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STAGGERED AND SYNCHRONIZED PRICE POLICIES UNDER INFLATION: THE MULTI–PRODUCT MONOPOLY CASE

by

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1. Introduction

The microeconomic background of an inflationary process is characterized by discrete jumps in individual prices. This observation has led to several studies on the aggregation of discrete pricing policies into a smooth time path for the aggregate price level. The feasibility of such aggregation is necessary for the overall consistency of individual pricing policies (Caplin and Spulber [1987]). A crucial issue for such an analysis is the interaction among individual price policies. If all firms follow identical real price cycles which are uniformly spread over time, then consistent aggregation is feasible (Sheshinski and Weiss [1977]). There may, however, be important reasons why such uniformity may not emerge as an equilibrium outcome. In oligopolistic markets, where each firm takes into account the actions of its rivals, pricing policies will be interdependent. In multiproduct monopolies, there is a further source for interdependence, namely, increasing returns in the cost of price adjustment. Even under competitive conditions, bunching over time may be caused by aggregate shocks, while idiosyncratic shocks are needed to maintain the spread.

Apart from the issue of consistent aggregation, the time pattern of individual price policies has important implications for the real costs of inflation. If individual price paths are staggered, then temporary shocks may be propagated over long periods. Synchronized price policies, on the other hand, may accelerate the adjustment process (see Blanchard [1983], Blanchard and Fischer [1989, Ch. 8], and Taylor [1980]). In addition, non-synchronized price policies lead to price variations across products and thereby to search costs incurred by consumers (Benabou [1987], Fishman [1987]).

In this paper we analyse the optimal price policy of a single profit maximizing decision maker, i.e., a multiproduct monopoly. This policy may be interpreted as the cooperative outcome of a duopoly game. Indeed, we view this analysis as a first step in the investigation of various non-cooperative equilibria of dynamic Bertrand duopoly games with differentiated products.

The main object of our paper is the determination of the conditions which lead to staggered or synchronized pricing policies, when the timing of price changes is endogenous.[2] Two aspects of the multiproduct monopoly decision problem influence this choice. First, the interaction in the profit function between the prices of the two goods. Generally, positive interactions enhance synchronization while negative interactions lead to staggering. Second, the form of the price adjustment costs. Here one may distinguish between '<u>menu costs</u>' and '<u>decision costs</u>'. Under menu costs, costs are independent of the number of items in the price list. This extreme form of increasing returns to scale ('economies of scope') leads to synchronization. Under decision costs, we consider a constant returns to scale technology, whereby each price change requires an adjustment cost. This provides an incentive for staggering, namely, the saving on the additional adjustment costs associated with joint price changes.

In this paper we devote our attention to the case of positive interactions and constant returns to scale in the costs of price adjustment. This choice is motivated by our interest in the duopoly problem, where these assumptions are likely to hold. A longer version (Sheshinski and Weiss [1989]) treats menu costs and negative or zero interactions.

Special consideration is given to steady-state (repetitive) pricing policies where the <u>same</u> real price is chosen at each adjustment. In staggered steady-states, price adjustments alternate. In a synchronized steady state, prices are adjusted simultaneously.

Our main results can be summarized as follows:

(1) The synchronized steady state and the symmetric staggered steady-state are unique.

(2) A positive rate of interest is required to sustain both types of equilibria under positive interactions. In particular, for the class of quadratic profit functions, when the rate of interest approaches zero, a staggered steady state is optimal if, and only if, the two prices are strategic substitutes, while a syncrhonized steady state is optimal if, and only if, prices are strategic complements.

(3) The synchronized steady state is locally stable. Specifically, if initial real prices are sufficiently close to each other, then a synchronized steady state is attained after the first price change. In addition, there is a broad class of initial conditions which lead to an immediate change in both prices, followed by a synchronized steady state.

(4) We provide a necessary and sufficient condition for the local stability of the staggered steady-state. For the class of quadratic profit functions, we show that the staggered steady-state is locally unstable. Moreover, under no circumstance will a joint price change be followed by a staggered steady state. That is, a staggered steady state can only be reached asymptotically (Sheshinski and Weiss [1989]).

(5) We derive explicit solutions for the case of quadratic profit functions when the rate of interest approaches zero. As in the single good

case, we find that an increase in the costs of adjustment or a reduction in the rate of inflation reduce the frequency of price changes. A stronger positive price interaction reduces the frequency of price changes in the synchronized steady-state.

The analysis in this paper applies beyond the price adjustment problem to other multiproduct inventory models. From this point of view, we extend the work by Bensoussan and Proth [1982] and Sulem [1986] who analysed an optimal reordering policy in a multiproduct case. Our work differs from theirs by allowing for interactions in demands. However, Sulem discusses a more general cost of adjustment structure.

2. The Model

Consider an economy subject to an inflationary trend where the aggregate price level grows at a constant rate, g $(g > 0)$. We analyse a monopoly who sells two related products whose demands depend on the current real prices of the two goods. The monopoly controls the nominal price of each good and there is a fixed real cost of nominal price adjustments.

Let $z_i(t)$ denote the log of the real price of good i at time t, $t \in [0,\infty)$. The real profit function of the monopoly, denoted by $F(z_1,z_2)$, is assumed to be time invariant and symmetric in its arguments, $F(a,b) = F(b,a)$. In addition, it is assumed to be strictly quasi-concave and twice differentiable for all (z_1,z_2) for which $F(z_1,z_2) > 0$. Naturally, we assume that $F(z_1,z_2) > 0$ for some (z_1,z_2). However, there exist $\underline{z}$ and $\bar{z}$ $(\bar{z} > \underline{z})$, such that $F(z_1,z_2) \leqslant 0$ for all (z_1,z_2) not satisfying $\underline{z} \leqslant z_i \leqslant \bar{z}$, $i = 1,2$. These assumptions imply the existence of a unique maximum for $F(z_1,z_2)$, which, by symmetry, satisfies $z_1 = z_2 = \bar{S}$ and $F(\bar{S},\bar{S}) > 0$. The assumption that positive profits are attained on a compact set of real prices is intended to ensure the existence of a well-defined pricing policy. The set of prices for which profits are non-positive need not be compact if the firm has the option of non-production at prices below variable costs. The class of functions satisfying all of these conditions is denoted $\mathcal{F}$.

The problem facing the monopoly is a choice of price paths, $(z_1^*(t), z_2^*(t))$, which maximize the present value of real profits over an infinite horizon, given some initial condition $(z_1(0), z_2(0))$.

The salient feature of our model is the discontinuous pattern of nominal price adjustments. This widely observable phenomenon is generated in our model by the presence of non-convex costs of price adjustment: any nominal price change, no matter how small, requires non-negligible costs of adjustment. Specifically, the real cost of any nominal price change is assumed to be a constant denoted by β ($\beta > 0$).

The main question which the paper addresses is the following: will the monopoly adopt a <u>synchronized</u> policy of price adjustments, whereby both prices are changed simultaneously, or a <u>staggered</u> policy whereby the two nominal prices are changed at different points in time.

Any pricing policy can be described by two pairs of sequences, $\{S_\tau^i\}_{\tau=0}^\infty$ and $\{t_\tau^i\}_{\tau=0}^\infty$, $i = 1,2$, where S_τ^i is the real price of good i set at time t_τ^i by adjusting the nominal price of good i, keeping it unchanged during the interval $[t_\tau^i, t_{\tau+1}^i)$. Special attention will be given to repetitive price paths satisfying

$$S^i_{\tau+1} = S^i_\tau \quad \text{and} \quad t^i_{\tau+1} = t^i_\tau + \varepsilon^i , \qquad \tau = 0,1,2,... \tag{1}$$

where ε^i $(\varepsilon^i > 0)$, $i = 1,2$, are constants denoting the time intervals between subsequent price changes. Thus, on such paths, the real prices chosen at the beginning of each interval and the duration until the next price change remain constant. We shall refer to such paths as steady-states. A symmetric steady-state is defined by the additional restriction

$$S^1_\tau = S^2_\tau = S \quad \text{and} \quad \varepsilon^1 = \varepsilon^2 = \varepsilon , \qquad \tau = 0,1,2,... \tag{2}$$

where S $(S > 0)$ and ε $(\varepsilon > 0)$ are constants. Along such a path, the real price of each good follows the same cycle. Among the symmetric steady-states we can identify a synchronized steady-state by the added requirement that

$$t^1_0 = t^2_0 , \tag{3}$$

that is, the prices of both goods are always changed at the same time. Finally, a (symmetric) staggered steady-state is defined by

$$|t^1_0 - t^2_0| = \frac{\varepsilon}{2} , \tag{4}$$

that is, the prices of the two goods are changed alternately and the time distance between any two price changes is equal.

The time pattern of the monopolist's optimal price policy, in particular, whether price changes will be synchronized, depends crucially on two features of the model. The first relates to the technology of price adjustments, and the second to the form of the profit function. One issue of concern is the degree of returns to scale when both prices are changed simultaneously. Under constant returns to scale in the costs of price adjustment the monopoly incurs a cost of 2β whenever prices are changed jointly. Under increasing returns to scale these costs will be less than 2β , possibly as low as β . The degree of returns to scale depends on the distinction between '*menu costs*' and '*decision costs*' of price adjustment. By menu costs we refer to costs such as advertizing and updating of price lists. By decision costs we refer to costs of acquiring information on the production and demand of different products and to costs related to the organization and computation of coordinated price changes in multiproduct firms. If the costs of price adjustment are interpreted as menu costs, one would expect these costs to be β , independently of the number of items in the menu. If, however, these costs are interpreted as decision costs, one would expect that the complexity of the choice, and thus the costs, will depend on the number of items involved, suggesting that constant returns to scale is the more appropriate assumption. Indeed, a typical organizational solution to this problem is decentralization, whereby separate divisions are allowed to follow separate pricing policies, maximizing objective functions set by the center. The overall outcome of this process is that adjustment costs for the monopoly are the sum of the costs incurred by the separate 'price centers'. A similar distinction in the inventory adjustment context was made by Sulem [1986].[3]

The other issue of concern is the interaction in demand and possibly in the production of the two goods. In general, an increase in one price may increase or decrease the marginal profitability of an increase in the other price. For instance, in the absence of costs, a positive (negative) interaction arises when an increase in one price raises (reduces) both the quantity demanded and the slope of the demand curve for the other good. One would expect that if the goods are *strategic complements*, i.e., raising z_i increases the marginal profits of z_j , $j \neq i$, then synchronization is more likely, and *vice versa*.

The focus of this paper is on the case of constant returns to scale in the costs of adjustment and positive price interactions. These assumptions and symmetry appear to be more appropriate for the duopoly case. We retain them in the analysis of the multiproduct monopoly to support the interpretation of this model as a cooperative duopoly equilibrium. Specifically, we assume

A1. Complementarity

For any (z_1, z_2) ,

(5) $$F_{ij}(z_1, z_2) > 0 , \qquad i \neq j , \quad i,j = 1,2 .$$

An additional assumption which will be used in subsequent analysis is:

A2. Non-Reversibility

For any (z_1, z_2) and x $(x > 0)$,

(6) $$F_i(z_1, z_2) > 0 \implies F_i(z_1 - x, z_2 - x) > 0 , \qquad i = 1,2 .$$

Assumption A2 imposes the natural requirement that if a price increase is profitable at (z_1, z_2) , then it is also profitable after these real prices are eroded by inflation to $(z_1 - x,\ z_2 - x)$.

In some cases we will need a stronger version of A2:

A3. Monotonicity

For any (z_1, z_2) ,

(7) $$F_{ii}(z_1, z_2) + F_{ij}(z_1, z_2) < 0 , \qquad i \neq j , \quad i,j = 1,2$$

A3 in conjunction with A1, ensure that over any time interval with fixed nominal prices, the profitability of a price increase rises with time. Observe also that A1 and A3 imply that $F(z_1,z_2)$ is strictly concave. On the other hand, A2 and A1 do not imply concavity.

