TEL-AVIV

WP No. 1-90

THE FOERDER INSTITUTE FOR ECONOMIC RESEARCH

TEL-AVIV UNIVERSITY

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TECHNOLOGICAL PROGRESS AND INCOME INEQUALITY

by

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Working Paper No.1-90

January, 1990

* The Johns Hopkins University
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This research was partially supported by funds granted to the Foerder Institute for Economic Research by the JOHN RAUCH FUND

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1. Introduction

Technological changes alter the distribution of incomes over time directly — through their effect on productivity — and indirectly — by affecting the rate of accumulation of factors of production. In this paper we study the relationship between changes in the technology of production and the resulting variations in income inequality, the aggregate stock of capital, and the aggregate supply of labor. We do not intend to examine the historical sources of income inequality. Rather, taking the stochastic process generating income inequality as given, we try to evaluate the changes in this inequality resulting from the introduction of new production technologies.

The stochastic process generating the inequality in the distribution of income in our model represents random variations in tastes of individual agents. These variations in tastes induce inequality in the distribution of intergenerational transfers and, consequently, income inequality. The effect of the intergenerational transfers, however, is mitigated by variations in the supply of labor. Casting our model in these terms does not mean that we regard other sources of inequality, such as differences in talent and education or pure luck (see, for example, Loury (1981)) as less important. However, disregarding these factors enables us to isolate the effect of intergenerational transfers. A more comprehensive treatment may be built upon the analysis presented here.

We conduct our analysis within the framework of a competitive equilibrium in an overlapping generations economy with endogenous labor supply and a bequest motive. Each individual in this economy lives for two periods. During the first period he works, consumes, and saves some of his income. Saving is intended in part for bequest and in part to pay for consumption during the second period of the individual's lifetime. At the end of the first period of his life, each individual gives birth to a single offspring and at the same time makes the bequest transfer. During the second period he engages solely in consumption. Thus, all the relevant decisions – the consumption–saving decision, the labor–leisure decision, and the decision concerning the allocation of saving between the second period consumption and bequest – are made in the first period. Bequests are motivated by the "joy of giving," and together with the variations in preferences constitute the source of heterogeneity among individuals in each generation. The distribution of incomes in each generation, however, is determined in part by the amount of labor supplied by different individuals. The technology of production is characterized by constant returns to scale.

Our analysis involves comparative dynamic experiments in which a permanent shift of the production function occurs at a given point in time. **We trace the resulting changes in the distribution of income during the period in which the new technology is introduced and in every period thereafter.** We examine the consequences for income inequality of three types of shifts in the production function known in the growth literature as Hicks–neutral, Harrod–neutral, and Solow–neutral technological changes. We found that (a) in all cases the aggregate capital stock tends to increase in the aftermath of the introduction of the new production technologies, (b) the effect of the technological changes on the aggregate labor supply depend on the specific nature of the technological change as well as on the elasticity of substitution in production, and (c) the effect of the technological changes on income inequality depend solely on the elasticity of substitution in production. The last point merit further elaboration. Our analysis shows that if the the technological improvement is Hicks–neutral or Harrod–neutral then the income inequality decreases (increases) if and only if the elasticity of substitution is larger than one (smaller than one). If the technological change is Solow–neutral then the income inequality increases (decreases) if and only if the elasticity of substitution is larger than one (smaller than one). The inequality in the distribution of incomes is unaffected by the aforementioned technological changes if and only if the elasticity of substitution is equal to one.

In the next section we specify the model. In section 3 we analyze the effects of technological changes on the distribution of incomes. A summary of the main results and concluding remarks appear in section 4.

2. The Model

2.1 *Preferences and Technology*

Consider an overlapping generations economy with no population growth. Each individual in this economy lives for two periods – a working period followed by a retirement period. At the end of the first period every individual gives birth to one offspring. We denote by G_t the set of individuals born at the outset of period t and refer to these individuals as generation t. The economy starts at $t = 0$, where G_{-1} live their retirement period. Their only source of income is their savings. Denote by Ω the set of families in each generation; it is time–independent since there is no population growth. Although our analysis can be carried out for any finite Ω, to simplify our notations we assume there is a continuum of individuals (or families) in each period, hence we may assume that $\Omega = [0, 1]$ with some, time–independent, density function μ on $[0, 1]$.

Preferences: The preferences of individual $\omega \in \Omega$ of generation t are represented by

$$U = c_{1t}^{\alpha_1}(1 - \ell_t)^{\alpha_2} b_t^{\alpha_3} c_{2t}^{\alpha_4} \tag{1}$$

where c_{it} $i = 1,2,$ denotes the consumption spending of individuals in generation t during the first and second periods of their lives; ℓ_t denotes the labor supply of individuals in generation t (for simplicity of exposition we assume that $0 \leq \ell_t \leq 1$, so that $(1 - \ell_t)$ represents the amount of leisure during the working period of the individual's lifetime); b_t denotes the bequest transfer of an individual of generation t to his offspring, which, in our model, is motivated by the "joy of giving," $\alpha_i > 0$, $i = 1, 2, 3, 4$, are parameters. We assume that α_1 and α_2 are constants and for each $\omega \in \Omega$, α_3 and α_4, the parameters representing the inclination of parents to support their offspring at the expense of their own second period consumption, are functions of a random variable ξ_ω which takes values in some compact interval. We assume that for each family ω the random variable $\{\xi_{\omega t}\}_{t=0}^{\infty}$ are independently and identically distributed and that the common distribution is that of ξ_{ω}. Furthermore, we assume that $\{\xi_{\omega}\}_{\omega \in \Omega}$ are independently and identically distributed and have a density η. For all $\omega \in \Omega$ the realization of ξ_{ω} in period t, denoted $\xi_{\omega t}$, determines $\alpha_3(\xi_{\omega t})$ and $\alpha_4(\xi_{\omega t})$ and **these values are known to individual ω at the outset of the first period of his life.** This implies that insofar as the individuals in this model are concerned they make their decisions under certainty. To simplify the exposition we assume that $\alpha_1 + \alpha_2 + \alpha_3(\xi) + \alpha_4(\xi) = 1$ for all ξ.

The utility function of each $\omega \in \Omega$ depends upon the realizations of $\xi_{\omega t}$. This heterogeneity in tastes in conjunction with the inequality of the bequests results in dispersion in the transfers to the coming generation, independently of whether or not the economy is in steady state. Finally, to simplify the notation, when there is no danger of confusion we shall write ξ instead of $\xi_{\omega t}$.

