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THE HOTELLING MODEL UNDER UNCERTAINTY:
A NOTE

by

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In a recent study, Fishelson (1989) examined the implication of continuous changes on the classical (static) Hotelling (1931) model. The changes took place in the demand for the resource, the costs of extraction of the resource and the costs of a backstop technology that produces a substitute for the resource. The common simplifying assumption for these three changes was that the rates of change were constants and a priori known. The purpose of this note is to examine the effects of random changes in the parameters of the Hotelling model. One way to do it is to return to the Hotelling static model and examine the effects of uncertainty in its parameters on the market path.

The Role of Uncertainty

The role of uncertainty in the economics of exhaustible resources was examined previously. Long (1975) allowed the reserves to change (via nationalization) sometimes in the future. Heal (1979) allowed for a single discovery of a random size at a random point in time. Kemp (1976) and Loury (1978) look at exploitation when reserves are unknown while Deshmukh and Pliska (1982) and Arrow and Change (1982) introduce stochastic discoveries, i.e., allowing the reserves to increase by a random (positive) quantity. The role of uncertainty is more pronounced in the analysis of markets of renewable resources (see Pindyck (1984) and references therein). Again the uncertainty is applied to the rate at which the resource is renewed, i.e., to the reserves that would be available.

The uniqueness of the present note is that the uncertainty is not limited to the reserves but shows up at any of the parameters determining the instantaneous market outcome and thus the market path. For the sake of simplicity we assume that the various random effects are independent over parameters and over time. This enables a separate analysis for each parameter. The setting is of a competitive market, i.e., all mine owners are aware of the uncertainty and are identical in their attitude towards risk.

The effect of uncertainty of any of the parameters of the system on the market path depends upon the relationship between the parameter and the two endogenous, although not independent, variables of the system, the initial price and the length of the extraction period. We recall that under certainty once one of these two is determined the price at each point in time and the quantity extracted are determined.

We start the analysis for the random world by looking at a non random non changing world. We assume that the demand for the resource is linear and stable: $Q = a - bP$ $(b > 0)$. Optimality at each point in time, denoted by 0, requires that from then on

$$(1) \qquad (P_t - c) = (P_0 - c)e^{rt}$$

$$(2) \qquad R_T = 0 \ , \quad \text{or} \quad \int_0^T Q_t dt = R_0$$

where c is the constant marginal (average) extraction costs and r a constant interest rate.

Employing these two conditions for solving for the two unknowns: the initial price, P_0, and the length of the remaining extraction period, T, yields respectively two equations each with one of two unknowns.

(3) $$-(a - bc)(\ln \frac{bP_0}{a}) + bP_0 + \frac{cP_0}{a} - a - bc - rR_0 = 0 ,$$

and

(4) $$rT(a - bc) + ae^{-rT} + bce^{rT} - a - bc - rR_0 = 0 .$$

One might view equations (3) and (4) as the reduced form equations of the market system. Thus, one can solve them for changes in the (exogenous) parameters, dP_0/dx, dT/dx, as long as there is no structural change in the market. The inference with regard to the instantaneous output, Q, is derived from the solution for P given the relation between P and Q, the market demand.

In order to find the effect of a change of the demand parameter "a" on P_0, equation (3) is totally differentiated w.r.t. "a" and P_0 from which the sign of dP_0/da is determined. Then, the term dP_0/da is again differentiated w.r.t. "a" which yields the d^2P_0/da^2. The same is done with equation (4) which yields dT/da and d^2T/da^2.

Given uncertainty the objective function of each of the mine owners changes from that of maximizing the present value of the stream of incomes to that of maximizing the expected utility from that present value. As is however well known if the firm is risk neutral the maximization of the expected value of the present value is identical to that of maximizing the expected utility. Thus for the sake of simplicity we assume, at this stage, that each of the mine owners is risk neutral. Hence, the market's objective function is