3. Characterization of the Optimal Policy and the Associated Value Function

Let $V(z_1,z_2)$ be the value function associated with an optimal policy starting at real prices (z_1,z_2) at time 0. The existence of such a function is guaranteed by our assumption that $F(z_1,z_2)$ has a well-defined maximum and by assuming that the real interest rate, r, is positive. The value function is defined recursively:[4]

$$(8) \qquad V(z_1,z_2) = \underset{t \geq 0}{\text{Max}}\Big\{\int_0^t e^{-rx}F(z_1-gx,\ z_2-gx)dx + \\ + e^{-rt}\text{Max}\big[\underset{S_1,S_2}{\text{Max}}\ V(S_1,S_2) - 2\beta,\ \underset{S_1}{\text{Max}}\ V(S_1,z_2-gt) - \beta, \\ \underset{S_2}{\text{Max}}\ V(z_1-gt,S_2) - \beta\big]\Big\} .$$

where t is the time of the subsequent price change and (S_1,S_2) are the real prices chosen at that time (i.e., nominal prices are set so as to attain these real prices). If the optimal t is $t = 0$, then a price change occurs immediately; otherwise the current nominal prices will be kept unchanged, with real prices decreasing at the rate of inflation, g, over the interval $[0,t)$. For any initial (z_1,z_2), a well-defined price change is optimal after a finite lapse of time. That is, the R.H.S. of (8) actually achieves the maximum (see Appendix A).

We begin our analysis by stating some properties of the value function which will be used subsequently:

(i) $\frac{1}{1 - e^{-r(\bar{z}-\underline{z})/g}} [\int_0^{(\bar{z}-\underline{z})/g} e^{-rx} F(\bar{z}-gx,\ \bar{z}-gx)dx - 2\beta] \leqslant V(z_1,z_2) \leqslant \frac{F(\bar{S},\bar{S})}{r}$,

(ii) $V(z_1,z_2)$ is symmetric,

(iii) $V(z_1,z_2)$ is continuous,

(iv) $V(z_1,z_2)$ is differentiable, except possibly at some boundary points.

The upper and lower bounds on $V(z_1,z_2)$ can be easily demonstrated. The upper bound is the present dicounted value of the flow of maximum profits, $F(\bar{S},\bar{S})$, which would be attained in the absence of adjustment costs, $\beta = 0$. The lower bound is the present discounted value of a feasible repetitive policy where real prices vary between $\bar{z}$ and $\underline{z}$. (Recall that outside these bounds profits are negative.) We assume throughout that costs of price adjustment are relatively small, specifically, that $\int_0^{(\bar{z}-\underline{z})/g} e^{-rx} F(\bar{z}-gx,\ \bar{z}-gx)dx - 2\beta > 0$. This ensures that for any initial condition, $V(z_1,z_2) > 0$.

Symmetry of $V(z_1,z_2)$ follows directly from the assumed symmetry of the profit function, $F(z_1,z_2)$. Starting from $z_1 = a$ and $z_2 = b$ or $z_1 = b$ and $z_2 = a$, the monopoly can obtain the same present value of future profits simply by exchanging the optimal price sequences of the two products.

Continuity of $V(z_1,z_2)$ can be established by noting that (8) is a fixed point of a contraction mapping which maps continuous functions into continuous functions (see Stokey and Lucas [1989], ch. 3, pp.49-55).

Differentiability of $V(z_1,z_2)$ can be established whenever the choice of the controls in (8) is unique and thus continuous in (z_1,z_2) . In Appendix A we show that after the first price change, the time of the subsequent price increase and the value of the real prices chosen at that time are uniquely determined.

Let

$$M(z_1,z_2) = \text{Max}\{V^* - 2\beta,\ \underset{S_1}{\text{Max}}\ V(S_1,z_2) - \beta,\ \underset{S_2}{\text{Max}}\ V(z_1,S_2) - \beta\} \tag{9}$$

where $V^* = \underset{S_1,S_2}{\text{Max}}\ V(S_1,S_2)$.

Since it is always feasible to change prices immediately, that is, to set $t = 0$ in (8), we have

$$V(z_1,z_2) \geqslant M(z_1,z_2) . \tag{10}$$

Similarly, since it is always feasible not to change any price in the time interval $[0,t)$, we must have

$$V(z_1,z_2) \geqslant \int_0^t e^{-rx}F(z_1-gx,z_2-gx)dx + e^{-rt}V(z_1-gt,\ z_2-gt) . \tag{11}$$

Equation (11) must hold for all $t \geqslant 0$. Expanding the R.H.S. by a Taylor expansion, we can rearrange (11), divide by gt , and take the limit as $t \to 0$, to obtain

$$F(z_1,z_2) \leqslant g\nabla V(z_1,z_2) + rV(z_1,z_2) \tag{12}$$

where ∇V , the directional derivative of V , is defined

$$\nabla V(z_1,z_2) = \lim_{t\to 0} \frac{V(z_1,z_2) - V(z_1-gt,\ z_2-gt)}{gt} . \tag{13}$$

Clearly, when $V(z_1,z_2)$ is differentiable, then $\nabla V(z_1,z_2) = = V_1(z_1,z_2) + V_2(z_1,z_2)$. It can be shown (see Sulem [1986] and Bensoussan, Crouhy and Proth [1983]) that inequalitites (10) and (12) are related by the complementary slackness condition

$$(14) \qquad [V(z_1,z_2) - M(z_1,z_2)]\,[g\nabla V(z_1,z_2) + rV(z_1,z_2) - F(z_1,z_2)] = 0 .$$

The solution of the monopoly's problem is now described with the aid of four distinct sets:

$$(15) \qquad \begin{aligned} C &= \{z_1,z_2 \mid V(z_1,z_2) > M(z_1,z_2)\} \\ T_0 &= \{z_1,z_2 \mid V(z_1,z_2) = M(z_1,z_2) = V^* - 2\beta\} \\ T_1 &= \{z_1,z_2 \mid V(z_1,z_2) = M(z_1,z_2) = \operatorname*{Max}_{S_1} V(S_1,z_2) - \beta\} \\ T_2 &= \{z_1,z_2 \mid V(z_1,z_2) = M(z_1,z_2) = \operatorname*{Max}_{S_2} V(z_1,S_2) - \beta\} \end{aligned}$$

The set C is the <u>continuation set</u>, where no price change occurs. The set T_0 <u>triggers a change in both prices</u>, while T_i , $i = 1,2$, is the set which triggers a change in the price of good i only.

Condition (14) implies that for $(z_1,z_2) \in C$, we have

$$(16) \qquad rV(z_1,z_2) = F(z_1,z_2) - g\nabla V(z_1,z_2) .$$

Equation (16) can be interpreted as an asset pricing formula. The imputed value of a state which does not generate a price change, $rV(z_1,z_2)$, is given by the current flow of profits, $F(z_1,z_2)$, less the depreciation caused by the inflationary erosion in real prices, $g\nabla V(z_1,z_2)$. In subsequent analysis we shall refer to equation (16) as the '<u>valuation formula</u>'.

With each point in the trigger sets T_0 , T_1 and T_2 is associated a choice of an optimal pair of new real prices. Specifically, for any $(z_1,z_2) \in T_i$, $i = 1,2$, there is a <u>unique</u> real price chosen for good i ,

S_i^* , whose value depends on z_j , $j \neq i$. We write $S_i^* = S(z_j)$, $j \neq i$, $i,j = 1,2$. These '<u>reaction functions</u>' are symmetric and stationary (see Appendix A). In contrast, for (z_1,z_2) T_0 , in view of the symmetry imposed by our assumptions, if (S_1^*,S_2^*) is an optimal choice, so is (S_2^*,S_1^*) . Hence, in general, uniqueness cannot be expected. It is, however, easy to show that with <u>positive</u> interactions, $F_{12} > 0$, any point $(z_1,z_2) \in T_0$ triggers a unique action $S_1^* = S_2^* = S_a^*$. To see this, observe that for a symmetric profit function with positive interactions

$$(17) \qquad F(z_1,z_2) \leqslant \frac{1}{2} F(z_1,z_1) + \frac{1}{2} F(z_2,z_2) = f(z_1) + f(z_2)$$

for all (z_1,z_2) , where $f(z_i) \equiv \frac{1}{2} F(z_1,z_2)$, $i = 1,2$.[5] That is, a mixture of the profits at the extremes exceeds profits at any midpoint. Hence, the value of the optimal program associated with an additive profit function of the form $f(z_1) + f(z_2)$ provides an upper bound for the optimal value of the program associated with $F(z_1,z_2)$. An optimum for an additive profit function is attained when an (S,s) policy is followed for each of the two goods (see Sheshinski and Weiss [1989]). When the initial conditions are subject to choice the firm can attain this upper bound by selecting at time 0 the same real price, S_a^* , for both goods, followed by a synchronized (S,s) policy for both goods thereafter.

The chosen pair of real prices triggered by $(z_1,z_2) \in T_0$, must be in the interior of C . Clearly, an immediate subsequent price change cannot be optimal since the same outcome can be obtained without incurring the additional adjustment cost. For the same reason, with $(z_1,z_2) \in T_i$, $i = 1,2$, the chosen prices cannot be in T_i . Nor can the chosen prices be in T_j , $j \neq i$, unless (z_1,z_2) is also in T_0 , in which case a joint change into C is triggered. Moreover, at the chosen point, $V(z_1,z_2)$ is differentiable. This follows from the fact that the subsequent optimal date of price adjustment, $t^*(z_1,z_2)$, and the real prices chosen at that time are uniquely determined (see Appendix A). Accordingly, setting $V_1(S_a^*,S_a^*) = V_2(S_a^*,S_a^*) = 0$ in the 'valuation fomula', (16), we obtain the following equation:

(18) $$rV(S_a^*,S_a^*) = F(S_a^*,S_a^*) .$$

At the time of a joint price change, current profits reflect the full imputed value of the new state, since depreciation is locally negligible.

However, when only one price is chosen optimally, the depreciation of the other price has to be taken into account. Specifically, we have

(19) $$rV(S_1^*,z_2) = F(S_1^*,z_2) - gV_2(S_1^*,z_2) ,$$

and

(20) $$rV(z_1,S_2^*) = F(z_1,S_2^*) - gV_1(z_1,S_2^*) .$$

Equations (19) and (20) are obtained from (16) by setting $V_1(S_1^*,z_2) = 0$ and $V_2(z_1,S_2^*) = 0$, respectively.

Recall that points in the trigger sets are related to the corresponding chosen points via the relationship $V(z_1,z_2) = M(z_1,z_2)$. Thus, if (z_1,z_2) is in the interior of, say, T_1, we have $V(z_1,z_2) = V(S(z_2),z_2) - \beta$.

Since this relationship holds for all (z_1,z_2) in the interior of T_1, we can differentiate to obtain

(21) $$V_1(z_1,z_2) = 0$$

and

$$V_2(z_1,z_2) = V_2(S(z_2),z_2) \ . \tag{22}$$

Thus, using equations (12) and (19), we obtain

$$F(z_1,z_2) \leqslant F(S(z_2),z_2) - r\beta \ , \quad \text{for} \quad (z_1,z_2) \in T_1 \ . \tag{23}$$

By a similar argument:

$$F(z_1,z_2) \leqslant F(z_1,S(z_1)) - r\beta \ , \quad \text{for} \quad (z_1,z_2) \in T_2 \tag{24}$$

$$F(z_1,z_2) \leqslant F(S_a^*,S_a^*) - 2r\beta \ , \quad \text{for} \quad (z_1,z_2) \in T_0 \ . \tag{25}$$

The economic interpretation of equations (23)-(25) is clear. The R.H.S. of each equation is the cost of a delay in a price change, consisting of foregone profits at the new real prices net of adjustment costs, while the L.H.S. is the benefit of such delay, consisting of profits at the old prices. To trigger an immediate price change it is necessary (though not sufficient) for a short delay to be unprofitable.

FIGURE 1

Figure 1 describes the continuation and trigger sets and the corresponding choice (reaction) functions, $S(z)$. Observe, first, that T_0 consists of four distinct subsets: For all $(z_1,z_2) \in T_0^{++}$, a price-increase of both goods to (S_a^*,S_a^*) is triggered. For all $(z_1,z_2) \in T_0^{--}$, a reduction of both prices is triggered, while for $(z_1,z_2) \in T_0^{-+}$ (or $\in T_0^{+-}$), one price is increased and the other decreased. Similarly, each T_i, $i = 1,2$, consists of two regions, one, denoted by T_i^+, calls for a price increase in good i, while T_i^- calls for an immediate price reduction.

Generally, a joint price change is triggered whenever both real prices are distant from the level, $(\bar{S},\bar{S})$, which yields maximum profits. A single price change is triggered whenever one price is distant from $\bar{S}$ while the other is close. Continuation occurs when both prices are close to $(\bar{S},\bar{S})$.