Technology: Production in this economy is carried out by competitive firms that use labor and capital to produce a single commodity. The commodity serves for consumption and investment. Following Diamond (1965), we assume that the stock of capital in each period, K_t, is determined by the level of saving in the preceding period. The aggregate production function $F(K_t, L_t)$ is assumed to exhibit constant returns to scale, where K_t is the aggregate level of capital and L_t the aggregate labor input. We also assume that $F_{KK} < 0$, $F_{LL} < 0$ and $F_{KL} > 0$.

2.2 *Equilibrium*

In each period the economy features three markets; two factor markets, namely, labor and capital, and a commodity market. To define competitive equilibrium we begin by considering the state of the economy at the outset of period t. Each family $\omega \in \Omega$ is composed of two members, the "old" member belongs to G_{t-1} and the "young" member belongs to G_t. Suppose that the distribution of the bequests received by individuals of generation G_t is given by the function $b_{t-1}: \Omega \to [0, m]$, where $m < \infty$. (We suppress the dependence of $b_{t-1}(\omega)$ on $\xi_{\omega(t-1)}$ for notational convenience.) Given his inheritance, $b_{t-1}(\omega)$, the wage rate, w_t, and the rates of interest, r_t and

r_{t+1}, each individual $\omega \in G_t$ with $\xi_{\omega t} = \xi$ chooses the level of saving, $s_t(\omega, \xi)$, the level of bequest, $b_t(\omega, \xi)$ and the level of labor supply, $\ell_t(\omega, \xi)$, so as to maximize his utility given in equation (1) with $\alpha_3 = \alpha_3(\xi)$ and $\alpha_4 = \alpha_4(\xi)$ subject to

(2) $$c_{1t} = b_{t-1}(1 + r_t) + w_t\ell_t - s_t - b_t,$$

and

(3) $$c_{2t} = s_t(1 + r_{t+1}).$$

Note that in period $t = 0$ individual ω in G_{-1} consumes $c_{2,-1}(\omega) = (1+r_0)s_{-1}(\omega)$.

Definition 1: Given K_0, s_{-1} and b_{-1}, a **competitive equilibrium** is a sequence of functions, $\{c_{1t}(\omega, \xi), c_{2t}(\omega, \xi), \ell_t(\omega, \xi), s_t(\omega, \xi), b_t(\omega, \xi)\}_{t=0}^{\infty}$, and a sequence of prices $\{w_t, r_t\}_{t=0}^{\infty}$, such that for all t, t = 0,1,2,...,

(a) For all (ω, ξ), $(c_{1t}, c_{2t}, \ell_t, s_t, b_t)$ is the solution to the maximization problem (1)–(3).

(b) $$\iint \ell_t(b_{t-1}(\omega)(1+r_t), w_t, r_{t+1}, \xi)\mu(\omega)\eta(\xi)d\omega d\xi = F_L^{-1}(K_t, w_t),$$

(c) $$K_t = F_K^{-1}(L_t, r_t)$$

(d) $$K_{t+1} = \iint [b_t(b_{t-1}(\omega)(1+r_t), w_t, r_{t+1}, \xi) + s_t(b_{t-1}(\omega)(1+r_t), w_t, r_{t+1}, \xi)]\mu(\omega)\eta(\xi)d\omega d\xi.$$

Condition (a) asserts that the various demand functions in the economy are derived from optimal consumer behavior assuming that all consumers are price takers. Conditions (b) and (c), are the equilibrium conditions in the labor and capital markets, respectively. The specification of the demand functions is based on the assumption that firms are price takers in the factor markets. Condition (d) describes the dynamic adjustment of the aggregate capital stock in the economy assuming full depreciation of the capital stock in each period. These conditions, in conjunction with the constraints (2) and (3) imply the material balance condition:

$$(4) \qquad \iint [c_{1t}(\omega,\xi) + c_{2(t-1)}(\omega,\xi)]\mu(\omega)\eta(\xi)d\omega d\xi + K_{t+1} = F(K_t, L_t), \text{ for } t = 0,1,\ldots,.$$

The existence of a competitive equilibrium in this economy can be shown using standard methods. We do not prove it here.

2.3 *Demand Functions and Income*

For each given value ξ of $\xi_{\omega t}$, solving the maximization problem (1)–(3) with the utility function

$$(5) \qquad U(\xi) = c_{1t}^{\alpha_1}(1-\ell_t)^{\alpha_2} b_t^{\alpha_3(\xi)} c_{2t}^{\alpha_4(\xi)}$$

and denoting the optimal values of the variables by asterisks, *we obtain the reduced form solution* of c_{1t}^*, s_t^*, b_t^* and $1-\ell_t^*$; namely,

$$(6) \qquad c_{1t}^*(\omega) = \alpha_1(1+r_t)[\frac{w_t}{1+r_t} + b_{t-1}(\omega)]$$

$$(7) \qquad s_t^*(\omega,\xi) = \alpha_4(\xi)(1+r_t)[\frac{w_t}{1+r_t} + b_{t-1}(\omega)]$$

$$(8) \qquad b_t^*(\omega,\xi) = \alpha_3(\xi)(1+r_t)[\frac{w_t}{1+r_t} + b_{t-1}(\omega)]$$

$$(9) \qquad 1-\ell_t^*(\omega) = \alpha_2[1 + \frac{1+r_t}{w_t} b_{t-1}(\omega)].$$

Observe that since α_1 and α_2 are constant and $\alpha_1 + \alpha_2 + \alpha_3(\xi_{\omega t}) + \alpha_4(\xi_{\omega t}) = 1$ for all values of $\xi_{\omega t}$, c_{1t}^* and ℓ_t^* are independent of $\xi_{\omega t}$. However, the bequest transfer b_t^* and s_t^* depend on the realization of $\xi_{\omega t}$. Thus we obtain a nondegenerate distribution of intergenerational transfers in each period. Furthermore, the income, $y_t(\omega)$ of individual $\omega \in G_t$ in period t, defined by:

(10) $$y_t(\omega) = w_t\ell_t(\omega) + (1+r_t)b_{t-1}(\omega),$$

is independent of $\xi_{\omega t}$. (The dependence of income on past realization of ξ_ω is summarized in $b_{t-1}(\omega)$). Next we express income in reduced form,

(11) $$y_t(\omega) = (1 - \alpha_2)(1 + r_t)[\frac{w_t}{1+r_t} + b_{t-1}(\omega)].$$

The aggregate level of income in period t is given by:

(12) $$Y_t = \int y_t(\omega)\mu(\omega)d\omega = (1 - \alpha_2)(1 + r_t)[\frac{w_t}{1+r_t} + B_{t-1}],$$

where $B_{t-1} = \iint b_{t-1}(\omega, \xi)\mu(\omega)\eta(\xi)d\omega d\xi$.

