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by

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ABSTRACT

This paper presents a general analysis of money demand when net disbursements follow a Wiener process and overdrafting is allowed at some penalty rate. Using recent developments in optimal control theory, the "impulse control" method, a solution for both the target and trigger money levels is presented. This relaxes the assumption that overdrafts are excluded and the trigger level is exogenously fixed at level zero and thus extends the works of Frenkel and Jovanovic (1980) and others. By allowing for a variety of interest rates, the model generates rich dynamics of the money stock. It is shown, for instance, that the short-run interest elasticity of money demand is probably large (in absolute value) and negative, but in the long run this elasticity is much smaller or even positive. It is also argued that inappropriate current definitions of the monetary aggregates, which exclude unused credit, may spuriously generate instability of the money demand. An alternative definition of money stock is suggested which seems to be conceptually more satisfying.

1. Introduction

The proposition that unused credit should count as money goes back at least to Keynes (1930), who wrote (p.42),

> "There exists in unused overdraft facilities a form of Bank-Money of growing importance, of which we have no statistical record... the Cash Facilities, which are truly cash for the purposes of the Theory of the Value of Money, by no means correspond to the Bank Deposits which are published. The latter... take no account of something which is a Cash Facility, in the fullest sense of the term, namely, unused overdraft facilities."

Although many economists, both before and after Keynes, have expressed similar views,[1] no explicit derivation of the demand for money with overdrafts has been made. This paper is remedying this omission. In addition, it generalizes previous models by allowing some components of the money supply, such as NOW accounts, to bear interest. This generates rich dynamics of the response of the money stock to changes in interest rates.

According to the transactions theory of money demand,[2] the optimal rule of money holding is a trigger-target rule; that is, the money stock is adjusted to the target only when it falls below the trigger.[3]

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[1] Two of the numerous examples are Lavington (1921) and Laffer (1970).

[2] The transactions theory of money demand originated in a deterministic framework due to Baumol (1952) and Tobin (1956) and a stochastic version by Miller and Orr (1966). Some of the recent examples which extended these original works are Milbourne, Buckholtz and Wasan (1983) and Romer (1986, 1987). The most general solution, and the one which is closest to our paper, is that of Frenkel and Jovanovic (1980).

[3] The reason for the optimality of this rule is a fixed transaction cost in converting bonds to money.

However, an assumption which is common to virtually all papers in the field is to constrain the trigger to an exogenous value, which is usually zero.[4] Optimization is then carried out on the target level only. In the solution presented here, both the target and the trigger are chosen optimally. This is accomplished using impulse control, which is a new technique of optimal control.[5]

Some of the predictions of the model are fairly surprising. For example, it is shown that when banks lower the interest rate they charge on approved credit lines, both the target and the trigger levels will probably fall. This will lead to a reduction in the volume of assets and liabilities held by banks when they adopt this more liberal credit policy. This conclusion implies that, we might observe exceptionally high interest rates on approved credit lines sustained for long periods.

A new definition of the money supply, which is potentially better than the standard one, is suggested. By including outstanding credit, the new definition seems a conceptually more appropriate measure of money as describing the quantity of the medium of exchange. This definition of monetary aggregate makes it possible to retain the framework of a representative agent even in the general case.

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[4]Equating the trigger level to zero is implicitly equivalent to excluding the possibility of overdrafting.

[5]Foundations of impulse control can be found in Bensoussan and Lions (1982).

The structure of the paper is as follows. Section 2 presents the problem of optimal money holdings with a stochastic disbursements process, while section 3 describes the solution. Section 4 discusses some of the implications of this solution, and section 5 elaborates on the consequences for the definition of money. A brief summary is presented in section 6.

