{cases}$$

then $g \succsim f$.

This axiom says that the decision-maker (row player) is (i) indifferent or worse off if any two rows are both substituted with their (arbitrary) weighted average, and (ii) indifferent or better off if any two columns are both substituted with their weighted average. The idea is that control on strategies is

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[1] Another special case of axioms expressing attitude toward substitution of risk for uncertainty in a neo[2]bayesian version of Anscombe-Aumann model are the axioms of *uncertainty aversion* and *uncertainty appeal* introduced in Schmeidler (1989): $f \succsim g \Rightarrow \alpha f + (1-\alpha) g \succsim g$ and $f \succsim g \Rightarrow f \succsim \alpha f + (1-\alpha) g$ (respectively). (Drèze (1987) introduced the axiom of uncertainty appeal, in a slightly different context.) Axiom 3.2(i) has the flavour of uncertainty appeal and 3.2.(ii) has the flavour of uncertainty aversion (see Proposition (x) in Schmeidler (1998).)

less effective the less strongly outcomes depend on them: in 3.2(i) dependence on S^1 is 'averaged out' in g , and the decision maker (row player) is (weakly) worse off; and the opposite is the case in 3.2(ii), where weaker dependence of outcomes on S^2 makes the decision maker (weakly) better off.

3.2.1 Remark. Notice the special cases of axiom 3.2 where $\alpha = 0$ or $\alpha = 1$. In such cases act g of 3.2(i) is obtained from act f by eliminating a row and putting in its place a duplication of another row. The analogous goes for 3.2(ii) with columns.

3.2.2 Remark. This is the only axiom that links, in this context, the decision theoretic model with two-person zero-sum games. It will imply (together with the axioms of section 2) that the decision maker behaves 'as if' he were playing two-person zero-sum games against an opponent (Theorem 3.3 below). It is implicit in the result that the decision maker believes that such an opponent exists, but such existence is not dealt with explicitly in the model.

3.2.3 Remark. It is obvious that by interchanging the roles of S^1 and S^2 in the above axiom we would get 'player 2's viewpoint'.

The result of this section can now be stated:

3.3 THEOREM. *Let a binary relation $\succsim$ on L be given and S satisfy the structural assumption 3.1. Then the following two statements are equivalent.*

(i) The binary relation $\succsim$ satisfies transitivity and completeness 2.1, certainty-independence 2.2, continuity 2.3, monotonicity 2.4, nondegeneracy 2.5, and SR-uncertainty 3.2.

(ii) There exists an affine, non-constant function $u : \Delta(X) \to \mathbb{R}$, unique up to positive linear transformations such that the functional $f \mapsto V(u \circ f)$ represents $\succsim$ on L (i.e. $f \succsim g$ iff $V(u \circ f) \geq V(u \circ g)$), where V is defined in 3.1.1.

To prove the theorem, we need a lemma which follows by induction from the SR-uncertainty axiom 3.2.

3.4 LEMMA. *Given $f \in L$ and $p \in \Delta(S^1)$ (respectively $q \in \Delta(S^2)$) define $g \in L$: $g(s^1,s^2) = \sum_{\widetilde{s}^1 \in S^1} p(\widetilde{s}^1) f(\widetilde{s}^1, s^2)$, (respectively $g(s^1,s^2) = \sum_{\widetilde{s}^2 \in S^2} q(\widetilde{s}^2) f(s^1, \widetilde{s}^2)$), for all $(s^1,s^2) \in S$. Then $f \succsim g$ (respectively $g \succsim f$.)*