Finally, from (9), the aggregate labor supply in period t, $L_t = \int \ell_t(\omega)\mu(\omega)d\omega$, is given by:

(13) $$L_t = 1 - \alpha_2 - \alpha_2\frac{1+r_t}{w_t}B_{t-1}.$$

3. Aggregative and Distributional Effects of Technological Innovations

3.1 *The Measurement of Income Inequality*

A formal analysis of the distributional effects of technological changes requires a formal measure of income inequality. To define such a measure we need the following notation. Let X and Z be two random variables with values in a bounded interval in $\mathbb{R}$, and let m_x and m_z denote their respective means. Define $\hat{X} = X/m_x$ and $\hat{Z} = Z/m_z$ and denote by F_x and F_z the cumulative distribution functions of $\hat{X}$ and $\hat{Z}$, respectively. Let $[a, b]$ be the smallest interval containing the supports of $\hat{X}$ and $\hat{Z}$.

Definition 2: F_x **is more equal than** F_z if, for all $t \in [a, b]$, $\int_a^t [F_x(s) - F_z(s)]ds \leq 0$.

This definition, due to Atkinson (1970), is equivalent to the requirement that the Lorentz curve corresponding to X is everywhere above that of Z. Thus, if F_x is more equal than F_z according to definition 2, then it has a lower Gini index. We say that X is more equal than Z if the c.d.f. of $\hat{X}$ and $\hat{Z}$ satisfy: F_x is more equal than F_z. Henceforth the relation X is more equal than Z is denoted $X \gg Z$. X is equivalent to Z, $X \approx Z$, if $X \gg Z$ and $Z \gg X$.

The following result concerning the relation $\gg$ between two random variables will be needed in the sequel.

Lemma 1: Let Z and Z' be bounded random variables, then $A > B$ implies $A + Z \gg B + Z$.

PROOF (i) Given Z let $\mathscr{A}(Z) = \{H(A) \equiv \frac{A + Z}{A + m_z} \mid A \in \mathbb{R}\}$. Then, for each ω, $\text{sgn}\frac{\partial H}{\partial A}(\omega) = \text{sgn}(m_z - Z(\omega))$. Thus, $B < A$ implies that, for every ω such that $Z(\omega) \leq m_z$ $H(B)(\omega) \leq H(A)(\omega)$, and for every ω such that $Z(\omega) > m_z$ $H(B)(\omega) > H(A)(\omega)$. But for all s $F_{H(A)}(s) = \int_{\{\omega \mid Z(\omega) \leq s\}} \mu(\omega)d\omega$. Thus, for $s \leq 1$ $F_{H(A)}(s) \leq F_{H(B)}(s)$, and for $s > 1$ $F_{H(A)}(s) > F_{H(B)}(s)$. Hence, $F_{H(A)}(s)$ is more equal than $F_{H(B)}(s)$. ◇

Applying Definition 2 to income inequality in the model of section 2 we observe that, by equations (11) and (12), and using the above notation,

$$\hat{y}_t(\omega) = \frac{w_t/(1+r_t) + b_{t-1}(\omega)}{w_t/(1+r_t) + B_{t-1}}. \quad (14)$$

Consequently, given the distribution of bequests $b_{t-1}(\cdot)$ an increase in $w_t/(1+r_t)$ leads to a greater equality in the distribution of income in period t. Thus, the immediate distributional effects of technological changes depend on the effects of these changes on the relative factor prices. In the long–run the relative factor prices depend also on the changes in the capital–labor ratio induced by the new technologies and on the effects of the changing technology on the intergenerational transfers.

3.2 *Technological Changes — Definitions*

To examine the effects of improved technology on income inequality we conduct the following comparative dynamics analysis. We take the distribution of incomes at the point at which the technological innovation is introduced as given. We also assume that the new technology is unanticipated. We consider three kinds of exogenous changes in period $t=0$ representing permanent shifts in the production technology. These shifts are represented parametrically by pairs $(\gamma_{1t}, \gamma_{2t})$, $t \geq 0$, and are defined by $F(\gamma_{1t} K_t, \gamma_{2t}L_t)$, where $F(\cdot, \cdot)$ is the production function. A **Hicks–neutral** technological improvement is characterized by: $\gamma_{1t} = \gamma_{2t} = \gamma_t$, $\gamma_t = 1$ for $t < 0$, $\gamma_t = \gamma > 1$ for $t \geq 0$. A **Harrod–neutral** technological improvement is characterized by $\gamma_{1t} = 1$ for all t, $\gamma_{2t} = 1$ for $t < 0$ and $\gamma_{2t} = \gamma > 1$ for $t \geq 0$. A **Solow–neutral** technological improvement is characterized by $\gamma_{2t} = 1$ for all t, $\gamma_{1t} = 1$ for $t < 0$ and $\gamma_{1t} = \gamma > 1$ for $t \geq 0$.

We denote by prime superscript the values of the various variables following the introduction of the technological innovation, and by σ the (constant) elasticity of substitution (For a definition of the elasticity of substitution see Allen (1938).) It will become apparent in the sequel that this assumption involves no essential loss of generality. Finally, for all t let ζ_t be defined by $B'_{t-1} = \zeta_t B_{t-1}$ and note that, by equations (7), (8), and (11), $K'_t = \zeta_t K_t$. The following result is essential for the subsequent analysis.

Lemma 2: Suppose that a technological change represented by $(\gamma_{1t}, \gamma_{2t})$ occurs in period $t = 0$ then, for all $t \geq 1$,

(i) $\sigma > 1$ implies $(\frac{\gamma_{1t}K'_t}{\gamma_{2t}L'_t} \gtreqless \frac{K_t}{L_t} \Leftrightarrow \zeta_t X'_t \gtreqless X_t)$,

(ii) $\sigma = 1$ implies $\zeta_t X'_t = X_t$,

(iii) $\sigma < 1$ implies $(\frac{\gamma_{1t}K'_t}{\gamma_{2t}L'_t} \gtreqless \frac{K_t}{L_t} \Leftrightarrow \zeta_t X'_t \lesseqgtr X_t)$.