Having assumed a positive inflation, $g > 0$, price reductions can occur only once, at the outset $(t = 0)$. In contrast, the regions T_0^{++} and T_i^{+}, $i = 1,2$, are reachable from C after some delay and revisited periodically. At any point on the boundaries of T_i^{+} and T_0^{++}, which are reachable from the reaction curves $S(z_1)$ and $S(z_2)$, $V(z_1,z_2)$ is differentiable. As shown in Appendix A, following a price increase, the timing and the prices chosen for the subsequent change are uniquely determined, which implies that V is differentiable along a path starting at any point on $S(z_1)$ or $S(z_2)$. Using the 'valuation formula', (16), which holds in C, it follows that (23)-(25) hold with equality at points where such paths reach the boundaries between C and the trigger sets. This observation helps to determine the boundaries between the continuation and the trigger sets. For instance, at the boundary between C and T_1^{+}, where an increase in the price of good one is triggered, the firm is indifferent between holding nominal prices unchanged, obtaining the current level of profits, $F(z_1,z_2)$, and raising the price of good one to $S(z_2)$, obtaining $F(S(z_2),z_2) - r\beta$. The term $r\beta$ is the imputed interest on adjustment costs. We denote by $s(z_i)$ the boundary points between C and T_j^{+}, i.e. points which trigger and immediate increase in z_j, $j \neq i$, $i,j = 1,2$ (see Figure 1).

Somewhat different considerations apply to price reductions. For instance, at the boundary between T_0^{--} and C, point k in Figure 1, the firm is indifferent between an immediate price reduction to (S_a^*,S_a^*) and holding both prices unchanged until their real values erode to point f on the boundary between C and T_0^{++}. This discontinuity in the action leads to non-differentiability of $V(z_1,z_2)$ at point k. On the diagonal above point k, $V_1(z,z) = V_2(z,z) = 0$. However, on the diagonal below this point, the gradient must be strictly negative.[6] Let $\hat{S}(z_i)$ denote the boundary points between C and T_j, i.e. points which trigger an immediate decrease in z_j, $j \neq i$, $i,j = 1,2$. Generally, $V(z_1,z_2)$ is not differentiable along this boundary.

We now turn to a description of the boundary between C and T_0. This boundary is determined by the condition $F(z_1,z_2) = F(S_a^*,S_a^*) - 2r\beta$. That is the firm is just indifferent between raising both prices to (S_a^*,S_a^*) and holding them unchanged instantly. In particular, consider point h in Figure 1, whose coordinates are $(z_1,s(z_1))$. This point is on the boundary of C, T_0^{++} and T_2^{+}. At such a point, changing only z_2 to $S(z_1)$ is equivalent to changing both prices to (S_a^*,S_a^*). Consistency requires that $S(z_1) = S_a^*$ and, in addition, following the change in z_2, the firm should be willing to change z_1 immediately. That is, the best response to $(z_1,s(z_1))$ is (z_1,S_a^*), which in turn has to be a point in T_1^{+}. Furthermore, (z_1,S_a^*) must be on the boundary of C and T_1^{+}, for otherwise points in the interior of T_2^{+} would also lead to a joint price increase. Thus,

(26) $$F(z_1,s(z_1)) = F(z_1,S_a^*) - r\beta ,$$

and

(27) $$F(z_1,s(z_1)) = F(S_a^*,S_a^*) - 2r\beta ,$$

which implies that

(28) $$F(z_1, S_a^*) = F(S_a^*, S_a^*) - r\beta \ .$$

Under the assumption of positive interactions, $F_{12} > 0$, and using the quasi-concavity of $F(z_1, z_2)$, conditions (26) and (28) imply that $z_1 > s(z_1)$. That is, point h must be strictly to the right of the diagonal. Consequently, there is a non-degenerate segment (gh) on the boundary between C and T_0^{++}, where a joint price increase is strictly preferable to a single price increase. This feature is special to the case of positive interactions. The segment gh degenerates to a single point when $F_{12} = 0$ and disappears when $F_{12} < 0$ (see Sheshinski and Weiss [1989]). Intuitively, under positive interactions, when the two prices are not too far from their maximum profit position, an anticipated change in the price of good j at some future date, $t^*(z_1, z_2)$, creates an incentive to postpone the change in the price of good i , $i \neq j$, to that same date.

It can be shown that with positive interaction in the profit function, the reaction curves must have a positive slope in the neighborhood of (S_a^*, S_a^*) . That is, an increase in z_i leads to a higher chosen real price for good j, i.e., $S'(S_a^*) > 0$.[7] Finally, it can be shown that $S'(z) < 1$ for all z , and that whenever $V(z_1, z_2)$ is differentiable at the boundary, s(z), then $s'(z) < 1$.[8] The former is the local stability condition in the static Bertrand model. The latter is consistent with the requirement that any path emanating from S(z) intersects a trigger set once.

Any path satisfying the first-order conditions can be portrayed by a trajectory which moves smoothly inside the continuation set where both real prices erode at the same rate, g , and then jumps to the reaction curves whenever the corresponding trigger sets are met. In Figure 2 we present three such paths. Consider first the repetitive paths, indicated by (e,f) and (a,b,c,d) . The first represents a synchronized steady-state and the latter a (symmetric) staggered steady-state.

FIGURE 1

The conditions determining these repetitive paths are as follows. A synchronized steady state is characterized by a pair, (S_a^*, ε) , satisfying:

$$(30) \qquad F(S_a^*, S_a^*) = rV(S_a^*, S_a^*) = \frac{r}{1 - e^{-r\varepsilon}} \left[\int_0^{\varepsilon} e^{-rx} F(S_a^* - gx,\ S_a^* - gx)dx - 2\beta e^{-r\varepsilon}\right]$$

and

$$(31) \qquad F(S_a^* - g\varepsilon,\ S_a^* - g\varepsilon) - F(S_a^*, S_a^*) + 2r\beta = 0 \ ,$$

where S_a^* is the initial level of real prices at the beginning of a cycle and ε is the duration of each cycle.

The first equality in condition (30) follows, using the 'valuation formula', (16), from the maximization of V w.r.t. the chosen initial prices. The second equality is derived from the definition of V , (7), under a stationary policy. Condition (31) is necessary for the optimality of the timing of a price change.

Similarly, a symmetric staggered steady state is characterized by a pair (S_b^*, t) satisfying

$$(32) \qquad F(S_b^*,\ S_b^* - gt) = rV(S_b^*,\ S_b^* - gt) + gV_2(S_b^*,\ S_b^* - gt) =$$

$$= \frac{r}{1 - e^{-rt}} \left[\int_0^t e^{-rx} F(S_b^* - gx,\ S_b^* - gt - gx)dx - \beta e^{-rt}\right] +$$

$$+ g\int_0^t e^{-rx} F_2(S_b^* - gx,\ S_b^* - gt - gx)dx \ ,$$

and

(33) $$F(S_b^*-2gt,\ S_b^*-gt) - F(S_b^*,\ S_b^*-gt) + r\beta = 0\ ,$$

where S_b^* is the level of the real price chosen when only one price is raised and t is the time interval between successive price changes (i.e. the duration between subsequent price changes of the same good is $2t$).

Using integration by parts, (30) and (31) imply

(34) $$\int_0^{\varepsilon} e^{-rx} F_1(S_a^*-gx,\ S_a^*-gx)dx = 0\ .$$

Similarly, (32) and (33) imply

(35) $$\int_0^{t} e^{-rx} F_1(S_b^*-gx,\ S_b^*-gt-gx)dx + \int_t^{2t} e^{-rx} F_1(S_b^*-gx,\ S_b^*+gt-gx)dx = 0\ .$$

The above conditions can be easily interpreted: in steady-state, the discounted value of marginal profits over a typical cycle is zero. In addition, at a point of price change, marginal benefits and losses from postponing the price adjustment are equal.

Under assumptions A1 and A2, it follows from equations (34) and (35), that marginal profits, F_i $(i = 1,2)$, are negative at the beginning and positive at the end of each price interval. Thus, a typical price cycle starts at a price which exceeds the level which maximizes profits and terminates at a price below that level. Such oscillations strike a balance between the loss in profits and the costs of price adjustment. It can be shown that the same pattern also obtains outside steady-state (Sheshinski and Weiss [1989], Lemma 2).

Not every repetitive path that satisfies conditions (30) and (31) or (32) and (33) is optimal. It is easy to find examples which satisfy equations (32)-(33) and are sub-optimal (see Section 6). In terms of Figure 2, a non-optimality of the staggered steady-state will be indicated by tangency of this path with the boundaries of C and T_i, $i = 1,2$. Such tangency implies that continuation towards the boundary of C with T_0 also satisfies the first-order conditions. The choice between these alternative policies cannot be determined from the figure alone.

It is easy to check that both paths (e,f) and (a,b,c,d) satisfy the first order conditions at all times, provided that the initial state is on one of these paths. That is, the staggered and synchronized steady states, defined in Section 2, provide a solution of the first order conditions for some S, ε and t. In Appendix B we prove the following:

Proposition 1. There is a unique optimal synchronized steady-state. Under A1 and A2, there is a unique optimal symmetric staggered steady-state.

The proof of uniqueness of the two steady states relies on the assumption that these are optimal. Thus, in addition to (30)-(33), we assume that the appropriate second-order conditions hold.

4. Stability Analysis

We have identified two repetitive paths which satisfy the necessary conditions for an optimum, provided the initial conditions are on one of these paths. For other initial conditions, the optimal path is, in general, non-repetitive. That is, different real prices are chosen at successive points of price change and the time intervals between such changes. This dynamic adjustment reflects history-dependence. That is, the optimal level for the new price of good i depends on the real price of good j, $j \neq i$. Of course, history does not matter if it is optimal to change both prices.

It is thus natural to inquire whether such dynamic paths converge asymptotically to a steady-state. We restrict our inquiry to paths that initiate in the neighborhood of one or the other steady-state and provide two local-stability results.

Proposition 2. The synchronized steady-state is locally stable.

Proof: Starting in the neighborhood of the synchronized steady-state path, it is seen in Figure 1 that the trajectory reaches a point on the boundary of T_0^{++} on the nondegenerate segment gh. This triggers a joint price increase to the synchronized steady-state at point e. ||

In Appendix B we provide a necessary and sufficient condition for local stability of the staggered steady-state. This condition depends, in general, on the values of (S_b^*, t) which solve equations (32)-(33). For specific profit functions, however, these conditions can be verified a-priori. In particular, consider the class of (symmetric) quadratic profit functions:

$$F(z_1, z_2) = a(z_1+z_2) - b(z_1^2+z_2^2) + cz_1z_2 , \tag{36}$$

where, to guarantee strict quasi-concavity, we set $b > 0$ and $4b^2 - c^2 > 0$. For this important class of functions we prove the following:

Proposition 3. If $F(z_1, z_2) \in \mathcal{F}$ is quadratic, given by (36), then the staggered steady-state is locally unstable.

<u>Proof</u>: Appendix C.

The path a'b'c'd' in Figure 2 illustrates the local instability of the staggered steady-state. Starting at an initial point near the staggered steady-state path, a' , the optimal trajectory converges to the synchronized steady-state.

Propositions 2 and 3 present a sharp contrast between the two types of steady-state. For positive interactions, $F_{12} > 0$, the staggered path is followed only if the initial price configuration is on this path. On the other hand, the synchronized steady-state is attained from a wide range of initial conditions, i.e., an immediate jump to the synchronized steady-state path occurs whenever the two prices are far away from their maximum profits level. If prices are not too far from that point and from each other, then it is optimal to postpone the jump to the synchronized steady-state. In contrast, if only one price is far away from its profit maximizing level, then this price will be changed immediately, but a steady-state does <u>not</u> follow. Instead, the firm will adopt a non-repetitive path. We have seen that with quadratic profits, this path does not converge to a staggered steady-state. Thus, it remains to be determined whether it converges to the synchronized steady-state. This is a question of <u>global stability</u>.

The analysis of global stability requires a complete solution for the value function. An approximate solution can be obtained from the knowledge of $V(z_1,z_2)$ around the staggered and synchronized steady-states (see Sheshinski and Weiss [1989]).

5. An Example

In this section we derive, for certain cases of our model, explicit solutions to the steady-state values in terms of the underlying parameters of the model and provide some comparative statics. For this purpose we use two simplifying assumptions common in the related literature on optimal inventory policy (e.g. Costantinides and Richard [1978] and Sulem [1986]).

The first assumption is a quadratic instantaneous profit function, of the form (36). The second assumption is the maximization of average profits per price cycle, which is equivalent to the limit of discounted real profits in steady-state when the rate of interest approaches zero.