Proof: Let $m = \#S^1$ and denote the elements of S^1 as $s_1^1, s_2^1, \ldots, s_m^1$ where $p(s_1^1) > 0$. Define $f_2 \in L$ by: $f_2(s_1^1, s^2) = f_2(s_2^1, s^2) = [p(s_1^1)f(s_1^1,s^2) + p(s_2^1)f(s_2^1,s^2)]/(p(s_1^1)+p(s_2^1))$ and $f_2(s_k^1,s^2) = f(s_k^1,s^2)$ if $k \neq 1,2$. We proceed by induction. Suppose that f_j, for $2 \leq j < m$, has been defined. Now define $f_{j+1} \in L$ as follows: $f_{j+1}(s_1^1,s^2) = f_{j+1}(s_{j+1}^1,s^2) = [\sum_{i=1}^{j} p(s_i^1)/\sum_{i=1}^{j+1} p(s_i^1)] f_j(s_1^1,s^2) + [p(s_{j+1}^1)/\sum_{i=1}^{j+1} p(s_i^1)] f_j(s_{j+1}^1,s^2)$ and $f_{j+1}(s_k^1,s^2) = f_j(s_k^1,s^2)$ if $k \neq 1, j+1$. By axiom 3.2(i), $f \succsim f_2$ and $f_j \succsim f_{j+1}$ for $2 \leq j < m$. Hence, $f \succsim f_m$. Note also that for j as above, $f_j(s_{j+1}^1,s^2) = f(s_{j+1}^1,s^2)$. So by our definition $f_m(s_1^1,s^2) = \sum_{i=1}^{m} p(s_i^1) f(s_i^1,s^2)$ for all $s^2 \in S^2$, i.e. the first row of f_m coincides with the rows of g, which are all identical.

We now apply consecutively the special case of axiom 3.2.(i) with $\alpha = 1$ (see remark 3.2.1). Specifically, we replace all rows of f_m with its first row, thus obtaining act g and $f_m \succsim g$. By transitivity, $f \succsim g$. The proof for S^2 is analogous and omitted. □

Proof of Theorem 3.3. The direction (i)⇒(ii). Lemma 2.6 guarantees the existence of the required utility $u : \Delta(X) \to \mathbb{R}$. By Lemma 2.7 it suffices to prove that for all $f \in L : I(u \circ f) = V(u \circ f)$. Let $q \in \Delta(S^2)$ be a minmax strategy of player 2 in the game $u \circ f$. Define $g \in L$ as follows: $g(s^1,s^2) = \sum_{\widetilde{s}^2 \in S^2} q(\widetilde{s}^2) f(s^1,\widetilde{s}^2)$, for all $(s^1,s^2) \in S$, thus g has constant rows, i.e. identical columns. By Lemma 3.4, $g \succsim f$ so by Lemma 2.7(i)

$I(u \circ g) \geq I(u \circ f)$. From the von Neumann (1928) **minmax theorem** $(u \circ g)(s^1, s^2) \leq V(u \circ f)$ for all $(s^1, s^2) \in S$. By Lemma 2.7(ii) and (iii), $I(u \circ g) \leq V(u \circ f)$. Hence $I(u \circ f) \leq V(u \circ f)$.

To prove the other inequality, $I(u \circ f) \geq V(u \circ f)$, let $p \in \Delta(S^1)$ be a maxmin strategy of player 1 in the same game $u \circ f$. This time define $g(s^1, s^2) = \sum_{s^1 \in S^1} p(\tilde{s}^1) f(\tilde{s}^1, s^2)$ for all $(s^1, s^2) \in S$ and apply Lemma 3.4. The same arguments as previously, except the use of the minimax theorem, complete the proof of the other inequality. (The lack of symmetry in the use of the minmax theorem reflects the lack of symmetry in our definition of V in 3.1.1.) So (i)$\Rightarrow$(ii).

The proof of the direction (ii)$\Rightarrow$(i) is straightforward, hence omitted. (It uses elementary properties of the value and the trivial direction of the von Neumann-Morgenstern expected utility theorem.) $\square$

4. Conditional Acts and Matrix Games.

In the previous section all games or game forms considered were of fixed dimension, i.e. with fixed number of strategies for each player. In this section we recast the theory to deal simultaneously with all finite game (forms) in the framework of conditional acts.

Let Θ^1, Θ^2 be two infinite sets and let $\Lambda = \{S = S^1 \times S^2 \mid S^i \subset \Theta^i\ ,\ i = 1, 2 \text{ and } 0 < \#S < \infty\}$ be the set of events or *conditions*. *Conditional acts* are elements of the set $\Gamma = \{(f, S) \mid S \in \Lambda \text{ and } f : S \to \Delta(X)\}$, and our primitive in this context is a binary relation $\succsim$ on Γ.

Let Γ_S denote all acts in Γ conditioned on a given $S \in \Lambda$, and $\succsim_S$ the restriction of $\succsim$ on Γ_S. For each $S \in \Lambda$, we shall impose on $\succsim_S$ the axioms of section 2. On $\succsim$ we do not impose completeness, which is a very restrictive axiom when applied to comparisons of acts conditioned on different events. We shall impose transitivity, and an axiom which allows comparisons between different but not too different conditions. In a sense to be made precise in Proposition 4.3, this axiom is the counterpart of axiom 3.2, SR-uncertainty, in the framework of conditional acts. It says that eliminating a column is (weakly) advantageous for the decision maker, whereas eliminating a row is (weakly) disadvantageous for him; and that furthermore, the decision maker is indifferent to addition of a row (or a column) which is a convex combination of two existing rows (or columns).