PROOF (i) Suppose that $\sigma > 1$ and, for some $t \geq 1$, $\beta_t \equiv \frac{\gamma_{1t}K'_t}{\gamma_{2t}L'_t} / \frac{K_t}{L_t} > 1$ and $\zeta_t X'_t \leq X_t$. By equation (13) $L'_t \geq L_t$. Since $K'_t = \zeta_t K_t$ we have, $\gamma_{1t}K'_t/\gamma_{2t}L'_t \leq \gamma_{1t}\zeta_t K_t/\gamma_{2t}L_t$. Thus, $\gamma_{1t}\zeta_t/\gamma_{2t} \geq \beta_t$. Note that $\frac{\gamma_{2t}}{\gamma_{1t}}X'_t = \frac{F_K(\gamma_{1t}K'_t/\gamma_{2t}L'_t,\ 1)}{F_L(\gamma_{1t}K'_t/\gamma_{2t}L'_t,\ 1)}$. Hence, by definition of σ, the fact that $\beta_t > 1$, and the preceding inequality,

$$1 < \frac{\beta_t - 1}{(\gamma_{1t}X_t/\gamma_{2t}X'_t) - 1} \leq \frac{(\gamma_{1t}\zeta_t/\gamma_{2t}) - 1}{(\gamma_{1t}X_t/\gamma_{2t}X'_t) - 1}.$$

Hence, $\zeta_t X'_t > X_t$, a contradiction.

Suppose, that $\beta_t < 1$. By equation (13) $\zeta_t X'_t > X_t$ implies $L'_t < L_t$ and, consequently, $\gamma_{1t}K'_t/\gamma_{2t}L'_t > \gamma_{1t}\zeta_t K_t/\gamma_{2t}L_t$. By definition of σ, the fact that $\beta_t < 1$, and the preceding inequality, which implies $\beta_t > \gamma_{1t}\zeta_t/\gamma_{2t}$ we have,

$$1 < \frac{\beta_t - 1}{(\gamma_{1t}X_t/\gamma_{2t}X'_t) - 1} < \frac{(\gamma_{1t}\zeta_t/\gamma_{2t}) - 1}{(\gamma_{1t}X_t/\gamma_{2t}X'_t) - 1}.$$

Since both the numerator and the denominator are negative we get in this case $\zeta_t X'_t < X_t$, a contradiction.

Next suppose that $\beta_t = 1$. Since F_K and F_L are continuous functions and since as $\beta_t \to 1$ and $\beta_t > 1$, we have $\zeta_t X_t' > X_t$ and for $\beta_t \to 1$ and $\beta_t < 1$, we have $\zeta_t X_t' < X_t$, we must have $\beta_t = 1 \Leftrightarrow \zeta_t X_t' = X_t$. This completes the proof of (i). The proof of (ii) and (iii) is similar. ◇

Corollary 1: For Hicks–neutral, Harrod–neutral, or Solow–neutral technological changes lemma 2 holds with the appropriate parameter configurations (γ, γ), $(1, \gamma)$, and $(\gamma, 1)$, respectively.

3.3 *The Effects of Hicks–Neutral Technological Changes*

Let $Q_t = F(\gamma K_t, \gamma L_t)$ $\gamma > 1$. Then, competitive equilibrium implies

(15) $$X_t \equiv \frac{1+r_t}{w_t} = \frac{\partial Q_t/\partial K_t}{\partial Q_t/\partial L_t}.$$

Thus, in the case of Hicks–neutral technological change,

(16) $$X_t = \frac{\gamma F_K(K_t,L_t)}{\gamma F_L(K_t,L_t)} = \frac{F_K(K_t,1 - \alpha_2 - \alpha_2 B_{t-1} X_t)}{F_L(K_t,1-\alpha_2 - \alpha_2 B_{t-1} X_t)},$$

where the second equality follows from equations (13) and (15). Consequently, $\partial X_t/\partial\gamma = 0$. Hence, since for all ω $b_{-1}(\omega)$ is predetermined, it follows from equation (14) that a Hicks–neutral technological change does not affect the distribution of income during the period in which the change occurs.

For all $t \geq 1$, the effect of an increase in K_t and B_{t-1} on X_t may be inferred from equation (16), i.e., since $F_{KL} > 0$

(17) $$A \frac{\partial X_t}{\partial K_t} = \frac{F_L F_{KK} - F_K F_{KL}}{F_L^2} < 0 \quad \text{for all } t,$$

where $A = 1 + \frac{F_{KL}F_L - F_{LL}F_K}{F_L^2} \alpha_2 B_{t-1} > 0$. In addition, for all t,

$$\frac{\partial X_t}{\partial B_{t-1}} = \frac{F_K F_{LL} - F_L F_{KL}}{F_L^2} \alpha_2 X_t < 0. \tag{18}$$

The effects of an Hicks–neutral technological change on the aggregate capital stock, the aggregate labor supply, and the inequality in the distribution of incomes are summarized in the following theorem.

Theorem 1: Given the economy in section 2, if an unanticipated Hicks–neutral technological improvement is introduced in period $t = 0$ then:

(a) In period $t = 0$ the distribution of income is unaffected, (i.e., $y_0' \approx y_0$,), the aggregate labor supply remains unchanged, (i.e., $L_0' = L_0$).

(b) For all $t \geq 1$, the inequality in the distribution of income decreases, increases, remains unchanged, if and only if the elasticity of substitution is larger than one, smaller than one, or equal to one, respectively.

(c) For all $t \geq 1$, the aggregate labor supply decreases, increases, remains unchanged, if and only if the elasticity of substitution is, respectively, larger than one, smaller than one, or equal to one.

(d) For all $t \geq 1$, the aggregate stock of capital increases.

PROOF Part (a) was proved by the argument preceding Lemma 2. To prove (b) – (d) let $\sigma > 1$, and $t = 1$. Since $X_0' = X_0$ and $(1 + r_0') = \gamma(1 + r_0)$ equations (7) and (8) imply $B_0' = \gamma B_0$ and $S_0' = \gamma S_0$. Thus, by definition 1(d) that $K_1' = \gamma K_1 > K_1$. Moreover, $L_1' < \gamma L_1$. (To see this suppose that $L_1' \geq \gamma L_1$, then using equation (13), $L_1' - \gamma L_1 = (1 - \alpha_2)(1 - \gamma) - \alpha_2 \gamma B_0(X_1' - X_1) \geq 0$. Since $\gamma > 1$, the first expression on the righthand side of the last equation is negative. This implies that $X_1' < X_1$, or explicitly, $F_K(K_1'/L_1', 1)/F_L(K_1'/L_1', 1) < F_K(K_1/L_1, 1)/F_L(K_1/L_1, 1)$.

Thus, $K_1'/L_1' > K_1/L_1$, a contradiction.) Hence, we have $K_1'/L_1' > K_1/L_1$. By lemma 2(i) this implies $\gamma X_1' > X_1$. Consequently, by equation (13), $L_1' < L_1$. Since y_1' is proportional to $[(\gamma X_1')^{-1} + b_0]$ and y_1 is proportional to $[(X_1)^{-1} + b_0]$, $\gamma X_1' > X_1$ and lemma 1 imply $y_1 \gg y_1'$.