2. Optimal Money Holding: Formulation of the Problem

An individual or a business firm have a choice of two assets to include in their portfolio. They can hold either money, the medium of exchange, or another asset, called "bonds", which cannot be used as a means of payment. Hence, people must hold money to complete their transactions even though they implicitly pay a liquidity premium for doing so, which results from the lower interest rate paid on money as compared to bonds. The amount of money held is determined by minimizing the expected costs associated with money holding. These costs take the following form:

(i) Cost of holding a money balance m, denoted by h(m). When $m > 0$, the cost is the foregone interest on bonds relative to money; when $m < 0$, the cost is a shortage cost which is the excess interest paid on overdrafts relative to bonds or any other penalty paid on negative account balances. Assume that both the holding and penalty costs are linear to get,

$$(1) \qquad I(m) = \begin{cases} rm & \text{for } m \geq 0 \\ -pm & \text{for } m < 0 \end{cases}$$

where $r > 0$ is the difference between the interest rate on bonds and the one on money balances and $p > 0$ is the cost per dollar of holding negative money balances.

(ii) The cost of transfer of u dollars from bonds to money, denoted by $C(u)$. These costs might include two terms: a fixed cost K per transfer which is independent of the transaction size and arises from time and effort of making the transfer decision and implementing it, and a proportional brokerage fee c per dollar. This gives,

$$(2) \qquad C(u) = \begin{cases} K + cu & \text{for } u > 0 \\ 0 & \text{for } u = 0 \end{cases}$$

with $K, c > 0$.[6]

At each point in time t, when the money stock is $m(t)$, the agent decides whether to convert bonds to money. Suppose he decides for such a transfer of size u_i dollars at time t_i. This transfer

[6] Implicit in equation (2) is the assumption of prohibiting transfer from money to bonds $(u < 0)$ which arises from very large cost of transfer in this direction. The assumption is made for computational reasons by allowing for only one trigger point, from bonds to money, and excludes the upper point which might trigger a transfer from money to bonds. The implications of this assumption are less important when the downwards drift of the money stock is large relative to the standard deviation. See Frenkel and Jovanovic (1980), footnote 3.

costs $C(u_i)$ and is carried out promptly to yield the money stock $m(t_i^+) = m(t_i) + u_i$. The money stock at hand is also changed by the random net expenditure flow, according to the following stochastic differential equation,

$$(3) \qquad dm(t) = -gdt + \sigma dw(t) + \sum_{i\geq 1} u_i \delta(t-t_i).$$

Positive values of mean disbursements g denote net cash outflow. The stochastic part of the expenditures is described by the Wiener process $w(t)$ with mean zero and variance t.[7] The last term in (3) denotes the discrete[8] increases of size u_i of the money stock made at times t_i where $\delta(t)$ is Dirac's delta function.[9]

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[7]Frenkel and Jovanovic (1980) identify g as representing the transactions motive for holding money while σ^2 stands for the precautionary motive. Miller and Orr (1966), on the other hand, interpret σ^2 loosely as transactions term (p.425). I think that the latter interpretation is more appropriate because there is no room for precautionary motives in this framework. Since the analysis is done in continuous time and the disbursements flow is finite, there is zero probability that money holding will overshoot beyond the thresholds. In this case, even when σ^2 is large, the agent can control his money holding by choosing the right thresholds without worrying about holding money as a precaution against unexpected low stock. In order to study the precautionary motive the analysis should be done in discrete time, when there might be a finite probability of overshooting beyond the trigger levels.

[8]Financial transactions will be made infrequently because of the fixed cost $K > 0$ which accompanies any transaction. In this case a continuous transfer during any finite period of time results in an infinitely high cost.

[9]Delta function is defined by

The optimizing agent chooses a sequence of financial transfers u_i made at t_i in order to minimize the expected discounted cost over an infinite horizon:

$$(4) \qquad V(m) = \min_{\{u_i, t_i\}} E_o \left[\int_0^\infty I(m(t)) e^{-\alpha t} dt + \sum_{i \geq 1} (K + cu_i) e^{-\alpha t_i} \right]$$

subject to the stochastic process described in equation (3). E_o denotes the expectations operator given information known at time zero and α is the interest rate on bonds. The cost of holding (positive or negative) money stock $m(t)$ is accumulated continuously at a rate $I(m(t))$ given in equation (1). The transfer cost $K + cu$ is accumulated discretely.