(5) $$\text{Max} \quad E \int_0^T (P_t - c_t) Q_t e^{-rt} dt$$

subject to

(5a) $$\int_0^T Q_t = E(R)$$

and

(5b) $$P_T = a^0/b^0$$

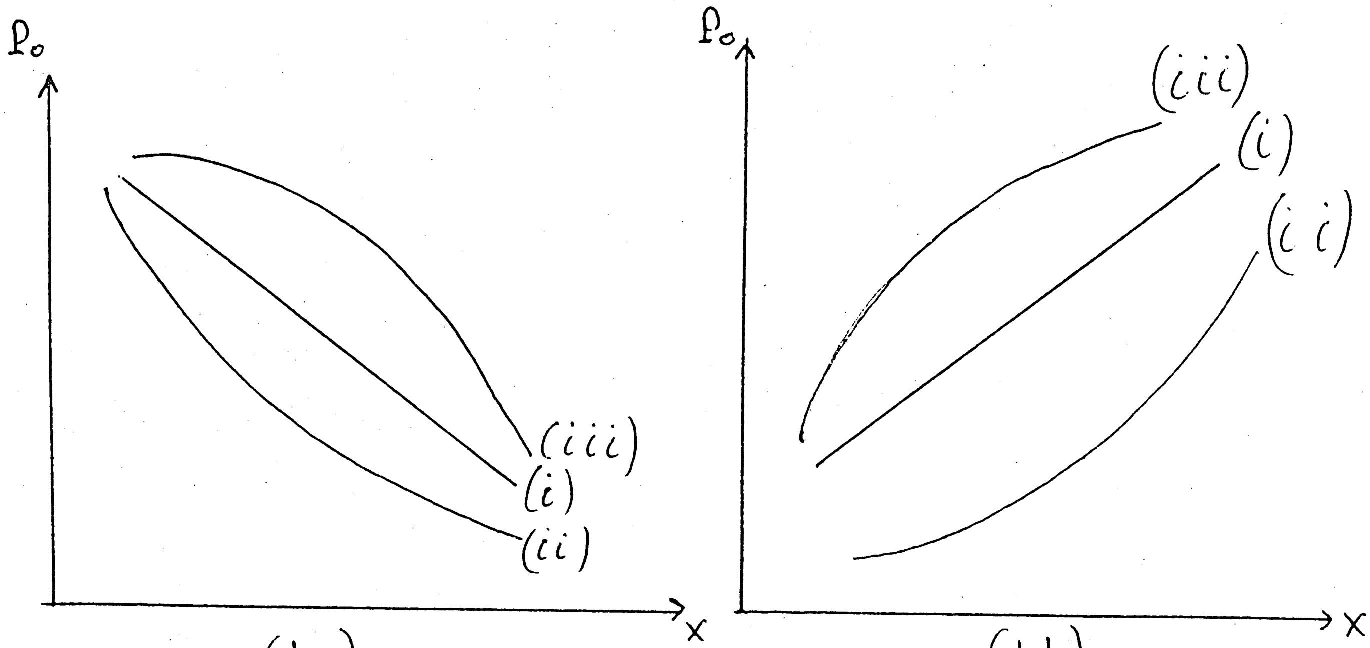
given that the demand function is stable and its expected expression is

(5c) $$Q_t = a^0 - b^0 P_t \quad \text{where} \quad a^0 = E(a) \quad \text{and} \quad b^0 = E(b) .$$

Uncertainty implies that either the cost of extraction, c_t, or the demand parameters, a and b, are random. This uncertainty is crucial since the quantity to be mined has to be determined before the market is realized. Furthermore, due to uncertainty the market realization would differ from one period to another. If the random effects on the demand parameters and the cost per unit are independent over time there is also no way to infer from the realization in one period on the realization next period or at any future period. Thus, when solving the problem stated in equations (5), in spite of the uncertainty, each firm decides on the path of quantities it would mine, at each point in time, corresponding to the expected prices that emerge from the process. Thus, also under uncertainty the initial price, P_0, and the planned extraction period, T, determine the path of quantities, Q_0 to Q_T, that would be mined.

The solution, however, with regard to P_0 and to the corresponding planned T, are affected by the uncertainty of the demand and costs parameters. The directions of the effects depend upon the relations between the random variables and P_0 and T. In order to identify the relations one has to know the signs of the first and second derivatives of P_0 and T w.r.t. the random variables. This takes us back to what was said above regarding the effect of a change in a parameter on P_0 and T.

Hence, to evaluate the effects of uncertainty upon P_0 and T one has to find whether they are linear, convex or concave in the random variables. Figures 1a and 1b below look at any parameter, X.



(1a) (1b)

Curves (i) imply linearity, thus there is no effect of the uncertainty of X upon P_0 ($E(P_0) = P_0|X = E(X)$). Curves (ii) imply convexity, thus the determined P_0 would be larger due to uncertainty of X ($E(P_0) > P_0|X = E(X)$). Curves (iii) imply concavity thus the determined P_0 will be lower due to uncertainty ($E(P_0) < P_0|X = E(X)$). The characteristics of the six curves are:

(i) $$\frac{\partial P_0}{\partial X} < 0 \quad \text{or} \quad \frac{\partial P_0}{\partial X} > 0 \quad \text{and} \quad \frac{\partial^2 P_0}{\partial X^2} = 0$$

(ii) $$\frac{\partial P_0}{\partial X} < 0 \quad \text{or} \quad \frac{\partial P_0}{\partial X} > 0 \quad \text{and correspondingly}$$

$$\frac{\partial^2 P_0}{\partial X^2} > 0 \quad \text{and} \quad \frac{\partial^2 P_0}{\partial X^2} < 0$$

(iii) $$\frac{\partial P_0}{\partial X} < 0 \quad \text{or} \quad \frac{\partial P_0}{\partial X} > 0 \quad \text{and correspondingly}$$

$$\frac{\partial^2 P_0}{\partial X^2} < 0 \quad \text{and} \quad \frac{\partial^2 P_0}{\partial X^2} > 0$$

In the following section the terms $\frac{\partial P_0}{\partial X}$, $\frac{\partial^2 P_0}{\partial X}$, $\frac{\partial T}{\partial X}$ and $\frac{\partial^2 T}{\partial X^2}$ are determined using equations (3) and (4).