We proceed by induction. Suppose that in period $t \geq 1$ $y_t' \gg y_t$, $K_{t+1}' = \zeta_{t+1}K_{t+1}$, $\zeta_{t+1} > 1$, $L_t' < L_t$ (and consequently, $K_t'/L_t' > K_t/L_t$). Equations (17) and (18) imply $X_{t+1} > X_{t+1}'$. Thus, $F_K(K_{t+1}'/L_{t+1}', 1)/F_L(K_{t+1}'/L_{t+1}', 1) < F_K(K_{t+1}/L_{t+1}, 1)/F_L(K_{t+1}/L_{t+1}, 1)$, hence, $K_{t+1}'/L_{t+1}' > K_{t+1}/L_{t+1}$. By lemma 2(i) $\zeta_{t+1}X_{t+1}' > X_{t+1}$. Hence, by equation (13) $L_{t+1}' < L_{t+1}$. Thus, $\sigma > 1$ implies $L_t' < L_t$ for all $t \geq 1$.

Equation (14), the fact that $\zeta_{t+1}X_{t+1}' > X_{t+1}$, and lemma 1 imply $y_{t+1}' \gg y_{t+1}$. Hence, $\sigma > 1$ implies $y_t' \gg y_t$ for all $t \geq 1$.

Suppose, by way of negation, that $K_{t+2} > K_{t+2}'$. By equations (7), (8) and definition 1(d), this implies $B_{t+1} > B_{t+1}'$. But integrating equation (8) and using the fact that $X_{t+1}'B_t' = X_{t+1}'\zeta_{t+1}B_t > X_{t+1}B_t$ implies $B_{t+1}/w_{t+1} = \alpha_2[1 + X_{t+1}B_t] < \alpha_2[1 + X_{t+1}'B_t'] = B_{t+1}'/w_{t+1}'$. Thus, it follows that $w_{t+1} > w_{t+1}'$. But the last inequality implies $K_{t+1}'/L_{t+1}' < K_{t+1}/L_{t+1}$. By the induction assumption, however, $K_{t+1}' \geq K_{t+1}$ and as was established $L_{t+1}' < L_{t+1}$, a contradiction. Thus, $\sigma > 1$ implies that for all $t \geq 1$, $K_t' \geq K_t$.

If $\sigma < 1$, by a similar argument we get, for all $t \geq 1$, $L_t' < L_t$, $y_t \gg y_t'$, and $K_t' \geq K_t$. Notice, in particular, that while the argument is the same as in the case $\sigma > 1$, the conclusions regarding the income inequality and the aggregate labor supply are reversed since, by lemma 2(iii) $\sigma < 1$, implies $(K_t'/L_t' > K_t/L_t \implies \zeta_t X_t' < X_t)$. The conclusions follow from the fact that in this case $\zeta_t X_t' < X_t$ by equations (13) and (14).

Let $\sigma = 1$. Since $B_0' = \gamma B_0$ lemma 2 implies $\gamma X_1' = X_1$. Thus, by equation (14) and lemma 1, we get $y_1' \approx y_1$. Proceeding by induction, suppose that $\zeta_{t+1} > 1$, $L_t' = L_t$, and $y_t' \approx \hat{y}_t$. As before this implies $B_t' = \zeta_{t+1}B_t$ and, by lemma 2(ii) $\zeta_{t+1}X_{t+1}' = X_{t+1}$. Therefore, $(X_{t+1}B_t)^{-1} = (X_{t+1}'B_t')^{-1}$, and by equation (14) and lemma 1 $y_{t+1} \approx y_{t+1}'$. The proof that $L_t' = L_t$ follows immediately from equation (13) and the fact that $X_{t+1}B_t = X_{t+1}'B_t'$. To show that $K'_{t+2} \geq K_{t+2}$ suffices it to note that (by integration of equation (8) and simple manipulation) $B'_{t+1} = w'_{t+1}[1 + X'_{t+1}B'_t] > w_{t+1}[1 + X_{t+1}B_t] = B_{t+1}$, where the inequality follows from the fact that $X_{t+1}B_t = X'_{t+1}B'_t$ and the configuration of inputs $L_{t+1} = L'_{t+1}$ and $K'_{t+1} \geq K_{t+1}$ that imply $w'_{t+1} > w_{t+1}$. ◇

3.4 *The Effects of Harrod–Neutral Technological Changes*

Unlike Hicks–neutral technological changes, Harrod–neutral technological changes affect the relative factor prices and, as a consequence, the income distribution as soon as they occur. Subsequent effects depend on the elasticity of substitution, and are described in theorem 2 below.

Theorem 2: Given the economy in section 2, if an unanticipated Harrod–neutral technological improvement occurs in period $t = 0$ then:

(a) For all $t \geq 0$, the inequality in the distribution of income decreases, increases, remains unchanged, if and only if the elasticity of substitution is larger than one, smaller than one, or equal to one, respectively.

(b) For all $t \geq 1$, the aggregate labor supply increases, decreases, remains unchanged, if and only if the elasticity of substitution is, respectively, larger than one, smaller than one, or equal to one.

(c) If $\sigma \geq 1$ then, for all $t \geq 1$, the aggregate stock of capital increases.

Remark Note that when $\sigma < 1$ the effect of Harrod–neutral technological change on the aggregate capital stock is ambiguous. For σ sufficiently close to one the capital stock increases. However, if the elasticity of substitution is small enough the aggregate capital stock may actually decline in the aftermath of the introduction of the new technology. This is a result of the fact that we may have situations in which $L'_t < L_t$ while $\gamma L'_t > L_t$.

PROOF Let $Q_t = F(K_t, \gamma L_t)$, $\gamma > 1$, $t = 0, 1, \ldots$. Then, by definition,

$$X_t = \frac{F_K(K_t, (1-\alpha_2)\gamma - \alpha_2\gamma B_{t-1}X_t)}{\gamma F_L(K_t, (1-\alpha_2)\gamma - \alpha_2\gamma B_{t-1}X_t)}. \tag{19}$$

Holding K_t and B_{t-1} constant and differentiating X_t with respect to γ we get:

$$\frac{\partial X_t}{\partial\gamma} = \frac{F_K F_L}{\gamma^2}\left[\frac{\gamma L_t\left[\frac{F_{KL}}{F_K} - \frac{F_{LL}}{F_L}\right]K_t^{-1} - 1}{F_L^2 + (F_{KL}F_L - F_K F_{LL})\alpha_2 B_{t-1}/K_t}\right]. \tag{20}$$