Common assumption made in studies of money demand is that the optimal transfer policy $\{u_i, t_i\}$ takes a form of simple trigger-target rules. The existence of such a rule for the problem (3)-(4) was proved by Constantinides and Richard (1978). They proved that the optimal policy is of the (S,s) type studied in the inventories literature: when the money stock is below the trigger point s, a sale of bonds will be made such that the money stock will increase to the target level S; otherwise no financial transaction will be made. We now proceed to the evaluation of these trigger and target levels.

$$\int_a^b f(x)\delta(x-c)dx = \begin{cases} f(c) & \text{if } a \leq c \leq b \\ 0 & \text{otherwise} \end{cases}$$

for any continuous function $f(x)$.

3. Solution

The optimization problem described in the former section generalizes previous works on money demand in several ways. The most important is the consideration of overdrafts. This option was implicitly excluded in other works; instead, the trigger level s was assumed to have a certain value (usually zero) and the solution procedure has been to find the target S, given the exogenous value of s.

Allowing overdrafting at a finite penalty rate makes the trigger s a control variable which is chosen optimally. Hence the solution to the optimization problem requires finding both s and S. This can be accomplished by using an optimal control theory, the "impulse control", which analyzes optimal behavior in continuous time with fixed cost of taking an action.[10] This type of control characterizes the behavior of the cash manager in the presence of fixed transaction costs. Sulem (1986) used this apparatus to solve the optimization problem (3)-(4) as follows. At each point in time t, when the money stock is at a level m, the cash manager can either sell bonds immediately to increase his money stock or postpone his transaction at least up to time $t+\tau$. In the latter case

[10] Constantinides and Richard (1978) have used this theory to prove the optimality of the (S,s) as the money rule to the problem (3)-(4).

(5) $$V(m) \leq \int_t^{t+\tau} I(m(x))e^{-\alpha(x-t)}dx + V(m(t+\tau))e^{-\alpha\tau}.$$

The first term on the r.h.s. of equation (5) is the cost of money holding between t and $t+\tau$. Using Bellman's principle of optimality, the second term in (5) is the minimum expected cost from time $t+\tau$ on. Expanding equation (5) as a Taylor series around time t, where $m(t+\tau) = m - g\tau + \sigma dw(t)$ from equation (3) and letting $\tau \to 0$ yield the following differential equation[11]

(6) $$-\frac{1}{2}\sigma^2 \frac{d^2V}{dm^2} + g\frac{dV}{dm} + \alpha V \leq I.$$

When bonds are sold at time t the money stock is increased immediately to level $m+u$. Since the transaction size u is chosen optimally we get

(7) $$V(m) \leq K + \min_{u \geq 0} (cu + V(m+u)).$$

Since either equation (6) or (7) must hold as an equality, $V(m)$ is the solution of the following set of equations,

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[11]The properties of mean zero and variance t of the Wiener process $w(t)$ are also used in the derivation of equation (6).

$$(8) \qquad AV \leq I$$

$$(9) \qquad V \leq BV$$

$$(10) \qquad (AV-I)(V-BV) = 0$$

where

$$AV(m) \equiv -\frac{1}{2}\sigma^2 \frac{d^2V}{dm^2} + g\frac{dV}{dm} + \alpha V$$

$$BV(m) \equiv K + \min_{u\geq 0}\,(cu + V(m+u)).$$

The system (8-10), called a quasi-variational inequality in the impulse control literature, allows us to solve for the expected cost V as a function of the money stock m. The outline of the solution is as follows. Since the (S,s) rule is the optimal policy for this problem, $V(m)$ can be defined over two regimes. When the money stock is below the trigger point s, denoted by μ from now on, the money stock is increased to the target level S (new notation: M) and equation (9) holds as an equality:

$$(11) \qquad V(m) = K + c(M-m) + V(M) \qquad \text{for } m < \mu.$$

When $m \geq \mu$ no financial transaction is made and equation (8) holds as an equality. In this case the solution of the linear differential equation (8) is,

$$(12) \qquad V(m) = -\frac{r}{\alpha}(\frac{g}{\alpha}-m) + D_1 e^{\lambda_1 m} + D_2 e^{\lambda_2 m} \qquad \text{for } m \geq 0.$$

$$(13) \qquad V(m) = \frac{p}{\alpha}(\frac{g}{\alpha}-m) + E_1 e^{\lambda_1 m} + E_2 e^{\lambda_2 m} \qquad \text{for } m < 0.$$