Consider a synchronized steady-state path satisfying equations (30)-(31). These necessary conditions implicitly define S_a^* and ε as functions of the rate of interest, r. The limit of these values as $r \to 0$ must satisfy

$$a - (2b-c)(S_a^* - g\,\frac{\varepsilon}{2}) = 0 , \tag{37}$$

and

$$\varepsilon = \left[\frac{12\beta}{g^2(2b-c)}\right]^{\frac{1}{3}} . \tag{38}$$

These equations are obtained by taking the limits of (30)-(31) as $r \to 0$. The limit of $rV(S_a^*,S_a^*)$ as $r \to 0$, denoted μ_1, is

$$\mu_1 = \frac{1}{\varepsilon}\left[\int_0^{\varepsilon} F(S_a^*-gx,\ S_a^*-gx)dx - 2\beta\right] , \tag{39}$$

where $F(z_1,z_2)$ is given by (36). The R.H.S. of (39) is seen to be the average profits associated with the limit price cycle.

Similarly, consider a staggered steady-state path satisfying equations (32)-(33). As $r \to 0$, the solution (S_b^*,t) approaches a limit satisfying

$$a - (2b-c)(S_b^* - gt) = 0 , \tag{40}$$

and

$$t = \left[\frac{3\beta}{g^2(4b-c)}\right]^{\frac{1}{3}} . \tag{41}$$

The corresponding limit of $rV(S_b^*, S_b^*-gt)$ as $r \to 0$, denoted μ_2, is

$$\mu_2 = \frac{1}{t}\left[\int_0^t F(S_b^*-gx, S_b^*-gt-gx)dx - \beta\right] , \tag{42}$$

where $F(z_1,z_2)$ is given by (36).

As we have already indicated in Section 3, conditions (30)-(31) and (32)-(33) are necessary but, in general, not sufficient for optimality. To determine the optimality of (S_a^*,ε) and (S_b^*,t), we shall refer to:

Proposition 4. If $F(z_1,z_2) \in \mathcal{F}$ is a quadratic function given by (36), then

$$\mu_1 - \mu_2 \gtrless 0 \Longleftrightarrow F_{12} = c \gtrless 0 . \tag{43}$$

Proof: Appendix D.

Proposition 4 implies that with positive interactions, $F_{12} = c > 0$, average profits associated with a synchronized price policy exceed average profits associated with a staggered policy. It follows that in this case, whenever a price change is contemplated, a move to the synchronized steady-state which costs 2β, will dominate a move to the staggered steady-state, which costs only β. The reason is that the undiscounted difference between the value of the two programs exceeds any finite β. We therefore conclude:

<u>Corollary 1:</u> For a quadratic profit function, (36), with positive interaction, $F_{12} = c > 0$, the staggered steady-state is never optimal whenever $r \to 0$.

Corollary 1 is not true if $r > 0$, since in this case the gain from moving to the synchronized steady-state may be outweighed by the additional costs of adjustment.

Expressions (37)-(38) and (40)-(41) provide a convenient framework for comparative statics. It is seen that in both steady-states, an increase in the costs of adjustment, β , or a reduction in the rate of inflation, g , increase the duration between successive price changes. For the synchronized steady-state, an increase in the concavity of the profit function, i.e. an increase in $2b-c$, decreases the duration of fixed nominal price intervals. The reason is that as profits decline faster around the maximum it becomes more costly to keep the nominal price unchanged. It is seen from (38) that a stronger positive interaction, i.e. higher c , reduces the frequency of price changes in the synchronized steady-state. Similarly, as seen from (41), a stronger negative interaction, i.e. a decrease in c , reduces the frequency of price changes in the staggered steady-state. The effects on the initial real price, S_a^* and S_b^* , are generally in the same direction as the effect on duration, as seen in equations (37) and (40). This is also true with regard to the effect of an increase in g on initial prices. This is because the elasticities determined by (38) and (41) are less than unity. The terminal prices, $S_a^* - g\varepsilon$ and $S_b^* - 2gt$ in the synchronized and staggered steady-states, respectively, change in oppposite direction to ε or t .

For additive profits, $c = 0$, (38) and (41) imply that $\varepsilon = 2t$ and, using (37) and (40), that $S_a^* = S_b^*$. Thus, the synchronized and the symmetric staggered steady-states have the same upper and lower bounds on real prices.

Along the staggered path, price adjustments occur alternatingly at a frequency twice as high as that of joint price adjustment in the synchronized steady-state.

6. Conclusions and Their Robustness to Alternative Assumptions

The main finding of this paper is that, with positive interactions, a staggered price policy is unlikely. Since such a policy has been assumed in support of consistent aggregation (i.e., a continuous path for the aggregate price level associated with discrete price changes by individual firms) this negative result is worrisome. It is therefore important to examine its sensitivity to our assumptions concerning the costs of price adjustment and the interactions of the two prices in the monopolist's profit function.

Increasing returns to scale in the costs of price adjustment, e.g. menu costs, would further strengthen the tendency for synchronization of price changes. In fact, as shown in Sheshinski and Weiss [1989], in the menu costs case, the monopolist's optimum pricing strategy is fully synchronized following the first price change. In this case, the optimum policy for two goods reduces to the one-good case analysed in Sheshinski and Weiss [1977].

In the absence of positive price interactions in the profit function, the likelihood of a staggered policy increases substantially. In the case of additive profits (zero interactions) there is a continuum of non-synchronized steady-states, all with the same (S,s) values for both goods, which differ in the timing of the price change of the goods. The value of each steady-state increases as the difference between the time of price adjustment of the different goods decreases (see Sheshinski and Weiss [1989]). Thus, the synchronized steady state is the best. Nevertheless, for a wide range of initial conditions, it is preferable to change only one of the two prices, thereby saving on the costs of price adjustment.

Negative price interactions in the profit function fully exclude a synchronized price policy (Sheshinski and Weiss [1989]). This has been illustrated in the example discussed in Section 5. For a positive interest the interactions are non-negative. However, for the special case of quadratic profits, we have shown that in the limit, when the rate of interest approaches zero, a sharp classification exists: positive interactions eliminate staggering while negative interactions eliminate synchronization.

The important role of the interactions in determinining the timing of price changes, can be explained intuitively. Recall that the gain from postponing a price change is the sum of current flow of profits $F(z_1,z_2)$ and the interest gained on the delay in adjustment cost, $r\beta$. The loss from postponing a price change is the flow of profits evaluated at the new real price, S_1^*. Hence, the difference $F(z_1,z_2) + r\beta - F(S_1^*,z_2)$ is the net gain from postponing a price change of good 1. If the price of good 2 is to be raised now, then, with positive interactions, $F_{12} > 0$, the gains from postponing the price increase of good 1 diminish. This creates an incentive for synchronization. Conversely, if $F_{12} < 0$, then staggering is enhanced.

To further explore the role of different assumptions on the nature of the optimal program we use some numerical analysis. We restrict our attention to synchronized and staggered steady states which can be solved using equations (30)-(31) or (32)-(33).

In Table 1 we present such solutions for the case of a quadratic profit function. The Table highlights the following points.

TABLE 1: COMPARISON OF STEADY STATES FOR DIFFERENT RATES OF INFLATION, DIFFERENT COSTS OF ADJUSTMENT AND DIFFERENT INTERACTIONS.[1/]

Cost of Price Adjustment β	Inflation Rate g	Synchronized Steady State [2/]				Staggered Steady States [2/]											
						c = 1				c = .025				c = -.25			
		S_a^*	s_a	ϵ	rV_a	S_b^*	s_b	$2t$	rV_b	S_b^*	s_b	$2t$	rV_b	S_b^*	s_b	$2t$	rV_b
.001	0.05	1.041	.957	1.687	.9983	1.029	.967	1.243	.9976*	1.041	.957	1.667	.9982	1.048	.950	1.973	.9985
	0.10	1.052	.946	1.063	.9973	1.038	.960	.783	.9962*	1.051	.947	1.050	.9987	1.061	.937	1.243	.9977
	0.20	1.066	.932	.669	.9956	1.048	.950	.493	.9940*	1.065	.933	.661	.9955*	1.077	.783	.783	.9962
	0.30	1.076	.923	.511	.9943	1.058	.943	.376	.9921*	1.075	.924	.505	.9941*	1.089	.909	.598	.9950
	0.50	1.093	.909	.363	.9918	1.066	.932	.263	.9888*	1.089	.910	.359	.9917*	1.106	.893	.425	.9930
0.1	0.05	1.171	.778	7.856	.9708	1.110	.821	5.782	.9518*	1.168	.780	7.760	.9655	1.208	.748	9.193	.9717
	0.10	1.227	.733	4.939	.9486	1.154	.973	3.637	.9217*	1.222	.736	4.879	.9430	1.271	.694	5.778	.9527
	0.20	1.295	.673	3.108	.9131	1.207	.749	2.290	.8735*	1.291	.677	3.891	.9070	1.350	.623	3.637	.9225
	0.30	1.342	.630	2.372	.8832	1.243	.719	1.747	.8329*	1.337	.634	2.343	.8767	1.404	.572	2.774	.8967
	0.50	1.410	.566	1.687	.8320	1.295	.673	1.243	.7634*	1.406	.571	1.666	.8248	1.483	.497	1.973	.8528

1/ The interest rate is set at $r = 0.1$. The profit function is $F(Z_1,Z_2)=a(Z_1+Z_2)-b(Z_1^2+Z_2^2)+cZ_1\cdot Z_2)$. Throughout the Table we set $a = 1$ and $2b-c = 1$. The normalization implies that the maximal profits, 1, is attained at $Z_1=Z_2=1$.

2/ Subscripts a and b indicate synchronized and staggered steady states, respectively. S_i^* is the real price of each good just after a price increase; s_i is the real price of each good just before a price increase; rV_i is the imputed flow value of the optimal program, i = a,b. The time interval over which the nominal price of each good is held fixed is ϵ and $2t$, respectively, t is the time between price change of different goods at the staggered steady state.

* The staggered steady state is suboptimal.

1. The synchronized steady state provides a higher value than the staggered steady state whenever $c > 0$. This pattern is reversed for negative interactions. Although not presented in the Table, the synchronized steady state still dominates at zero and small negative interactions.

2. The staggered steady state may be optimal even if it is dominated by the synchronized steady state since it costs β to move from one steady state to the other. The asterisks in Table 1 indicate cases where such a move is profitable. It is seen that optimality of the staggered steady state is enhanced by high costs of price adjustment in conjunction with high rates of inflation.

3. Although the staggered steady state appears to be optimal under mild positive interactions, it is not locally stable (as stated in Proposition 3). For all the simulations in Table 1 we have verified that the stability test fails if $c > 0$ and passes if $c < 0$.

4. With positive interactions the staggered steady state involves more frequent price changes and smaller price increases for each of the two goods. This is in addition to the fact that one of the goods is changed every <u>half</u> period. With negative interactions, price changes for each product are less frequent in the staggered steady state.

5. As noted in our previous work, small costs of price adjustment are sufficient to generate time intervals with fixed nominal prices whose duration seems plausible. In the upper panel of Table 1 we use costs of adjustment which are one permill of the current flow of profits, $\beta = 0.001$. We see that at an inflation rate of 20 percent a year, prices are raised jointly every 8 months in a synchronized steady state and, in

the staggered steady state (with $c = 1$), each price is raised every 6 months.

6. The loss of potential profits due to adjustment costs increases with the rate of inflation. There is a magnification effect where even minute adjustment costs translate into a non-negligible loss of profits. This is due to the accumulation of two effects: as the rate of inflation increases, price adjustments occur more frequently and the firm is away from the maximal profits point for a longer period. For instance, with $g = .50$, $c = 1$, we see, in the upper panel of Table 1, that at a staggered steady state an adjustment cost of .001 translates into a loss of profits of .011. In the lower panel of Table 1, the numbers are .1 and .237, respectively.

7. In all the cases presented in Table 1, an increase in the inflation rate causes more frequent price adjustments and a larger variation in real prices. These patterns are consistent with the analysis discussed in Section 5.

We conclude with a brief remark on the duopoly case. If price changes are controlled by different firms, the likelihood of a staggered price policy increases. This is a consequence of two considerations. First, in the absence of cooperation, returns to scale in the costs of price adjustment cannot be exploited. Second, the monopoly, who internalizes all interactions, would change both prices even if this is not optimal for each duopolist separately. If the number of firms increases to the point where interactions become negligible, then idiosyncratic shocks are required to sustain a staggered steady-state (Ball and Romer [1989]).