It is easy to verify that, for all t,

$$\frac{\gamma L_t}{K_t}\left[\frac{F_{KL}(K_t, \gamma L_t)}{F_K(K_t, \gamma L_t)} - \frac{F_{LL}(K_t, \gamma L_t)}{F_L(K_t, \gamma L_t)}\right] = \frac{1}{\sigma}, \tag{21}$$

where σ is the elasticity of substitution. Since in period t K_t and B_{t-1} are given equations (20) and (21) imply

$$\frac{\partial X_t}{\partial\gamma} \gtreqless 0 \Leftrightarrow \sigma \lesseqgtr 1. \tag{22}$$

(a) Let $\sigma > (<, =)\ 1$. Since K_0 and B_{-1} are predetermined, equation (22) implies that $X_0' < (>, =)\ X_0$. Hence, by equation (14) and lemma 1, $y_0' \gg y_0$ ($y_0 \gg y_0'$, $y_0 \approx y_0'$). Moreover, by equation (13), $L_t' > (<, =)\ L_t$. Since $K_0'/\gamma L_0' < K_0/L_0$, $(1 + r_0') > (1 + r_0)$, together with $X_0' < X_0$ (or $X_0' = X_0$) this implies (integrating on both sides of equations (7) and (8),) that $K_1' > K_1$. This establishes the theorem for $t = 0$.

We proceed by induction. Suppose that $\sigma > 1$, and for some $t \geq 1$ $y_t' \gg y_t$, $K_{t+1}' > K_{t+1}$, $L_t' > L_t$, and $K_t'/\gamma L_t' < K_t/L_t$. (Note that by lemma 2(i) $\zeta_t X_t' < X_t$. But $K_t' \geq K_t$

implies $\zeta_t \geq 1$, hence, $X'_t < X_t$.)

Claim 1: $\gamma > \zeta_{t+1}$.

Proof of claim 1 $w'_t = \gamma F_L(K'_t/\gamma L'_t, 1) \leq \gamma F_L(K_t/L_t, 1) = \gamma w_t$. Moreover, $(1 + r'_t)\zeta_t \leq \gamma(1 + r_t)$. (Otherwise $w'_t \leq \gamma w_t$ would imply $\zeta_t X'_t > X_t$. But $\sigma > 1$ and by the induction assumption $K'_t/\gamma L'_t < K_t/L_t$, hence lemma 2(i) implies $\zeta_t X'_t < X_t$, a contradiction.) Hence, integrating equation (8), we have

$$\zeta_{t+1} \equiv \frac{B'_t}{B_t} < \gamma\frac{w_t + (1 + r_t)B_{t-1}}{w_t + (1 + r_t)B_{t-1}} = \gamma.$$

This completes the proof of claim 1.

Claim 2: $K'_{t+1}/\gamma L'_{t+1} < K_{t+1}/L_{t+1}$.

Proof of claim 2 Suppose that $K'_{t+1}/\gamma L'_{t+1} \geq K_{t+1}/L_{t+1}$, then,

$$\zeta_{t+1}X'_{t+1} = \frac{\zeta_{t+1}F_K(K'_{t+1}/\gamma L'_{t+1}, 1)}{\gamma F_L(K'_{t+1}/\gamma L'_{t+1}, 1)} < \frac{F_K(K'_{t+1}/\gamma L'_{t+1}, 1)}{F_L(K'_{t+1}/\gamma L'_{t+1}, 1)} \leq \frac{F_K(K_{t+1}/L_{t+1}, 1)}{F_L(K_{t+1}/L_{t+1}, 1)} = X_{t+1},$$

where the first inequality follows from claim 1 and the second from the supposition. However, by lemma 2(i), $\sigma > 1$ and the supposition imply $\zeta_{t+1}X'_{t+1} \geq X_{t+1}$, a contradiction.

Claim 2, $\sigma > 1$, and lemma 2(i) imply $\zeta_{t+1}X'_{t+1} < X_{t+1}$. Hence, by equation (14), $y'_{t+1} \gg y_{t+1}$ and, by equation (13), $L'_{t+1} > L_{t+1}$. Hence, for all $t \geq 0$, $\sigma > 1$ $y'_t \gg y_t$ and $L'_t > L_t$.

$K'_{t+2} \geq K_{t+2}$ follows directly from equations (7), (8), definition 1(d), the fact that $X'_{t+1} < X_{t+1}$, and the fact that, since $K'_{t+1}/\gamma L'_{t+1} < K_{t+1}/L_{t+1}$, $(1 + r'_{t+1}) > (1 + r_{t+1})$. To see this note that, by the above inequalities, $w'_{t+1} > w_{t+1}$. Hence,

$$\frac{B'_{t+1}}{B_{t+1}} = \frac{w'_{t+1} + (1 + r'_{t+1})B'_t}{w_{t+1} + (1 + r_{t+1})B_t} > 1.$$

But $K'_{t+2}/K_{t+2} = B'_{t+1}/B_{t+1}$. This establishes part (c).

The proof for the case $\sigma = 1$ is straightforward and therefore will not be provided here.

Suppose next that $\sigma < 1$ then we have,

Claim 3: If $\sigma < 1$, then for all $t \geq 1$, $\zeta_t < \gamma \Leftrightarrow \frac{K'_t}{\gamma L'_t} < \frac{K_t}{L_t} \Leftrightarrow \zeta_t X'_t > X_t$.

Proof of claim 3: Since $\sigma < 1$, the second equivalence follows from lemma 2(iii). To prove the first equivalence suppose that $\zeta_t < \gamma$ and $\frac{K'_t}{\gamma L'_t} \geq \frac{K_t}{L_t}$. Then, by lemma 2(iii) this implies $\zeta_t X'_t \leq X_t$. Hence, by equation (13), $L'_t \geq L_t$. Together with $\zeta_t < \gamma$ this implies $\frac{K'_t}{\gamma L'_t} = \frac{\zeta_t K_t}{\gamma L'_t} < \frac{K_t}{L_t}$, a contradiction. Hence, $\zeta_t < \gamma \Rightarrow \frac{K'_t}{\gamma L'_t} < \frac{K_t}{L_t}$.

Suppose that $\frac{K'_t}{\gamma L'_t} < \frac{K_t}{L_t}$. By lemma 2(iii) $\zeta_t X'_t > X_t$, and by equation (13) $L'_t < L_t$. But $\frac{K'_t}{\gamma L'_t} = \frac{\zeta_t K_t}{\gamma L'_t} < \frac{K_t}{L_t}$ we have $\gamma/\zeta_t L'_t > L_t$. Hence, $\zeta_t < \gamma$. This completes the proof of the claim.