The first term in (12) and (13) is a particular solution of the nonhomogenous part of equation (8) and the last two terms are the general solution to the homogenous part. D_i, E_i, $i = 1,2$ are constants to be determined and the roots of the characteristic equation, λ_1 and λ_2 are given by

$$(14) \qquad \lambda_1 = \sigma^{-2}[-(g^2 + 2\alpha\sigma^2)^{1/2} + g] \leq 0$$

$$(15) \qquad \lambda_2 = \sigma^{-2}[(g^2 + 2\alpha\sigma^2)^{1/2} + g] \geq 0.$$

Assume initially that $\mu \leq 0$. In this case equation (13) describes the expected cost $V(m)$ for $\mu \leq m < 0$. Complete characterization of the solution requires finding the values of D_1, D_2, E_1, E_2, μ, and M. These six parameers are solved using the following six conditions:

(a) continuity at $m = 0$,

(16) $$-\frac{rg}{\alpha^2} + D_1 + D_2 = \frac{pg}{\alpha^2} + E_1 + E_2$$

(b) continuous derivative at $m = 0$,

(17) $$\frac{r}{\alpha} + \lambda_1 D_1 + \lambda_2 D_2 = -\frac{p}{\alpha} + \lambda_1 D_1 + \lambda_2 D_2$$

(c) continuity at $m = \mu$,

(18) $$\frac{p}{\alpha}(\frac{g}{\alpha} - \mu) + E_1 e^{\lambda_1 \mu} + E_2 e^{\lambda_2 \mu} = K + c(M-\mu) + V(M)$$

(d) continuous derivative at $m = \mu$,

(19) $$-\frac{p}{\alpha} + \lambda_1 E_1 e^{\lambda_1 \mu} + \lambda_2 E_2 e^{\lambda_2 \mu} = -c$$

(e) M is the optimal target. Optimizing over M in equation (11) yields

(20) $$V'(M) = -c.$$

(f) $V(m)$ grows linearly at a rate $\frac{r}{\alpha}$ when $m \to \infty$.

(21) $$\lim_{m \to \infty} V'(m) = \frac{r}{\alpha}$$

which gives immediately

(22) $\qquad D_2 = 0.$

The rest of the parameters are:

(23) $$D_1 = \frac{1}{\lambda_1} e^{-\lambda_1 \mu}\left(\frac{p}{\alpha} - c\right) + \frac{1}{(\lambda_1-\lambda_2)}\left(\frac{p+r}{\alpha}\right)\left[\frac{\lambda_2}{\lambda_1} - e^{(\lambda_2-\lambda_1)\mu}\right]$$

(24) $$E_1 = \frac{1}{\lambda_1} e^{-\lambda_1 \mu}\left(\frac{p}{\alpha} - c\right) - \frac{1}{(\lambda_1-\lambda_2)}\left(\frac{p+r}{\alpha}\right) e^{(\lambda_2-\lambda_1)\mu}$$

(25) $$E_2 = \frac{\lambda_1}{\lambda_2(\lambda_1-\lambda_2)}\left(\frac{p+r}{\alpha}\right)$$

The solution for the target M and the trigger μ is given by the following equations

(26) $$M = (\alpha c+r)^{-1}\left[(\alpha c-p)\mu + \left(\frac{p+r}{\lambda_2}\right)\left(e^{\lambda_2 \mu} - 1\right) - \alpha K\right]$$

(27) $$e^{-\lambda_1 M} = (\alpha c+r)^{-1}\left[(\alpha c-p)e^{-\lambda_1 \mu} + \left(\frac{p+r}{\lambda_1-\lambda_2}\right)\left(\lambda_1 e^{(\lambda_2-\lambda_1)\mu} - \lambda_2\right)\right].$$

It is straightforward to see that the analogue of equations (16)-(21) for the case $\mu > 0$ yields the solution $\mu = M$, which is the solution of maximum (infinite) expected cost. We thus conclude that equations (26) and (27) determine the optimal levels of the target money level M and the trigger point μ, and that μ satisfies $\mu \leq 0$. We now turn to the implications for money demand which are suggested by equations (26) and (27).