Appendix A

The purpose of this appendix is to establish some basic features of the optimal pricing policy and its associated value function. First, we show that there exists an optimal program where $V(z_1,z_2)$, defined in (7), attains a maximum. Second, following the first price change, the timing of subsequent price increases and the associated real prices are uniquely determined. Finally, we note that uniqueness implies that $V(z_1,z_2)$ is differentiable at all (z_1,z_2) reachable after a price increase.

The optimal policy is a member of a class of policies which can be described as sequences of nominal prices for each of the two goods and time-intervals over which each nominal price prevails. This characterization follows from the existence of positive costs of adjustment for nominal price changes.

Let us define

$$\{\mu_\tau\}_{\tau=1}^{\infty} = \{t_\tau, S_{1\tau}, S_{2\tau}\}_{\tau=1}^{\infty} , \tag{A.1}$$

where t_τ are the dates of of <u>nominal</u> price changes (for at least one good) and $S_{j\tau}$, $j = 1,2$, are the logs of the <u>real</u> prices chosen at these dates. Clearly, $S_{j\tau} \in \mathbb{R}$ and $t_\tau \in \mathbb{R}^+$, $t_{\tau+1} \geqslant t_\tau$ and $t_1 \geqslant 0$. Without loss of generality, we can assume $t_{\tau+1} > t_\tau$. Due to the fixed costs of adjustment, the price of the same good will not be raised twice within a short interval. If two different prices are raised within a short interval, we can treat that as a simultaneous raise of two prices. We denote the set of all sequences satisfying the above restrictions by U.

To each sequence $\{\mu_\tau\}_{\tau=1}^{\infty} \in U$, we can associate the present discounted value of real profits

$$(A.2) \quad J(z_1,z_2,\{\mu_\tau\}_{\tau=1}^{\infty}) = \int_0^{t} e^{-rx}F(z_1-gx,\ z_2-gx)dx + \beta e^{-rt_1}\sum_{j=1}^{2}\delta(S_{j1}-z_j+gt_1) +$$

$$+ \sum_{\tau=1}^{\infty}\{\int_{t_\tau}^{t_{\tau+1}} e^{-rx}F(S_{1\tau}-g(x-t_\tau),\ S_{2\tau}-g(x-t_\tau))dx +$$

$$+ \ \beta e^{-rt_{\tau+1}}\sum_{j=1}^{2}\delta(S_{j\tau+1}-S_{j\tau}+g(t_{\tau+1}-t_\tau))\}$$

where

$$(A.3) \qquad \delta(y) = \begin{matrix} 0 & \text{if} \quad y = 0 \\ -1 & \text{if} \quad y \neq 0 \ . \end{matrix}$$

By definition (A.3), costs of adjustment are avoided if there is no nominal price change. That is, if the real price chosen at time t_τ coincides with the real price induced by the nominal price chosen at time $t_{\tau-1}$.

The value function is defined as

$$(A.4) \qquad V(z_1,z_2) = \sup_{\{\mu_\tau\}_{\tau=1}^{\infty} \ U} J(z_1,z_2,\{\mu_\tau\}_{\tau=1}^{\infty}) \ .$$

Observe that $J(z_1,z_2,\{\mu_\tau\}_{\tau=1}^{\infty})$ is <u>upper semi-continuous</u>.

Recall that under our assumptions, profits are positive only within a finite box, $\underline{z} \leqslant z_i \leqslant \bar{z}$, $i = 1,2$. Charging at some t_τ an initial price for good j , $S_{j\tau}$, which exceeds $\bar{z}$, yields non-positive profits for a period of $t_S = \frac{S_{j\tau} - \bar{z}}{g}$. The value of the program associated with this choice is at most $e^{-rt_S}\frac{F(\bar{S},\bar{S})}{r}$. On the other hand, the firm can obtain immediately $\frac{1}{1 - e^{-r(\bar{z}-\underline{z})/g}}\left[\int_0^{(\bar{z}-\underline{z})/g} e^{-rx}F(\bar{z}-gx,\ \bar{z}-gx)dx - 2\beta\right]$. By assumption, this value is positive. Hence, for a sufficiently large $S_{j\tau}$ (and t_S), the policy of repetitive adjustments from $\bar{z}$ to $\underline{z}$ dominates. Setting $S_{j\tau}$ below $\underline{z}$ or postponing the nominal price changes until some real price erodes below $\underline{z}$ is also suboptimal. Any such path can be replaced by a path which differs only in the initial phase but yields positive profits via continuous price adjustments. It follows that, without loss of generality, we can select the optimal policy from a <u>compact</u> subset of U.

We can now apply <u>Weierstrass'</u> <u>Theorem</u> that an upper semi-continuous function defined on a compact subset (of a normed linear space) achieves its maximum. We conclude:

<u>Proposition I</u>. There exists an optimal policy, $\{\mu_\tau^*\}_{\tau=1}^\infty \in U$, for which $V(z_1,z_2)$, defined in (A.4), attains its maximum.

Any path which maximizes $V(z_1,z_2)$ must satisfy a sequence of necessary first-order conditions. For example, consider a pair (z_1,z_2) where the optimal sequence is such that prices are raised alternatingly, first good 1, then good 2, then good 1 again, and so on. In this case the value function becomes

$$
\begin{aligned}
\text{(A.6)} \qquad V(z_1,z_2) = & \int_0^{t_1^*} e^{-rx}F(z_1-gx,\ z_2-gx)dx - \beta e^{-rt_1^*} \\
& + \int_{t_1^*}^{t_2^*} e^{-rx}F(S_{11}^*-g(x-t_1^*),\ x_2-gx)dx - \beta e^{-rt_2^*} \\
& + \int_{t_2^*}^{t_3^*} e^{-rx}F(S_{11}^*-g(x-t_1^*),\ S_{22}^*-g(x-t_2^*)) - \beta e^{-rt_3^*} \\
& + \int_{t_3^*}^{t_4^*} e^{-rx}F(S_{13}^*-g(x-t_3^*),\ S_{22}^*-g(x-t_2^*)) - \beta e^{-rt_4^*} \\
& + \ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

where $*$ superscripts indicate an optimal choice of the controls. The corresponding first-order conditions are

$$\text{(A.7)} \quad F(z_1 - gt_1^*,\ z_2 - gt_1^*) \leqslant F(S_{11}^*,\ z_2 - gt_1^*) - r\beta$$

$$F(S_{11}^* - g(t_2^* - t_1^*),\ z_2 - gt_2^*) = F(S_{11}^* - g(t_2^* - t_1^*), S_{22}^*) - r\beta$$

$$F(S_{11}^* - g(t_3^* - t_1^*),\ S_{22}^* - g(t_3^* - t_2^*)) = F(S_{13}^*, S_{22}^* - g(t_3^* - t_2^*)) - r\beta$$

$$\vdots \qquad\qquad \vdots$$

and

(A.8)

$$\int_{t_1^*}^{t_2^*} e^{-rx} F_1(S_{11}^* - g(x - t_1^*),\ z_2 - gx)dx + \int_{t_2^*}^{t_3^*} e^{-rx} F_1(S_{11}^* - g(x - t_1^*),\ S_{22}^* - g(x - t_2^*))dx = 0$$

$$\int_{t_2^*}^{t_3^*} e^{-rx} F_2(S_{11}^* - g(x - t_1^*),\ S_{22}^* - g(x - t_2^*))dx + \int_{t_3^*}^{t_4^*} e^{-rx} F_2(S_{13}^* - g(x - t_3^*),\ S_{22}^* - g(x - t_4^*))dx = 0$$

$$\vdots \qquad\qquad \vdots$$

Multiplying both sides of (A.6) by r, integrating by parts each integral in the sequence, and using the first-order conditions (A.7) and (A.8), one obtains

$$\text{(A.9)} \quad rV(z_1, z_2) = F(z_1, z_2) - g\int_0^{t_1^*} e^{-rx} F_1(z_1 - gx,\ z_2 - gx)dx -$$

$$- g\int_0^{t_1^*} e^{-rx} F_2(z_1 - gx,\ z_2 - gx)dx -$$

$$- g\int_{t_1^*}^{t_2^*} e^{-rx} F_2(S_{11}^* - g(x - t_1^*),\ z_2 - gx)dx\ .$$

Equation (A.9) corresponds to the '<u>valuation formula</u>', (16), in the text. It states that the imputed value of a given state equals the current flow of profits, $F(z_1,z_2)$, minus the reduction in profits which is caused by keeping the nominal price of good 1 fixed until the date t_1^* and the reduction in profits caused by keeping the nominal price of good 2 unchanged until t_2^* . All future information is incorporated in the choice of t_1^* , t_2^* and S_{11}^* . Similar 'valuation formulas' can be generated for any optimal pattern of price changes. It is important to note that no differentiability assumptions on $V(z_1,z_2)$ are required to derive this result. Similarly, equations (30) and (34) in the text, which identify the staggered steady-state, can be derived directly from equations (A.7)-(A.9), without the assumption that $V(z_1,z_2)$ is differentiable.

Differentiability of the value function is closely associated with the <u>uniqueness</u> of the optimal controls, $\{\mu_\tau^*\}_{\tau=1}^{\infty}$, for a given initial condition, (z_1,z_2). We cannot establish such uniqueness for all possible pairs of initial conditions. However, we can show that following the <u>first</u> price change, all subsequent choices of dates and real prices are uniquely determined.

To establish this fact, we first need to narrow down the possible patterns which constitute a solution to the optimization problem. This is done in the following two Lemmata.

<u>Lemma 1</u>. It is not optimal to delay a price reduction, i.e., if a price reduction occurs it must occur once, at time $t = 0$.

<u>Lemma 2</u>. If only one price is raised, it must be the price of the good with the lower real price.

Proof of Lemma 1. Suppose that for some (z_1, z_2), $t_1^* > 0$ and $S_{11}^* < z_1 - gt_1^*$, i.e. a reduction in z_1 occurs after a delay. The value of such policy can be written as

$$\int_0^{t_1^*} e^{-rx} F(z_1 - gx,\ z_2 - gx)dx + e^{-rt_1^*}[V(S_{11}^*, z_2 - gt_1^*) - \beta] .$$

Now consider a feasible alternative program where the price of good 1 is reduced at time 0 to $S_{11}^* + gt_1^*$ and kept unchanged up to t_1^*. The value of this alternative path is

$$-\beta + \int_0^{t_1^*} e^{-rx} F(S_{11}^* + gt_1^* - gx,\ z_2 - gx)dx + e^{-rt_1^*} V(S_{11}^*,\ z_2 - gt_1^*) .$$

We want to show that it exceeds the value of the original program. That is,

$$\text{(A.10)} \qquad \int_0^{t_1^*} e^{-rx}[F(S_{11}^* + gt_1^* - gx,\ z_2 - gx)dx - F(z_1 - gx,\ z_2 - gx)]dx > \beta(1 - e^{-rt_1^*}) .$$

By the assumption that $S_{11}^* + gt_1^* < z_1$, we have under monotonicity, (7), that the integral in (A.10) is strictly monotone decreasing in x. Since t_1^* is a maximizer, we have by (A.7) that

$$F(S_{11}^*,\ z_2 - gt_1^*) - F(z_1 - gt_1^*,\ z_2 - gt_1^*) \geq r\beta .$$

Hence,

$$F(S_{11}^* + gt_1^* - gx,\ z_2 - gx) - F(z_1 - gx,\ z_2 - gx) > r\beta .$$

for all $x \in [0, t_1^*)$, which establishes (A.10) ||.

Proof of Lemma 2. Suppose that $z_2 > z_1$, and z_2 is raised at $t = 0$. That is, $t_1^* = 0$, $S_{21}^* > z_2$ and $S_{11}^* = z_1$.

By (A.7),

(A.11) $$F(z_1, S_{21}^*) - r\beta \geqslant F(z_1, z_2) ,$$

because otherwise the firm would postpone the increase in z_2. By quasi-concavity and $S_{21}^* > z_2$, it follows that $F_2(z_1, z_2) > 0$. Suppose that at t_2^*, the price of z_1 is raised to S_{12}^*. Write the value of the proposed program:

$$\int_0^{t_2^*} e^{-rx} F(z_1 - gx,\ S_{21}^* - gx)dx + e^{-rt_2^*}[V(S_{12}^*,\ S_{21}^* - gt_2^*) - \beta] .$$

Consider now an alternative feasible program in which z_1 is raised at time 0 to S_{21}^*, and at time t_2^*, the price of z_2 is raised to S_{12}^*, yielding

$$\int_0^{t_2^*} e^{-rx} F(S_{21}^* - gx,\ z_2 - gx)dx + e^{-rt_2^*} [V(S_{21}^* - gt_2^*,\ S_{12}^*) - \beta] .$$

By symmetry, $V(S_{12}^*,\ S_{21}^* - gt_2^*) = V(S_{21}^* - gt_2^*,\ S_{12}^*)$.