To prove the theorem we assert that for all $t \geq 0$ $\zeta_t X'_t \geq X_t$. We saw that this is true for $t = 0$. Suppose, by way of negation, that this assertion is not true and let t be the first period where $\zeta_t X'_t < X_t$. By claim 3 this implies $\zeta_t \geq \gamma$, and $\frac{K'_t}{\gamma L'_t} \geq \frac{K_t}{L_t}$. By equation (13) $L'_t > L_t$. However,

$$\zeta_t X'_t = \frac{\zeta_t F_K(K'_t/\gamma L'_t,\ 1)}{\gamma F_L(K'_t/\gamma L'_t,\ 1)} \geq \frac{F_K(K'_t/\gamma L'_t,\ 1)}{F_L(K'_t/\gamma L'_t,\ 1)} \geq \frac{F_K(K_t/L_t,\ 1)}{F_L(K_t/L_t,\ 1)} = X_t,$$

which is a contradiction. Hence, $\zeta_t X'_t > X_t$ for all $t \geq 0$. Thus, for all $t \geq 0$, by equation (14) $y_t \gg y'_t$ and, by equation (13), $L'_t < L_t$. Hence, $\sigma < 1$ implies that $y_t \gg y'_t$ and $L'_t < L_t$ for all $t \geq 0$. ◇

3.5 *The Effects of Solow–Neutral Technological Changes*

Solow–neutral technological changes are similar to Herrod–neutral technological changes except that the role of labor and capital are interchanged. It is not surprising, therefore that the effects on the aggregate behavior display certain symmetries. These become obvious upon comparing Theorem 2 and Theorem 3 below. Note, however, that, qualitatively speaking, the income distribution effects are the same.

Theorem 3: Given the economy in section 2, if an unanticipated Solow–neutral technological improvement occurs in period $t = 0$ then:

(a) For all $t \geq 0$, the inequality in the distribution of income increases, decreases, remains unchanged, if and only if the elasticity of substitution is larger than one, smaller than one, or equal to one, respectively.

(b) For all $t \geq 1$, the aggregate labor supply decreases, increases, remains unchanged, if and only if the elasticity of substitution is, respectively, larger than one, smaller than one, or equal to one.

(d) For all $t \geq 1$, the aggregate stock of capital increases.

PROOF By definition,

$$X_t = \frac{\gamma F_K(\gamma K_t, L_t)}{F_L(\gamma K_t, L_t)}, \quad \text{for all } t \geq 0. \tag{23}$$

Fixing K_t and B_{t-1} and differentiating X_t with respect to γ, we get:

(24) $$\frac{\partial X_t}{\partial \gamma}M = F_K F_L\left[1 + \gamma K_t\left(\frac{F_{KK}(\gamma K_t, L_t)}{F_K(\gamma K_t, L_t)} - \frac{F_{LK}(\gamma K_t, L_t)}{F_L(\gamma K_t, L_t)}\right)\right],$$

where $M = F_L^2 + \gamma\alpha_2 B_{t-1}(F_L F_{KL} - F_K F_{LL}) > 0$. But,

(25) $$\frac{1}{\sigma} = -\frac{\gamma K_t}{L_t}\left[\frac{F_{KK}(\frac{\gamma K_t}{L_t}, 1)}{F_K(\frac{\gamma K_t}{L_t}, 1)} - \frac{F_{LK}(\frac{\gamma K_t}{L_t}, 1)}{F_L(\frac{\gamma K_t}{L_t}, 1)}\right] =$$

$$-\gamma K_t\left[\frac{F_{KK}(\gamma K_t, L_t)}{F_K(\gamma K_t, L_t)} - \frac{F_{LK}(\gamma K_t, L_t)}{F_L(\gamma K_t, L_t)}\right]$$

Hence, $\frac{\partial X_t}{\partial \gamma}M = F_K F_L[1 - \frac{1}{\sigma}]$ and

(26) $$\frac{\partial X_t}{\partial \gamma} \gtreqless 0 \Leftrightarrow \frac{1}{\sigma} \lesseqgtr 1.$$

If a Solow–neutral technological change is introduced in period $t = 0$ then, by (26), $X_0' \gtreqless X_0$ if and only if $\sigma \gtreqless 1$. Then, by equation (14), $\sigma > 1$ implies $y_0 \gg y_0'$, $\sigma < 1$ implies $y_0' \gg y_0$, and if $\sigma = 1$ then the income distribution is unchanged. By equation (13) this implies $\sigma \gtreqless 1$ if and only if $L_0' \lesseqgtr L_0$. Finally, $K_0' = K_0$. This completes the proof for $t = 0$. Note that $\sigma > 1$ implies $X_0' > X_0$ and, by lemma 2(i), this implies $\gamma K_0/L_0' > K_0/L_0$.

Proceeding by induction, suppose that $\sigma > 1$ and for some t $y_t \gg y_t'$, $L_t' < L_t$, $K_t' \geq K_t$, and $\gamma K_t'/L_t' > K_t/L_t$.

Claim 4: For all $t \geq 1$, $\gamma\zeta_t > 1 \Leftrightarrow \gamma K'_t/L'_t > K_t/L_t \Leftrightarrow \zeta_t X'_t > X_t$.

Proof of claim 4: Suppose that $\gamma\zeta_t > 1$ and, contrary to the assertion, $\gamma K'_t/L'_t \leq K_t/L_t$, then, by lemma 2(i), $\zeta_t X'_t < X_t$. But, given the supposition,

$$\zeta_t X'_t = \frac{\gamma\zeta_t F_K(\gamma K'_t/L'_t,\ 1)}{F_L(\gamma K'_t/L'_t,\ 1)} > \frac{F_K(\gamma K'_t/L'_t,\ 1)}{F_L(\gamma K'_t/L'_t,\ 1)} \geq \frac{F_K(K_t/L_t,\ 1)}{F_L(K_t/L_t,\ 1)} = X_t,$$

a contradiction. Thus, $\gamma\zeta_t > 1$ implies $\gamma K'_t/L'_t > K_t/L_t$.

Suppose that $\gamma K'_t/L'_t > K_t/L_t$ and $\gamma\zeta_t \leq 1$. By lemma 2(i), $\zeta_t X'_t > X_t$. But,

$$\zeta_t X'_t = \frac{\gamma\zeta_t F_K(\gamma K'_t/L'_t,\ 1)}{F_L(\gamma K'_t/L'_t,\ 1)} \leq \frac{F_K(\gamma K'_t/L'_t,\ 1)}{F_L(\gamma K'_t/L'_t,\ 1)} < \frac{F_K(K_t/L_t,\ 1)}{F_L(K_t/L_t,\ 1)} = X_t,$$

a conradiction. This completes the proof of the claim.