Results and Conclusions

Table 1 contains the signs of the first and second derivatives of P_0 with respect to the market parameters (equation (3)). Table 2 contains the signs of the first and second derivatives of T with respect to the market parameters (equation (4)).

The reaction of Q. with respect to the variability of R was found by others to be negative (Loury (1978), Heal (1979)). Pindyck (1984b) calls this result a "precautionary" effect and explains it by the fixed reserves (he found a positive effect of the variance of future reserves on production explaining it by the possibility to adjust production as reserves change). We reestablish this negative sign. The signs of the other first derivatives are also known (see Herfindahl (1967)), but noone looked at the second derivatives. As is well known, the sign of the second derivative determines the direction of the effect of uncertainty on the expected value of the variable. Hence, using the findings in Table 1 one can predict the direction of change of the expected value of P_0 once the market parameters become random, and the response of P_0 as the variances of the random component increases (a mean preserving spread experiment).

Correspondingly, the reaction of the initial price, P_0, and later prices, P_t (initial and later expected quantities) to randomness when compared to a world in which the parameters take values that equal their expected values of perfect certainty are:

Table 1:

Signs of $\frac{dP_0}{dX}$ and $\frac{d^2P_0}{dX^2}$ with respect to market parameter, X.

	Parameter				
Derivative	R	c	b	a	r
First	-	+	-	+	-
Second	+	-	-	-	-
Type of Figure and Curve	1a (ii)	1b (iii)	1a (iii)	1b (iii)	1a (iii)

Table 2:

Signs of $\frac{dT}{dX}$ and $\frac{d^2T}{dX^2}$ with respect to market parameters, X.

Derivative	R	c	b	a	r
First	+	+	-	-	-
Second	-	-	+	+	-
Type of Figure and Curve	1b (iii)	1b (iii)	1a (ii)	1a (ii)	1a (iii)

R - Reserves: the expected price is larger than the price that corresponds to the expected R and expected T is lower than the one corresponding to expected R.

c - Extraction Costs: the expected price is lower than the price that corresponds to expected costs and the expected extraction period is lower than the one corresponding to expected costs.

b - Demand Slope: the expected price is lower than the price that corresponds to the expected slope and the extraction time is lower than that for E(b).

"a" - Autonomous Demand: the expected price is smaller than that corresponding to expected "a" and the extraction period is larger than the one corresponding to E("a").

r - Interest: the expected price is smaller than the one corresponding to expected interest while the expected time is larger than the one corresponding to E(r).

Given these results one sees that the precautionary motives are working not only with respect to the randomness of the reserves, but also with respect to all other parameters that are involved in the determination of the equilibrium path of prices. Given uncertainty in extraction costs the expected price would be lower than the one at the same level but with certainty and the expected length of extraction is smaller. Similarly with uncertainty of interest rate the expected price is below the one that corresponds to the same interest level (equals the expected one) and the expected extraction period is larger than the corresponding one. Hence uncertainty in the net present value of the returns (royalties) per unit reserves lowers the initial price and shortens the extraction period.

REFERENCES

Arrow, K.J., and S. Chang, "Optimal Pricing, Use and Exploration of Uncertain Natural Resource Stocks," 1982, **Journal of Environment Economical Management**, 9: 1-10.

Deshmukh, S.D., and S.R. Pliska, "Optimal Consumption and Exploration of Nonrenewable Resources under Uncertainty," 1980, **Econometrica**, 48: 177-200.

Fishelson, G., "Continuous vs. A Single Change," 1989, Working Paper No. 16-89, Foerder Institute for Economic Research, Tel-Aviv University.

Heal, G.M., "Uncertainty and the Optimal Supply Policy for an Exhaustible Resource," 1979, in R.S. Pindyck (ed.), **Advances in the Economics of Energy and Resources**, 2: JAI Press, Greenwich, Conn.

Herfindahl, O.C., "Depletion and Economic Theory," 1967, in: M. Gaffney (ed.), **Extractive Resources and Taxation**, University of Wisconsin Press, Madison, Wisconsin.

Hotelling, H., "The Economics of Exhaustible Resources," 1931, **Journal of Political Economy**, 39: 137-75.

Kemp, M.C., "How to Eat a Cake of Unknown Size," 1976, in **Three Topics in the Theory of International Trade**, North Holland, Amsterdam.

Long, N.V., "Resource Extraction under the Uncertainty about Possible Nationalization," 1975, **Journal of Economic Theory**, 10: 42-53.

Loury, G.C., "The Optimal Exploitation of an Unknown Reserve," 1978, Rev. Econ. Stud., 45: 621-636.

Pindyck, R.S., "Uncertainty in the Theory of Renewable Resource Markets," 1984a, **Rev. Econ. Stud.**, 51: 289-303.

Pindyck, R.S., "On Monopoly Power in Extractive Resource Markets," 1984b, Working Paper MIT-EL 84-008WP, Energy Laboratory, MIT.