We want to show that

(A.12) $$\int_0^{t_2^*} e^{-rx}[F(S_{21}^* - gx,\ z_2 - gx) - F(z_1 - gx,\ S_{21}^* - gx)]dx =$$

$$= (z_2 - z_1) \int_0^{t_2^*} e^{-rx} F_2(S_{21}^* - gx,\ \xi - gx)dx > 0$$

where $z_1 \leqslant \xi \leqslant z_2$. By complementarity, (5), $F_2(z_1,z_2) > 0 \Rightarrow F_2(S^*_{21},z_2) > 0$, since $S^*_{21} > z_2 > z_1$. By monotonicity, (7), (and thus concavity of $F(z_1,z_2)$),

$$F_2(S^*_{21},z_2) > 0 \Rightarrow F_2(S^*_{21},\xi) > 0 \Rightarrow F_2(S^*_{21}-gx,\ \xi-gx) > 0 \ , \quad \text{for all} \quad x \in [0,t^*_2] \ .$$

Hence (A.12) holds, which implies that the alternative program is superior ||.

We are now ready to establish the following proposition:

<u>Proposition II</u>: Following the first price change, the date of subsequent price changes is <u>unique</u>.

<u>Proof</u>: Assume that at time 0 , the price of good 1 has changed and set at S^*_{11} and suppose that there are two dates, t^*_2 and t^{**}_2 ($> t^*_2$) which are both optimal dates for subsequent price changes. With each of these dates there is an associated future sequence of prices. Clearly, both sequences must satisfy the necessary conditions for a local maximum, (A.7)-(A.8), and must yield the same value of V . We do not need to specify these sequences, since one can use the 'valuation formula', (A.9), to evaluate each alternative, based only on actions taken in the next round.

There are several cases to consider depending on the pattern of price changes. In the first case, the price of good 2 is raised both at t^*_2 and at t^{**}_2 . Setting $t^*_1 = 0$ in (A.9), we have

$$\text{(A.13)} \qquad rV(S^*_{11},z_2) = F(S^*_{11},z_2) - g\int_0^{t^*_2} e^{-rx}F_2(S^*_{11}-gx,\ z_2-gx)dx =$$

$$= F(S^*_{11},z_2) - g\int_0^{t^{**}_2} e^{-rx}F_2(S^*_{11}-gx,\ z_2-gx)dx \ .$$

It follows that

(A.14) $$\int_0^{t_2^*} e^{-rx} F_2(S_{11}^* - gx,\ z_2 - gx)dx = \int_0^{t_2^{**}} e^{-rx} F_2(S_{11}^* - gx,\ z_2 - gx)dx \ .$$

Under the F.O.C. (A.7), we have

(A.15) $$F(S_{11}^* - gt_2^*,\ S_{22}^*) - F(S_{11}^* - gt_2^*,\ z_2 - gt_2^*) - r\beta \geq 0 \ ,$$

where S_{22}^* is the choice of z_2 at t_2^*. By Lemma 1, $S_{22}^* > z_2 - gt_2^*$ and therefore (A.15) and quasi-concavity imply that $F_2(S_{11}^* - gt_2^*,\ z_2 - gt_2^*) > 0$, i.e. marginal profits from raising the price of good 2 are positive just prior to the price increase. Under the irreversibility assumption, it follows that $F_2(S_{11}^* - gx,\ z_2 - gx) > 0$ for all $x \in [t_2^*, t_2^{**}]$. Hence, the two integrals in (A.14) can be equal only if $t_2^* = t_2^{**}$, which establishes uniqueness for this case.

A similar proof applies to the cases in which the price of good 1 is raised together with the price of good 2 and to the case in which the price of good 1 is raised at both t_2^* and t_2^{**}. The cases in which two <u>different</u> prices are raised at t_2^* and t_2^{**} are excluded by Lemma 1. Thus, the date of the subsequent price change is unique ||.

It remains to be shown that at the time of price change a unique action is taken.

<u>Proposition III</u>. Following the first price change, subsequent chosen real prices are unique.

<u>Proof</u>: Consider first the case where only the price of good 1 is raised at t_2^*. Assume, contrary to the Proposition, that there are two choices, say, S_{12}^* and S_{12}^{**}.

By the F.O.C., (A.7), setting $t_1^* = 0$, we must have

(A.16) $$F(S_{11}^*-gt_2^*, S_{21}^*-gt_2^*) = F(S_{12}^*,S_{21}^*-gt_2^*) - r\beta = F(S_{12}^{**},S_{21}^*-gt_2^*) - r\beta .$$

Hence,

(A.17) $$F(S_{12}^*,S_{21}^*-gt_2^*) = F(S_{12}^{**},S_{21}^*-gt_2^*) .$$

Let $S_{12}^* > S_{12}^{**}$. Then, by quasi-concavity, $F_1(S_{12}^{**},S_{21}^*-gt_2^*) > 0$, which under irreversibility, (A.1), contradicts the F.O.C. (A.7).

Finally, consider the case in which both prices are raised at t_2^*. As explained in the text, under positive interaction, the best choice, starting from $(z_1,z_2) \in T_0$ is to select a synchronized steady state. In Appendix B we shall prove that the synchronized steady state is unique ||.

Having established uniqueness, differentiability of the value function follows. First we note that uniqueness implies continuity of the optimal choice with respect to variations in the initial state. Now, let $\{\mu_\tau\}_{\tau=1}^{\infty}$ be an optimal choice for (z_1,z_2) and $\{\mu_\tau^h\}_{\tau=1}^{\infty}$ be an optimal choice for (z_1, z_2+h), where $h > 0$. By definition

(A.18) $$J(z_1,z_2,\{\mu_\tau\}_{\tau=1}^{\infty}) \geqslant J(z_1,z_2,\{\mu_\tau^h\}_{\tau=1}^{\infty})$$

(A.19) $$J(z_1,z_2+h,\{\mu_\tau^h\}_{\tau=1}^{\infty}) \geqslant J(z_1,z_2+h,\{\mu_\tau\}_{\tau=1}^{\infty})$$

or

(A.20)

$$J(z_1,z_2+h,\{\mu_\tau^h\}_{\tau=1}^{\infty}) - J(z_1,z_2,\{\mu_\tau^h\}_{\tau=1}^{\infty}) \geqslant J(z_1,z_2+h,\{\mu_\tau^h\}_{\tau=1}^{\infty}) - J(z_1,z_2,\{\mu_\tau\}_{\tau=1}^{\infty}) \geqslant$$

$$\geqslant J(z_1,z_2+h,\{\mu_\tau\}_{\tau=1}^{\infty}) - J(z_1,z_2,\{\mu_\tau\}_{\tau=1}^{\infty})$$

Consider a point (z_1, z_2) in the interior of C where, in (A.2), $\delta(S_{j1} - z_j + gt_j) = 0$, $j = 1,2$. In this neighborhood, $J(z_1, z_2(\mu_\tau)_{\tau=1}^{\infty})$ is differentiable w.r.t z_1 and z_2 . Dividing both sides of (A.20) by h , letting h approach zero and using the continuity of $\{\mu\}$ in z , we obtain the usual envelope relationship:

$$\frac{\partial V(z_1, z_2)}{\partial z_2} = \frac{\partial}{\partial z_2} J(z_1, z_2, (\mu_\tau)_{\tau=1}^{\infty}) \ . \tag{A.21}$$

A similar proof applies to the other partial derivative.

Appendix B

Proof of Proposition 1:

We first prove that the synchronized steady state is unique. Any maximizer of V must be in the interior of C. Otherwise an additional cost of β would be incurred to obtain the same value, V^*. Observe that for any point in the interior of C, equation (10) applies with strict inequality and V is differentiable. Combining these facts, we obtain

$$rV(S_1,S_2) = F(S_1,S_2) \tag{B.1}$$

where $(S_1,S_2) \in \arg\underset{z_1,z_2}{\text{Max}}\ V(z_1,z_2)$.

Now suppose that $S_1 \neq S_2$. Then, by symmetry, the points (S_1,S_2) and (S_2,S_1) are both maximizers of $V(z_1,z_2)$, yielding the same value V^*.

For any $0 < \gamma < 1$, define (S_1^γ,S_2^γ) as:

$$\begin{aligned} S_1^\gamma &= \gamma S_1 + (1-\gamma)S_2 , \\ S_2^\gamma &= \gamma S_2 + (1-\gamma)S_1 . \end{aligned} \tag{B.2}$$

Using recursive equation (7), the value associated with $V(S_1,S_2)$ is

$$V(S_1,S_2) = \int_0^{t^*} e^{-rx}F(S_1-gx,S_2-gx)dx + e^{-rt^*}[V^*-2\beta] \tag{B.3}$$

where $t^* = t^*(S_1,S_2)$ is the optimal time for the subsequent price change and V^* is the maximum value of V realized at t^*. Starting at (S_1^γ,S_2^γ), the same choices are still feasible. Hence,

$$V(S_1^\gamma,S_2^\gamma) \geq \int_0^{t^*} e^{-rx}F(S_1^\gamma-gx,S_2^\gamma-gx)dx + e^{-rt^*}[V^*-2\beta] . \tag{B.4}$$

By strict quasi-concavity, $F(S_1-gx,S_2-gx) < F(S_1^{\gamma}-gx,S_2^{\gamma}-gx)$ for all x, $t^* \geq x \geq 0$. Thus, equations (B.3) and (B.4) imply that $V(S_1^{\gamma},S_2^{\gamma}) > V(S_1,S_2)$, which contradicts the assumption that (S_1,S_2) maximizes V. This proves that $S_1 = S_2 = S$. To save on notation we shall omit here the superscript and subscript, writing $S = S_a^*$. To prove that S is unique we use again the quasi-concavity of F together with the 'valuation formula', (16).

Suppose there are two values, (S^a,S^a) and (S^b,S^b), $S^a \neq S^b$, that maximize V. Let $S^{\theta} = \theta S^a + (1-\theta)S^b$, $0 < \theta < 1$. Then,

(B.5) $$F(S^{\theta},S^{\theta}) > F(S^a,S^a) = F(S^b,S^b) = rV^* \geq rV(S^{\theta},S^{\theta}) .$$

Inequality (B.5) and the 'valuation formula', (16), imply that for any $(S^{\theta},S^{\theta}) \in C$, we must have

(B.6) $$gV_1(S^{\theta},S^{\theta}) + gV_2(S^{\theta},S^{\theta}) > 0 .$$

Letting $\gamma \to 0$ or $\gamma \to 1$, (B.6) implies that V can be increased in the neighborhood of (S^a,S^a) or (S^b,S^b), contrary to the assumption that these are local maxima.

We shall now prove the uniqueness of the symmetric staggered steady-state, (S_b^*,t). For brevity, we write $S_b^* = S$. Consider the point $(S,\hat{z}) \in C$, where the price of the first good has just been changed, and let t_2 be the timing of the subsequent price change. The F.O.C. satisfied at that point are:

(B.7) $$V_2(S-gt_2,\ S_2) = 0$$

and

$$\text{(B.8)} \qquad F(S-gt_2,\ \hat{z}-gt_2) - r(V(S-gt_2,\ S_2)-\beta) - gV_1(S-gt_2,\ S_2) = 0 .$$

At a symmetric steady state, $S_2 = S$, $t_2 = t$ and $\hat{z} = S - gt$. Evaluating the second-order conditions at this point, we have the requirement that the matrix A,

$$\text{(B.9)} \qquad A = \begin{bmatrix} V_{22}(S-gt,\ S) & -gV_{12}(S-gt,\ S) \\ -gV_{12}(S-gt,\ S) & \begin{array}{l} -g(F_1(S-gt,\ S-2gt) + F_2(S-gt,\ S-2gt) + \\ \quad + rV_1(S-gt,\ S) + gV_1(S-gt,\ S)) \end{array} \end{bmatrix}$$

be negative definite. Now consider the system

$$\text{(B.10)} \qquad V_2(S-gt,\ S) = 0$$

and

$$\text{(B.11)} \qquad F(S-gt,\ S-2gt) - r(V(S-gt,\ S)-\beta) - gV_1(S-gt,\ S) = 0$$

as two equations in the unknowns S and t.