4. Analysis of the Solution

4.1. Discussion

The first property of equations (26) and (27) worth mentioning is that the solution for M and μ is homogenous of degree one in the vector (g, σ, K). Hence money demand is demand for real balances.

The framework for analyzing money demand described in the previous two sections generalizes previous work in at least three aspects:

(i) Overdrafting is allowed at a penalty rate, p, hence the trigger level μ is a control variable and is not constrained to any given level.

(ii) The cost of holding money, r, is not necessarily equal to the discount rate, α.

(iii) The proportional cost of transferring bonds to money, c, is not restricted to zero.

The assumptions of constraining the trigger point μ to zero, common to virtually all analyses of money demand, has typically been made for analytical reasons. The general solution (26)-(27) shows that by adopting this restriction, economists implicitly assumed that the overdraft rate p was prohibitively high. For every finite p, however high, the transactions model of money demand implies that $\mu < 0$, i.e. optimizing agents make some use of the overdraft provision.[12] The empirical plausibility of this prediction of money demand theory might be questioned. However, it seems that economic agents do use the overdraft provision when it is not excessively expensive. Examples are accounts of business firms, trade credit, credit lines available through credit cards, and checking accounts in countries where it is possible for consumers to overdraft.[13] The existence of negative account balances has also some interesting implications for the aggregate money demand, implications which are discussed in the next section.

The assumption of the equality between r and α, which is also made very often in the literature, is a very limiting assumption. If the cost of holding money, r, assumed to equal the bond market rate, α, it must be the case that the interest paid on money is zero. This might have been a good assumption when the interest paid on demand

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[12] Numerical solutions for μ and M, some of them shown below (e.g. figures 1 and 2), show that for a wide range of parameters the convergence of μ to zero when p increases is fairly slow.

[13] Empirical support of a widespread use of overdrafts and trade credit appears in Kannianien (1978), Laffer (1970), and others.

deposits was legally so constrained. In the wake of the deregulation of the banking industry, the assumption $r = \alpha$ restricts the analysis to the demand for currency, which is not what the transaction theory of money demand is about. Since a large fraction of the monetary aggregates M_1 and M_2 bear positive interest rates, allowing for $r < \alpha$ is thus crucial for analyzing the demand for money. It also makes possible the analysis of the relative demands for different monetary measures.

The inclusion of both fixed transactions cost K and proportional cost c is an extension over previous work. In general, there is a dichotomy between models with fixed cost only, as in the Baumol-Tobin or Miller and Orr (1966) models, or analyses of proportional cost with no fixed cost, as in Eppen and Fama (1969). However, the consequences of the inclusion of both sources of costs are not as crucial as the other two generalizations mentioned previously. When the two kinds of costs are present, the important one is the fixed cost; then the optimal money holding rule is the trigger-target rule which is typical to models with lumpy transaction costs. We now proceed with some special cases of the general solution which allow analysis of each of the generalizations seriatim.

4.2. Very high overdraft rate

The most general analysis of money demand when net disbursements include both deterministic and stochastic elements is that of Frenkel and Jovanovic (1980). Their model is a special case of of ours with no proportional cost and with one interest rate instead of three, i.e. $c = 0$, $p \to \infty$ and $r = \alpha$. Assuming $c = 0$ and $p \to \infty$ we can concentrate on the importance of the generalization $r \neq \alpha$. In this case equations (26) and (27) yield the following approximate solutions for M and μ:[14]

$$(28) \qquad M \simeq \mu - \frac{K\alpha}{r} + \frac{p}{r}\lambda_2\mu^2$$

$$(29) \qquad e^{-\lambda_1 M} \simeq 1 - \lambda_1\mu - \frac{p}{r}\lambda_1\lambda_2\mu^2.$$

Equations (28) and (29) give the following solution for M:

$$(30) \qquad e^{-\lambda_1 M} \simeq 1 - \lambda_1 M - \frac{K\alpha\lambda_1}{r}.$$

Following Frenkel and Jovanovic, we expand equation (30) in Maclaurin series and ignore terms of third and higher order. The result is[15]

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[14]Equations (28) and (29) are derived by expanding equations (26) and (27), respectively, in Maclaurin series and ignoring terms of third and higher order for those expressions which multiply p/r and of second and higher order for those expressions which do not multiply p/r (which, by assumption, is infinitely high).