To state formally the new axiom we impose on $\succsim$, we need to consider the restriction of an act (f, S) to an event $T \subset S$. With slight abuse of notation, the resulting conditional act will be denoted by (f, T).

4.1 Conditional SR-Uncertainty.

(i) Let $(f, S) \in \Gamma$ and $T = (S^1 \backslash \{r^1\}) \times S^2$ for some $r^1 \in S^1$. Then $(f, S) \succsim (f, T)$.

(ii) Let $(f, S) \in \Gamma$ and $T = S^1 \times (S^2 \backslash \{r^2\})$ for some $r^2 \in S^2$. Then $(f, T) \succsim (f, S)$.

(iii) Let $(f, S) \in \Gamma$, $\alpha \in [0,1]$ and $r^i, t^i \in S^i$, $w^i \in \Theta^i$, $w^i \notin S^i$ for $i = 1, 2$. Define $(f_i, T_i) \in \Gamma$, $i = 1, 2$ by: $T_1^1 = S^1 \cup \{w^1\}$, $T_1^2 = S^2$, $f_1 = f$ on S and $f_1(w^1, s^2) = \alpha f(r^1, s^2) + (1 - \alpha) f(t^1, s^2)$ for $s^2 \in S^2$. Similarly $T_2^1 = S^1$, $T_2^2 = S^2 \cup \{w^2\}$, $f_2 = f$ on S and $f_2(s^1, w^2) = \alpha f(s^1, r^2) + (1 - \alpha) f(s^1, t^2)$ for $s^1 \in S^1$. Then $(f, S) \simeq (f_i, T_i)$ for $i = 1, 2$.

Notice the special case of 4.1(iii) with $\alpha = 0$ or 1, by which if two conditional acts are such that one is obtained from the other by eliminating one of two identical rows or columns, then they are indifferent. We will use this special case later, so we state it separately for future reference.

4.1.1 Irrelevance of Duplications. Let $(f, S) \in \Gamma$, and $r^i, t^i \in S^i$, $i = 1, 2$. If $f(r^1, s^2) = f(t^1, s^2)$ for all $s^2 \in S^2$ and $r^1 \neq t^1$, then $(f, S) \simeq (f, (S^1 \backslash \{r^1\}) \times S^2)$. Analogously, if $f(s^1, r^2) = f(s^1, t^2)$ for all $s^1 \in S^1$ and $r^2 \neq t^2$, then $(f, S) \simeq (f, S^1 \times (S^2 \backslash \{r^2\}))$.

The central result of this section is the following:

4.2 THEOREM. *Let a binary relation $\succsim$ on Γ be given. Then the following two statements are equivalent:*

(i) The binary relation $\succsim$ on Γ is transitive and satisfies conditional SR-uncertainty 4.1, and for each $S \in \Lambda$ the induced binary relation $\succsim_S$ on Γ_S satisfies completeness 2.1(i), C-independence 2.2, continuity 2.3, monotonicity 2.4 and non-degeneracy 2.5. (Transitivity 2(ii) of $\succsim_S$ is implied by that of $\succsim$).

(ii) There exists an affine non-constant function $u : \Delta(X) \to \mathbb{R}$, unique up to positive linear transformations, such that $(f, S) \mapsto V(u \circ f, S)$ represents $\succsim$ on Γ.

Remark. (a) The notation $(u \circ f, S)$ is self-explanatory. (b) Implicit in the theorem (4.2(ii)) is the fact that the preference relation $\succsim$ between conditional acts in Γ is complete. I.e., completeness is implied by other conditions of 4.2(i).

Proof of Theorem 4.2. The direction (ii) $\Rightarrow$ (i) is trivial (as in Theorem 3.3) and its proof is omitted.