To prove uniqueness, we shall show that the Jacobian, B,

$$\text{(B.12)} \qquad B = \begin{bmatrix} \begin{array}{l} V_{12}(S-gt,\ S) + \\ \quad + V_{22}(S-gt,\ S) \end{array} & -gV_{12}(S-gt,\ S) \\ \begin{array}{l} F_1(S-gt,\ S-2gt) + \\ \quad + F_2(S-gt,\ S-2gt) - \\ \quad - rV_1(S-gt,\ S) \\ \quad - gV_{11}(S-gt,\ S) \end{array} & \begin{array}{l} -g[F_1(S-gt,\ S-2gt) + 2F_2(S-gt,\ S-2gt) + \\ \quad + rV_1(S-gt,\ S) + gV_{11}(S-gt,\ S-2gt)] \end{array} \end{bmatrix}$$

is negative definite. The first diagonal term is, under A1 and A2,

(B.13) $$b_{11} = V_{12} + V_{22} = \frac{1}{g} F_2(S-gt, S) < 0 .$$

The other diagonal term, b_{22} , is equal to the lower diagonal term in (B.9), a_{22} minus $gF_2(S-gt, S-2gt)$. By A2, F_2 just prior to a price change has to be positive. Thus, the whole term is negative. The determinant condition can be written in the form

(B.14) $$b_{11}(-gF_2(S-gt, S-2gt)) + a_{22} V_{22} > 0$$ ||.

Appendix C

The purpose of this appendix is to analyse the stability of the staggered steady-state.

We begin by calculating the slope of the reaction curves $S_i^* = S(z_j)$, $i \neq j$, $i,j = 1,2$, evaluated at $S_i^* = S_b^*$ and $z_j = S_b^* - gt$, where S_b^* and t are determined by equations (32) and (33). To save on notation we shall omit again the superscript and subscript, writing S instead of S_b^*.

Suppose that z_1 has just been raised to S and that $z_2 = S - gt$. At this point we have

(C.1) $$V_1(S,z_2) = 0 ,$$

(C.2) $$gV_{11}(S,z_2) + gV_{21}(S,z_2) = F_1(S,z_2) ,$$

and

(C.3) $$V_2(S,z_2) = \int_0^t e^{-rx}F_2(S-gx,\ z_2-gx)dx ,$$

where (C.2) follows from the valuation formula (16) and (C.1). Differentiating (C.1) we obtain

(C.4) $$S'(z_2) = -\frac{V_{12}(S,z_2)}{V_{11}(S,z_2)} .$$

Using (C.2) to eliminate V_{11},

(C.5) $$gV_{12}(S,z_2) = \frac{S'(z_2)F_1(S,z_2)}{S'(z_2) - 1} .$$

Differentiating (C.3) we have

(C.6) $$V_{21}(S,z_2) = \int_0^t e^{-rx}F_{21}(S-gx,\ z_2-gx)dx + e^{-rt}F_2(S-gt,\ z_2-gt)\frac{\partial t}{\partial z_1} .$$

To find $\frac{\partial t}{\partial z_1}$, we note that in steady-state,

$$\text{(C.7)} \qquad (S,t) = \underset{S_2,t_2}{\operatorname{argmax}}\{\int_0^{t_2} e^{-rx}F(S-gx,\ z_2-gx)dx + e^{-rt_2}[V(S-gt_2,\ S_2) - \beta]\} .$$

Differentiating the F.O.C. for the maximization in (C.7), we obtain

$$\text{(C.8)} \qquad \begin{bmatrix} -g[F_1(S-gt,\ z_2-gt) + F_2(S-gt,\ z_2-gt) & -gV_{12}(S-gt,\ S) \\ -rV_1(S-gt,\ S) - gV_{11}(S-gt,\ S)] & \\ -gV_{21}(S-gt,\ S) & V_{22}(S-gt,\ S) \end{bmatrix} \begin{bmatrix} \frac{\partial t}{\partial z_1} \\ \\ \frac{\partial S}{\partial z_1} \end{bmatrix} =$$

$$= \begin{bmatrix} -F_1(S-gt,\ z_2-gt) + rV_1(S-gt,\ S) + gV_{11}(S-gt,\ S) \\ \\ V_{21}(S-gt,\ S) \end{bmatrix}$$

The second-order conditions for maximization require that the matrix in (C.6) be negative-definite. Using the valuation formula, we have

$$\text{(C.9)} \qquad rV_1(S-gt,\ S) + gV_{11}(S-gt,\ S) = F_1(S-gt,\ S) - gV_{12}(S-gt,\ S) .$$

Also,

$$\text{(C.10)} \qquad V_{22}(S-gt,\ S) = -\frac{V_{12}(S-gt,\ S)}{S'(S-gt)} .$$

Substituting (C.9)-(C.10) into (B.8), we can solve for $\frac{\partial t}{\partial z_1}$ in terms of $S'(S-gt)$, to obtain:

(C.11)

$$g \frac{\partial t}{\partial z_1} = \frac{F_1(S-gt,\ z_2-gt) - F_1(S-gt,\ S) - F_2(S-gt,\ S)S'(S-gt)}{F_1(S-gt,\ z_2-gt) + F_2(S-gt,\ z_2-gt) - F_1(S-gt,\ S) - F_2(S-gt,\ S)S'(S-gt)}$$

The denominator on the R.H.S of (C.11) is positive by second-order conditions. Combining (C.5), (C.6) and (C.11) and using symmetry, we obtain

(C.12)
$$\frac{S'(z_2)}{S'(z_2) - 1} = C + D\,\frac{A - S'(S-gt)}{B - S'(S-gt)}\ ,$$

where

$$A = \frac{F_1(S-gt,\ z_2-gt) - F_1(S-gt,\ S)}{F_2(S-gt,\ S)}$$

(C.13)
$$B = A + \frac{F_2(S-gt, z_2-gt)}{F_2(S-gt,\ S)}$$

$$C = \frac{g\int_0^t e^{-rx}F_{12}(S-gx,\ z_2-gx)dx}{F_2(S-gt,\ S)}$$

and
$$D = e^{-rt}\,\frac{F_2(S-gt,\ z_2-gt)}{F_2(S-gt,\ S)}\ .$$

Equation (C.12) determines the slope of the reaction curve for the first good in terms of the slope of the second good's reaction curve evaluated at the <u>subsequent</u> price change. Setting $z_2 = S - gt$, S' at this point is unknown, satisfying the non-linear difference equation:

(C.14)
$$\frac{x_n}{x_n - 1} = C + D\,\frac{A - x_{n+1}}{B - x_{n+1}}\ .$$

The optimality of the staggered path requires convergence of the sequence defined by (C.14) when solved forwards.

To analyse the behavior of the solutions to (C.14), we rewrite it as

$$\text{(C.15)} \qquad x_n = \frac{CB + DA - (C+D)x_{n+1}}{CB + DA - B + (1-C-D)x_{n+1}} \equiv f(x_{n+1}) \, .$$

Note first that the condition $x_n = x_{n+1} = x$ defines a quadratic equation which has at most two roots. In general, one root is characterized by $f'(x) < 1$ and the other by $f'(x) > 1$. The second, being unstable, cannot be obtained by iterations of the value function and therefore cannot represent the value of an optimal policy (Stokey and Lucas [1989, Ch. 4]). Henceforth, we shall consider only the root satisfying $f'(x) < 1$.

To find the roots of equation (C.15) it is convenient to rewrite (B.14):

$$\text{(C.16)} \qquad CB + AD - (C+D)x = \frac{x(B-x)}{x - 1} \equiv g(x) \, .$$

Under assumptions A1 and A2 in the text, $A > 0$, $C < 0$, $D < 0$ and $B - A < 0$. Second-order conditions (i.e., negative definiteness of the matrix in (C.8)) imply that $B < x < 1$. It follows from these restrictions that $g(x)$ is strictly convex. Furthermore, $g(0) = g(B) = 0$, $g'(0) = -B$ and $g'(B) = \frac{B}{B - 1}$.

Associated with the value of S' which solves (B.16) there is a corresponding value s' which is the slope of the boundary curve between the trigger sets T_i, $i = 1,2$, and C. This relation is obtained by differentiating (33) in the text, yielding

$$\text{(C.17)} \qquad s' = \frac{S' - A}{B - A} \, .$$

Note that since $B < S'$, by second-order conditions, $s' < 1$.

Consider a small perturbation around this path (a'b'c'd' in Figure 2). Let S' denote the slope of the reaction curve evaluated at the staggered steady-state (i.e. at $(S, S-gt)$). Similarly, let s' denote the slope of the boundary of the trigger sets T_i, $i = 1,2$, and C, evaluated at the staggered steady-state (i.e., at $(S-gt, S-2gt)$). For sufficiently small perturbations these slopes can be taken as constant. Let δ be the difference between the two paths along the reaction curve for the first good, measured in units of z_2. It can be seen in Figure 1, and rigorously proved, that the initial δ translates into a difference $\left(\frac{S' - 1}{1 - s'} \right)\delta$ along the trigger boundary for the second good and a difference of $\left(\frac{S' - 1}{1 - s'} \right)^2\delta$ along the trigger boundary for the first good. Thus, if and only if

$$\text{(C.18)} \qquad \left| \frac{S' - 1}{1 - s'} \right| < 1 ,$$

will the perturbed path return to the first good's curve, closer to the original point a. We therefore conclude that the necessary and sufficient condition for local stability of the staggered steady-state is condition (C.18).

It follows from (C.17) that if $S' < 0$, then $s' > 0$ and hence $\left| \frac{S' - 1}{1 - s'} \right| > 1$, i.e., the staggered steady-state is unstable. It remains to examine the case in which $S' > 0$. This can only occur if $AB + CD > 0$ and $B < 0$. It can be shown that

$$\text{(C.19)} \qquad S' > \frac{A}{1 - B + A} \Longleftrightarrow g\left(\frac{A}{1 - B + A} \right) < CB + AD - \frac{(C+D)A}{1 - B + A} .$$

It remains to discuss the case of negative interactions, where $F_{12} \leqslant 0$. In this case $C \geqslant 0$, $A \leqslant 0$ and $B < 0$. In contrast to the case with $F_{12} > 0$, (C.17) does <u>not</u> imply that if $S' < 0$ then $s' > 0$. Therefore, we need to consider the case $CB + AD < 0$ associated with negative S' . For this case too, it can be shown that (C.19) has to be satified. The condition in (C.19) can be simplified to

(C.20) $$A + C < CB + AD ,$$

which is the necessary and sufficient condition for local stability.

Appendix D

In this Appendix we prove Proposition 3 in the text.

Consider a quadratic profit function given by (36). The necessary conditions for a staggered steady-state, (32)-(33), are now:

(D.1) $$-2gt(a-(2b-c)(S-gt)) + r\beta = 0 ,$$

and

(D.2)

$$(a-(2b-c)S) \int_0^{2t} e^{-rx}dx + g(2b-c) \int_0^{2t} e^{-rx}xdx + cgt \left[\int_0^{2t} e^{-rx}dx - \int_0^{t} e^{-rx}dx\right] = 0$$

where $S = S_b^*$ in the text. Equations (D.1) and (D.2) uniquely determine the steady-state values for t and S. Given these values, one can calculate all the elements of the stability condition (D.20). In particular:

(D.3)
$$\begin{aligned}
F_1(S-gt,\ S-2gt) &= \Delta + (2b-c)gt \\
F_2(S-gt,\ S-2gt) &= \Delta + (4b-c)gt \\
F_2(S-gt,\ S) &= \Delta - cgt \\
F_1(S-gt,\ S) &= \Delta + 2bgt \quad ,
\end{aligned}$$

where $\Delta = a - (2b-c)S$. Using the definitions (C.13) in Appendix C ,

(D.4)
$$\begin{aligned}
F_2(S-gt,\ S)A &= -2cgt \\
F_2(S-gt,\ S)B &= \Delta + (4b-3c)gt \\
F_2(S-gt,\ S)C &= gc \int_0^{t} e^{-rx}dx \\
F_2(S-gt,\ S)D &= e^{-rt}(\Delta + (4b-c)gt)
\end{aligned}$$

The stability condition (C.20), $CB + AD > A + C$ is thus equivalent to:

(D.5) $$(\Delta+(4b-3c)gt) \int_0^{t} e^{-rx}dx - 2te^{-rt}(\Delta+(4b-c)gt) > (-2t+\int_0^{t} e^{-rx}dx)(\Delta-cgt) .$$

Substituting for Δ from (D.2) into (D.4) we obtain, after some manipulations, that (D.5) is equivalent to

$$c(2b+c) < 0 \,. \tag{D.6}$$

The concavity requirements $b > 0$ and $4b^2 - c^2 > 0$ imply that $2b + c > 0$ for all c. Hence, (D.6) always holds when $c < 0$ and never holds when $c > 0$. This proves Proposition 3.