[15]The approximation used in deriving equation (31) might be less accurate for finite p than that used in equations (28) and (29) because μ, but not M, approaches zero when $p/r \to \infty$.

$$(31) \qquad M \simeq \left(\frac{-2K\alpha}{\lambda_1 r}\right)^{1/2}.$$

Substituting for λ_1 we get

$$(32) \qquad M \simeq \left(\frac{2K\alpha\sigma^2}{r[(g^2 + 2\alpha\sigma^2)^{1/2} - g]}\right)^{1/2}$$

and the approximate solution for μ is

$$(33) \qquad \mu \simeq -\left(\frac{\lambda_2(\frac{K\alpha}{r} + M)}{\frac{p}{r}}\right)^{1/2} \rightarrow 0.$$

Assuming that $r = \alpha$, equation (32) reduces to the central result, the money demand equation, of Frenkel and Jovanovic.[16] Equation (32) can serve as a very convenient instrument in examining the significance of constraining the two interest rates to be equal. In fact, we can define at least four different elasticities which are of interest:

(a) Interest elasticity with $r = \alpha$. In this case we assume that no interest is paid on money; any change in the market interest rate

[16] Frenkel and Jovanovic call equation (32) with $r = \alpha$ "the optimal money holding." Although M is, in general, different from average money holding, the justification for this statement in their model is the existence of M as the only control variable.

results in an identical change in the cost of holding money. The proper empirical relevance of this case is the demand for currency, M_o. The interest elasticity in this case is the interest elasticity analyzed by Frenkel and Jovanovic, denoted by $\eta(M_0, r)$ and is derived from equation (32) to give,

$$(34) \qquad \eta(M_0, r) = (-\frac{1}{2})(r\sigma^2)[\mu^2 + 2r\sigma^2 - \mu(\mu^2 + 2r\sigma^2)^{1/2}]^{-1}$$

which satisfies $-1/2 \leq \eta(M_0, r) < 0$.

(b) Elasticity with respect to the market interest rate, α, given constant holding rate, r. In this case the interest paid on interest bearing money, such as NOW accounts, varies with the market interest rate point by point. For instance, when $\alpha = 7\%$ the NOW rate is 5% and when α increases to 7.5% the NOW rate increases to 5.5% so that $r = 2\%$ without change. In this case,

$$(35) \qquad \eta^{(1)}(M, \alpha) = \frac{1}{2} + \eta(M_0, r)$$

which is nonnegative. Hence, when the cost of holding money is constant, then an increase in the market interest rate leads to an increase in money demand. The intuition behind this apparently surprising result is simple: an increase in the nominal interest rate with no change in the nominal rate of money holding is in fact a decrease in the effective cost of holding money.[17]

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[17] The possibility of positive interest elasticities also arises in Romer's (1986) model.

(c) Elasticity with respect to the interest cost of money, r, given constant market rate, α. In this case the interest paid on the interest bearing money changes without a change in the market rate. For example, a fall in the interest paid on NOW accounts, with no other change, results in higher r with the same α. The interest elasticity is now

$$(36) \qquad \eta(M, r) = -\frac{1}{2}.$$

which is the well-known prediction of the square root rule of the Baumol-Tobin model. This is an interesting result. A well-established fact in the theory of money demand is that the crucial assumption in the Baumol-Tobin model is that of deterministic disbursements, i.e. $\mu/\sigma \to \infty$. What we find here is that their result is robust to a stochastic generalization as long as the rate α does not change. The reason for this is quite clear. What characterizes the Baumol-Tobin analysis is not only the deterministic nature of their model, but also the "steady state" assumption, which in fact means no discounting. Hence, we conclude that the assumption of constant discount rate is pivotal to the Baumol-Tobin model, a conclusion which could not be derived without the distinction between r and α.