By the induction assumption $\gamma K'_t/L'_t > K_t/L_t$. This implies $w'_t > w_t$, and (by claim 4,) $\zeta_t X'_t > X_t$. Thus,

$$\zeta_{t+1} = \frac{B'_t}{B_t} = \frac{w'_t(1\ +\ \zeta_t X'_t B_{t-1})}{w_t(1\ +\ X_t B_{t-1})} > 1.$$

Hence, $\gamma\zeta_{t+1} > 1$ and, by claim 4, $\zeta_{t+1}X'_{t+1} > X_{t+1}$. The last inequality implies $y_{t+1} \gg y'_{t+1}$, (by equation (14)), and $L'_{t+1} < L_{t+1}$ (by equation (13)). Finally, since $K'_{t+1} = \zeta_{t+1}K_{t+1}$, we have $K'_{t+1} > K_{t+1}$. Thus, for all $t \geq 0$, $\sigma > 1$ implies $y_t \gg y'_t$, $L'_t < L_t$, and $K'_t > K_t$.

The proof for $\sigma = 1$ is straightforward and is omitted.

Suppose that $\sigma < 1$. Condition (26), implies $X'_0 < X_0$ and consequently $L'_0 > L_0$. By lemma 2(iii), $\gamma K_0/L_0 > K_0/L_0$. Hence, $w_0' > w_0$. If $1 + r_0' > 1 + r_0$ then integrating equation (8) we get $B_0' > B_0$, which implies $K_1' > K_1$. If $1 + r_0' < 1 + r_0$ then, since $F(\gamma K_0, L_0) > F(K_0, L_0)$, if $K_1' < K_1$ we must have (see equations (4) and (6)), $(1 + r_0')S_{-1} > (1 + r_0)S_{-1}$, which is a contradiction. Thus, $K_1' > K_1$.

Next we claim that $\gamma K_0'/L_0' > K_0/L_0$. Suppose otherwise, then, by definition,

$$X_0'/\gamma = \frac{F_K(\gamma K_0'/L_0',\ 1)}{F_L(\gamma K_0'/L_0',\ 1)} > \frac{F_K(K_0/L_0,\ 1)}{F_L(K_0/L_0,\ 1)} = X_0,$$

a conradiction since $X_0' < X_0$ and $\gamma > 1$.

Suppose that, for some $t \geq 1$, $y_t \gg y_t'$, $L_t' > L_t$, $K_{t+1}' \geq K_{t+1}$, and $\gamma K_t'/L_t' > K_t/L_t$.

Claim 5: For all $t \geq 1$, $\gamma K_t'/L_t' \leq K_t/L_t \implies X_{t+1}' \geq \gamma X_{t+1}$.

<u>Proof of claim 5</u>. $\gamma K_t'/L_t' \leq K_t/L_t$ implies that $w_t' \leq w_t$ and $(1 + r_t') = \gamma F_K(\gamma K_t'/L_t', 1) > \gamma F_K(K_t/L_t, 1) = \gamma(1 + r_t)$. The conclusion follows from the definition of X.

Claim 6: For all $t \geq 1$, $\gamma K_t'/L_t' \leq K_t/L_t$ implies $\gamma\zeta_t \leq 1$.

<u>Proof of claim 6</u> The hypothesis implies $L_t' \geq \gamma\zeta_t L_t$. If $\gamma\zeta_t > 1$ then, using claim 5,

$$\gamma\zeta_t L_t = \gamma\zeta_t(1-\alpha_2) - \alpha_2\gamma\zeta_t X_t B_{t-1} \geq (\gamma\zeta_t - 1)(1-\alpha_2) + (1-\alpha_2) - \alpha_2 X_t' B_{t-1}' > L_t',$$

which is a contradiction, and claim 6 is proved.

Suppose that $\gamma K_{t+1}'/L_{t+1}' \leq K_{t+1}/L_{t+1}$. Since $\gamma > 1$, by claim 6 $\zeta_{t+1} < 1$. But, $K_{t+1}' = \zeta_{t+1}K_{t+1} \geq K_{t+1}$, where the inequality follows from the induction assumption. Hence, $\zeta_{t+1} \geq 1$, a conradiction, and therefore $\gamma K_{t+1}'/L_{t+1}' > K_{t+1}/L_{t+1}$. By lemma 2(iii) we get $\zeta_{t+1}X_{t+1}' < X_{t+1}$. Hence, by equation (14), $y_{t+1}' \gg y_{t+1}$ and, by equation (13), $L_{t+1}' > L_{t+1}$. This completes the proof of parts (a) and (b).

To prove part (c), note that if $K'_{t+2} < K_{t+2}$ while $K'_{t+1} > K_{t+1}$ then using equations (4), (6), and (3) we find that this implies $(1 + r'_t)S'_t > (1 + r_t)S_t$. Thus, $\zeta_{t+1}(1 + r_{t+1}) > (1 + r_{t+1})$. Integrating equation (8) we obtain,

$$1 > \zeta_{t+2} = \frac{B'_{t+1}}{B_{t+1}} = \frac{B'_t(1 + r'_{t+1})[(1/X'_{t+1}B'_t) + 1]}{B_t(1 + r_{t+1})[(1/X_{t+1}B_t) + 1]} > \frac{\zeta_{t+1}(1 + r'_{t+1})}{(1 + r_{t+1})} > 1,$$

a contradiction. ◇

4. Summary and Concluding Remarks

In this paper we raised the issue of the effects of technological improvements on the inequality in the distribution of incomes in a dynamic general equilibrium framework. We analyzed these effects as well as the effects on the aggregate capital stock and the aggregate labor supply within the context of a competitive overlapping generations economy with endogenous labor supply and a bequest motive, tracing the effects in each and every period following the introduction of the new technologies. Except in the case of Harrod–neutral technological changes when the elasticity of substitution is sufficiently small, the aggregate capital stock increases as a result of the introduction of the new technologies. The results concerning the effects of the the new technologies on the aggregate labor supply and the inequality in the distribution of income is summarized in the following table.

Elasticity of Substitution	Technological Change		
	Hicks–neutral	Harrod-neutral	Solow–neutral
$\sigma > 1$	$y_t' \gg y_t$ $L_t' < L_t$	$y_t' \gg y_t$ $L_t' > L_t$	$y_t \gg y_t'$ $L_t' < L_t$
$\sigma = 1$	$y_t' \approx y_t$ $L_t' = L_t$	$y_t' \approx y_t$ $L_t' = L_t$	$y_t' \approx y_t$ $L_t' = L_t$
$\sigma < 1$	$y_t \gg y_t'$ $L_t' > L_t$	$y_t \gg y_t'$ $L_t' < L_t$	$y_t' \gg y_t$ $L_t' > L_t$

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