We prove the direction (i) $\Rightarrow$ (ii). For any $S \in \Lambda$ and $y \in \Delta(X)$, denote by (y^*, S) the constant conditional act with $y^*(s) = y$ for all $s \in S$. The relation $\succsim_S$ induces a relation on $\Delta(X)$, also denoted by $\succsim_S$, defined by $y \succsim_S z$ iff $(y^*, S) \succsim_S (z^*, S)$, where $y, z \in \Delta(X)$. It is easy to see that to this relation we can apply Lemma 2.6, obtaining an affine non-constant $u_S : \Delta(X) \to \mathbb{R}$ such that $y \succsim_S z$ iff $u_S(y) \geq u_S(z)$.

We also have

4.2.1 CLAIM. *For all $y \in \Delta(X)$ and $R, T \in \Lambda$, $(y^*, R) \simeq (y^*, T)$.*

To prove this claim, apply repeatedly transitivity of $\succsim$ and irrelevance of duplications 4.1.1 (adding one row or column at a time) to show that both (y^*, R) and (y^*, T) are indifferent to $(y^*, (R^1 \cup T^1) \times (R^2 \cup T^2))$.

Claim 4.2.1 implies, by transitivity again, that for any $y, z \in \Delta(X)$ and $R, T \in \Lambda$, $(y^*, R) \succsim (z^*, R)$ iff $(y^*, T) \succsim (z^*, T)$. Hence by uniqueness of von Neumann-Morgenstern utility, u_R is a positive linear transformation of u_T. So we can choose an element from $\{u_S \mid S \in \Lambda\}$, say $u : \Delta(X) \to \mathbb{R}$, such that for $y, z \in \Delta(X)$, $u(y) \geq u(z)$ iff $(y^*, S) \succsim_S (z^*, S)$ for all $S \in \Lambda$.

To complete the proof, notice that by affinity of u and convexity of $\Delta(X)$, for any conditional act (f, S) there is $y \in \Delta(X)$ such that $u(y) = V(u \circ f, S)$. We will show that this in turn implies $(y^*, S) \simeq (f, S)$.

Let $q \in \Delta(S^2)$ be a minmax strategy of player 2 in the game $(u \circ f, S)$. Let $n = \#S^2$, denote the elements of S^2 as $s^2_1, s^2_2, \ldots, s^2_n$ and assume, without loss of generality, $q(s^2_1) > 0$. Let $T^2 = \{t^2_1, t^2_2, \ldots, t^2_n\} \subset \Theta^2$ be such that $T^2 \cap S^2 = \emptyset$. Define a conditional act $(g, S^1 \times (S^2 \cup T^2))$ as follows: $g(s) = f(s)$ for $s \in S$. For all $s^1 \in S^1 : g(s^1, t^2_1) = f(s^1, s^2_1)$, $g(s^1, t^2_2) = [q(s^2_1)/\sum_{j=1}^2 q(s^2_j)]g(s^1, t^2_1) + [q(s^2_2)/\sum_{j=1}^2 q(s^2_j)]f(s^1, s^2_2)$, $g(s^1, t^2_3) = [\sum_{j=1}^2 q(s^2_j)/\sum_{j=1}^3 q(s^2_j)]g(s^1, t^2_2) + [q(s^2_3)/\sum_{j=1}^3 q(s^2_j)]f(s^1, s^2_3), \ldots,$ $g(s^1, t^2_n) = [\sum_{j=1}^{n-1} q(s^2_j)]g(s^1, t^2_{n-1}) + q(s^2_n)f(s^1, s^2_n)$. By adding a column at a time and using 4.1(iii) and transitivity one gets $(f, S) \simeq (g, S^1 \times (S^2 \cup T^2))$. Eliminating the columns in $S^1 \times (S^2 \cup T^2)$ one at a time except the last one, t^2_n, and applying 4.1(ii) and transitivity one has $(g, S^1 \times \{t^2_n\}) \succsim (f, S)$. On the other hand, since for $s^1 \in S^1$, $g(s^1, t_n^2) = \sum_{s^2 \in S^2} q(s^2) f(s^1, s^2)$, by the **minimax theorem** $u(g(s^1, t_n^2)) \leq V(u \circ f, S) = u(y)$ for $s^1 \in S^1$, whence by monotonicity 2.4 $(y^*, S^1 \times \{t_n^2\}) \succsim (g, S^1 \times \{t_n^2\})$. By transitivity and claim 4.2.1 we then have $(y^*, S) \succsim (f, S)$. The parallel argument for rows, except again use of the minimax theorem, gives the reverse weak preference $(f, S) \succsim (y^*, S)$.