Appendix E

The purpose of this Appendix is to prove Proposition 4 in the text and its corollary.

Consider the staggered steady-state path starting at point b in Figure 2. At this point the staggered policy calls for a move to c . The value of the staggered steady-state path starting at c is given by

$$\text{(E.1)} \qquad V_b = \frac{1}{1 - e^{-rt}} \left[\int_0^t e^{-rx} F(S_b^* - gt - gx,\ S_b^* - gx)dx - \beta e^{-rt}\right]$$

If, instead, the monopolist would move to point e and follow thereafter the synchronized steady-state path, the value associated with this alternative path is

$$\text{(E.2)} \qquad V_a = \frac{1}{1 - e^{-r\varepsilon}} \left[\int_0^\varepsilon e^{-rx} F(S_a^* - gx,\ S_a^* - gx)dx - 2\beta e^{-r\varepsilon}\right]$$

where (S_b^*, t) and (S_a^*, ε) are the solutions of (32)-(33) and (30)-(31), respectively. By our assumptions, a move from b to c costs β while a move to e costs 2β . Therefore, a necessary condition for the optimality of the staggered program is that

$$\text{(E.3)} \qquad V_b - \beta > V_a - 2\beta .$$

Multiplying (E.3) by r , and taking the limit as $r \to 0$, the requirement is

$$\text{(E.4)} \qquad \frac{1}{t}\left[\int_0^t F(S_b^* - gt - gx,\ S_b^* - gx)dx - \beta\right] > \frac{1}{\varepsilon}\left[\int_0^\varepsilon F(S_a^* - gx,\ S_a^* - gx)dx - 2\beta\right] .$$

Using (31) and (33) in the text, (E.4) can be written

(E.5) $$F(S_b^*-gt,\ S_b^*) > g\int_0^t F_1(S_a^*-gt-gx,\ S_a^*-gx)dx + F(S_a^*, S_a^*)\ .$$

Substituting the quadratic formula (36) in the text into (E.5) and using (38)-(42) in the text, condition (E.5) is seen to be equivalent to

(E.6) $$\left[\frac{2b - c}{2b + \frac{c}{2}}\right]^{\frac{1}{3}} > 1\ .$$

This inequality holds if, and only if, $c < 0$.

Footnotes

[1] Sheshinski's research was partially supported by NSF grant SES-8821925 at Stanford University. The authors wish to thank Eyal Sulganik for helpful comments.

[2] A number of recent studies have analysed the dynamic interaction of pricing policies in oligopolistic markets (Maskin and Tirole [1988], Gertner [1986], and Benabou and Gertner [1988]). However, these studies take the time pattern as exogenous, and focus on the equilibrium price configuration.

The issue of staggering vs. synchronization has been taken up by Ball and Cecchetti [1987], Ball and Romer [1989] and by McMillan and Zinde-Walsh [1988]. The approach of Ball and Cecchetti emphasizes the informational gains from staggered pricing policies. McMillan and Zinde-Walsh consider a closed-loop equilibrium in an oligopolistic market for a homogeneous good. The homogeneity assumption eliminates price variation across products. Ball and Romer [1989] extend a model of Blanchard (Blanchard-Fischer [1989]), allowing each firm to choose whether to change prices at odd or even periods. This formulation permits both staggered and synchronized equilibria. The duration of the fixed price period is assumed to be determined exogeneously. In contrast, we treat the timing and the chosen real prices as endogeneous.

[3] We would like to thank Avner Bar-Ilan for the references to Bensoussan, Crouhy and Proth [1983], Bensoussan and Proth [1982], and Sulem [1986].

[4] For a reference on the methodology, see Stokey and Lucas [1989].

[5]
$$F(z_1,z_2) = F(z_1,z_1) + \int_{z_1}^{z_2} F_2(z_1,x)dx$$

$$F(z_1,z_2) = F(z_2,z_2) - \int_{z_1}^{z_2} F_1(x,z_2)dx = F(z_2,z_2) - \int_{z_1}^{z_2} F_2(z_2,x)dx$$

Adding up the above equalities we get

$$2F(z_1,z_2) = F(z_1,z_1) + F(z_2,z_2) + \int_{z_1}^{z_2} [F_2(z_1,x) - F_2(z_2,x)]dx$$

Assuming $F_{12} > 0$, the integral of the difference in marginal profits must be negative, which establishes the claim.

[6] The derivatives of V at point (z,z) on the diagonal just below k are $V_i(z,z) = \int_0^{t^*(z,z)} e^{-rx} F_i(z-gx,\ z-gx)dx$, $i = 1,2$. Note that by symmetry, $V_1(z,z) = V_2(z,z)$. By the optimality of S_a^*, we have $V_i(S_a^*,S_a^*) = \int_0^{t^*(S_a^*,S_a^*)} e^{-rx} F_i(S_a^*-gx,\ S_a^*-gx)dx = 0$, $i = 1,2$. But, $t^*(z,z) > t^*(S_a^*,S_a^*)$ and $F(z_1,z_2)$ is strictly quasi-concave. Hence, $V_i(S_a^*,S_a^*) = 0 \quad V_i(z,z_2) < 0$, for $i = 1,2$. To locate the boundary points between C and T_i^-, $i = 1,2$, we cannot use equations (23)-(25), since these do not hold with equality. Instead of marginal conditions we require that the firm be indifferent between holding prices constant for a non-negligible length of time, and changing prices instantly. For instance, on the diagonal,

$$V(z,z) = \int_0^{t^*(z,z)} e^{-rx} F(z-gx,\ z-gx)dx - e^{rt^*(z,z)}(V^*-2\beta) = V^* - 2\beta .$$

The solution to this equation is point k in Figure 1.

[7] Consider z_2 in the neighborhood of S_a^*, which triggers a change to $S(z_2)$. Observe that $S(z_2)$ is also in the neighborhood of S_a^*. From the properties of T_0^{++}, it follows that the subsequent price change at $t^*(S(z_2),z_2)$ will be a joint price increase. Hence,

$$(t^*,S(z_2)) = \underset{t,S}{\operatorname{argmax}} \int_0^t e^{-r} F(S-gx,\ z_2-gx)dx + e^{-rt}(V^*-2\beta) .$$

Differentiating the F.O.C. w.r.t. z_2, evaluating the derivatives at (S_a^*, S_a^*), we obtain

$$S'(S_a^*) = [\int_0^{t^*} e^{-rx} F_{11}(S_a^*-gx,\ S_a^*-gx)dx + \frac{e^{-rt^*}F_1(S_a^*-gt^*,\ S_a^*-gt^*)}{2g}] +$$

$$+ \int_0^{t^*} e^{-rx} F_{12}(S_a^*-gx,\ S_a^*-gx)dx + \frac{e^{-rt^*}F_1(S_a^*-gt^*, s_a^*-gt^*)}{2g} = 0$$

By second-order conditions, the term in square brackets is negative. The F.O.C. imply that $F_1(S_a^*-gt^*, S_a^*-gt^*) > 0$. Hence, with $F_{12} > 0$, $S'(S_a^*) > 0$.

[8] Differentiating the 'valuation formula', (16), we obtain

$$F_1(S(z_2),z_2) = gV_{11}(S(z_2),z_2) + gV_{12}(S(z_2),z_2) .$$

Since, $F_1(S(z_2),z_2) < 0$, $V_{11}(S(z_2),z_2) < 0$ and $V_1(S(z_2),z_2) = 0$ for all z_2, we obtain

$$S'(z_2) = -\frac{V_{21}(S(z_2),z_2)}{V_{11}(S(z_2),z_2)} > 1 .$$

Similarly, whenever V is differentiable at the boundary, the 'valuation formula' implies

$$F_1(s(z_2),z_2) = gV_1(s(z_2),z_2) + gV_{12}(s(z_2),z_2) .$$

Since $F_1(s(z_2),z_2) > 0$, $V_{11}(s(z_2),z_2) > 0$ and $V_1(s(z_2),z_2) = 0$ for all z_2, we obtain

$$s'(z_2) = -\frac{V_{21}(s(z_2),z_2)}{V_{11}(s(z_2),z_2)} < 1 .$$

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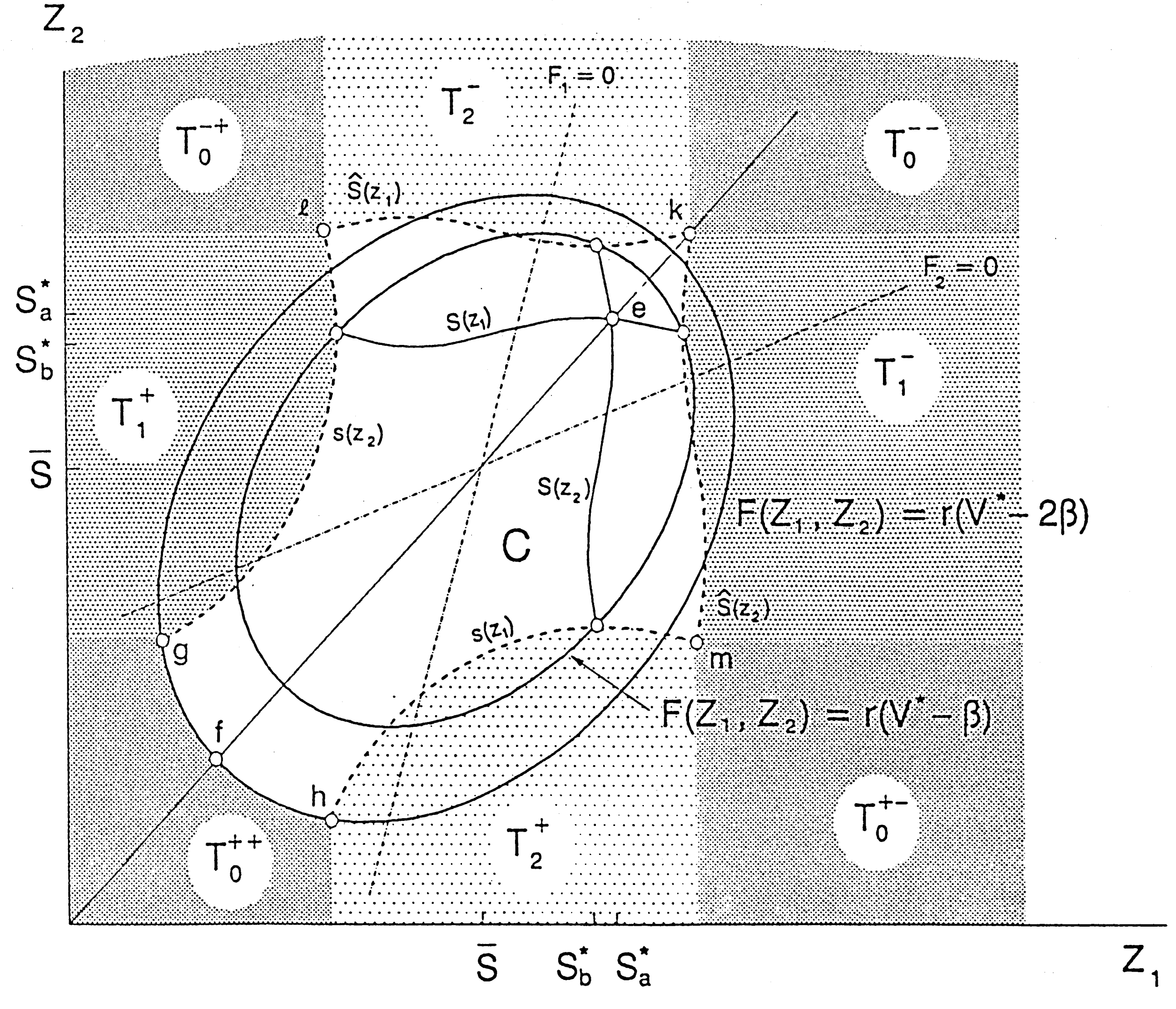
Figure 1



Z2
T0−+
T2−
F1 = 0
T0−−
Ŝ(z1)
ℓ
k
F2 = 0
Sa*
Sb*
e
s(z1)
T1+
T1−
s(z2)
S̄
S(z2)
F(Z1, Z2) = r(V* − 2β)
C
Ŝ(z2)
s(z1)
g
m
F(Z1, Z2) = r(V* − β)
f
h
T0+−
T0++
T2+
S̄
Sb*
Sa*
Z1