Given arbitrary conditional acts $(g, R), (h, T) \in \Gamma$, let $w, z \in \Delta(X)$ be such that $u(w) = V(u \circ g, R)$ and $u(z) = V(u \circ h, T)$.

Now suppose $V(u \circ g, R) \geq V(u \circ h, T)$. We will show that $(g, R) \succsim (h, T)$.

The inequality and the definitions just given imply: $V(u \circ w^*, R) = u(w) = V(u \circ g, R) \geq V(u \circ h, T) = u(z) = V(u \circ z^*, T)$. In turn this, claim 4.2.1 and the indifferences just proven imply: $(g, R) \simeq (w^*, R) \simeq (w^*, T) \succsim (z^*, T) \simeq (h, T)$.

On the other hand, weak inequalities and weak preferences can be replaced in the above arguments by their strict counterparts. Hence it is also true that if $(g, R) \succsim (h, T)$ then $V(u \circ g, R) \geq V(u \circ h, T)$. This concludes the proof. □

We presented the axiom of conditional SR-uncertainty 4.1 as a counterpart of axiom 3.2, SR-uncertainty, in the framework of conditional acts. In the following proposition we make explicit the formal relationship between the two axioms.

4.3 PROPOSITION. *Let a transitive binary relation $\succsim$ on Γ be given. Then the two following statements are equivalent:*

(i) The binary relation $\succsim$ on Γ satisfies conditional SR-uncertainty 4.1.

(ii) The binary relation $\succsim$ on Γ satisfies irrelevance of duplications 4.1.1, and for each $S \in \Lambda$ the induced binary relation $\succsim_S$ on Γ_S satisfies 3.2 SR-

uncertainty.

Proof: (We omit some details which are conceptually easy but notationally heavy to add.) "If": suppose $\succsim$ satisfies irrelevance of duplications and within each S SR-uncertainty. By the special case of the latter with $\alpha = 1$ (see Remark 3.2.1), replacing a row with another existing row makes the decision maker weakly worse off. By irrelevance of duplications we can eliminate one of the now two identical rows, obtaining 4.1(i). The same goes for columns (4.1(ii) from 3.2(ii) with $\alpha = 1$ and 4.1.1). To derive 4.1(iii), say for rows, i.e. for $i = 1$, let (f, S), α, r^1 and t^1 be given. Apply irrelevance of duplications twice to duplicate rows r^1 and t^1, obtaining an indifferent act. Then apply 3.2(i) for the given α and the added rows, obtaining a conditional act with two equal (new) rows which is weakly inferior to the original one. Eliminate one of the two new rows, obtaining the conditional act (f_1, T_1) of 4.1(ii), and observe that by irrelevance of duplications (f, S) is weakly preferred to it. Finally, apply 4.1(i) to obtain the weak preference in the opposite direction. This gives 4.1(iii) for rows, and again the parallel argument yields 4.1(iii) for columns.

"Only if": given conditional SR-uncertainty 4.1, we have already noticed that irrelevance of duplication is a special case of 4.1(iii) for $\alpha = 0$ or 1. We now prove SR-uncertainty for rows (3.2(i)). For any two rows and α, apply twice 4.1(iii) to add two identical rows each of which is the required convex combination. Then eliminate the two original rows, obtaining a weakly inferior conditional act, by 4.1(i). We now 'almost' have 3.2(i), in the sense that in the conditional act obtained the two original rows are 'empty' and the required convex combinations are in the newly created places. We then use 4.1.1 (already proved) to duplicate the new rows and put them in the 'empty' places, where they should be. Now we have two rows too many, which we just eliminate by 4.1.1 again, and this is 3.2(i). Once more, the analogous argument for columns gives 3.2(iii). □

Notice that in terms of the equivalent statements of Theorem 4.2 we have proved the following, which we state separately for the sake of completeness:

4.4 COROLLARY. *Condition (i) in Theorem 4.2 can be replaced by the following:*

(i′) The binary relation $\succsim$ on Γ is transitive and satisfies irrelevance of duplication 4.1.1, and for each $S \in \Lambda$ the induced binary relation $\succsim_S$ on Γ_S satisfies the conditions in statement (i) of Theorem 3.3 (i.e. completeness 2.1(i), C-independence 2.2, continuity 2.3, monotonicity 2.4, non-degeneracy 2.5 and SR-uncertainty 3.2).

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