(d) Elasticity with respect to the market interest rate, α, given constant rate on the interest bearing money, i. This case seems to be empirically plausible for short-run analysis since the interest paid on checking accounts i is much less volatile than competitive market rates.[18] Substituting $r = \alpha - i$, where i is fixed, in equation (32) we get

$$(37) \qquad \eta^{(2)}(M, \alpha) = \frac{-i}{2(\alpha - i)} + \eta(M_0, r)$$

which reduces to case (a) when $i = 0$. Notice, however, that the interest elasticity can now be much larger than one half in absolute value if i is close to α. For example, when $i = 5.25\%$ and $\alpha = 7\%$ than $\eta^{(2)}(M, \alpha) = -1.5 + \eta(M_0, r)$ which falls in the $(-2, -1.5)$ region. The reason for this high short-run interest elasticity is that a percentage rise in the market rate is translated to a much larger increase in the cost of holding money.

The analysis of interest rate elasticities can be summarized as follows. By retaining the assumptions of no proportional cost $(c = 0)$ and prohibitively expensive overdraft rate $(p \to \infty)$, we can study the effect of relaxing the assumption of zero interest rate paid on money. The implications of this generalization seem to be fairly important. We can now distinguish between different monetary aggregates in their response to interest rate changes. The interest elasticity of currency

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[18] The rigidity of i can arise from institutional rigidities and from the "menu costs" of administrating an interest rate change.

demand (case (a)), which is the case that is usually investigated in the literature, is negative and larger than -(1/2). However, the analysis of interest elasticity of interest bearing assets is much richer and depends on the way in which interest rates vary. In the short run, when the interest paid on checking accounts is fixed, we expect to see a large drop in the demand for these accounts when the market rate increases (case (d)). However, in the long run, when the interest paid on NOW and similar accounts adjusts to the new, higher interest rates, the drop in the money demand will be milder, and we might even see a rise in the demand (case (b)). Case (c) studies circumstances in which the interest paid on checking accounts is more volatile than market rates. This might have been the case when the legal restrictions on payment of interest on checking accounts were removed in recent years to produce an abrupt drop in the cost of money holding, not fully accompanied by a similar drop in other interest rates. In this case our model predicts an interest rate elasticity of (-1/2), the Baumol-Tobin result even in a stochastic framework.

The above analysis can give us a crude rule-of-thumb about the relative demand for different monetary aggregates. Suppose that people make independent decisions about the amount of currency and checking deposits they would like to hold.[19] Equation (32) means that if the proportional cost c equals zero and the fixed cost K is identical for both means of payment, the overdraft rate p is very high, and the stochastic processes are similar, then the ratio of the demand for interest-bearing checking accounts, $M_1 - M_0$, to the demand for currency, M_0, satisfies the following simple relation,

$$(38) \qquad \frac{M_1 - M_0}{M_0} \simeq \left(\frac{\alpha}{r}\right)^{1/2}$$

Substituting, say, 7% for the interest rate α and 5.25% as the interest paid on NOW accounts $(r = 1.75\%)$, equation (38) indicates that the ratio of the two assets will be 2. This rule of thumb can be applied to any two components of monetary aggregates:

$$(39) \qquad \frac{M_i}{M_j} \simeq \left(\frac{r_j}{r_i}\right)^{1/2}$$

where M_i and M_j denote demand for different components of money stock and r_i and r_j are the costs of holding these assets, respectively. With these definitions of the interest rate, and with the

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[19]This will be the case if certain transactions must be made using currency while other transactions have to be completed with checks, and when the transactions costs (c and K) for these means of payment are independent of each other.

previous reservations, the famous Baumol-Tobin square-root rule might still serve as a crude benchmark.

4.3. Deterministic disbursements

When net disbursements are determnistic ($\sigma^2 = 0$, $g > 0$), the quasi-variational inequality (8)-(10) leads to a first-instead of a second-order differential equation for $V(m)$, $m \geq \mu$. The straightforward solution for M and μ, assuming $c = 0$ for simplicity, is given by the following equations:[20]

$$rM + p\mu = -\alpha K \tag{40}$$

$$re^{\alpha M/g} + pe^{\alpha\mu/g} = r + p \tag{41}$$

When the overdraft rate p satisfies $p \to \infty$ the solution for M and μ becomes,[21]

$$M = \left(\frac{2gK}{r}\right)^{1/2}. \tag{42}$$

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[20]See Sulem (1986). Notice that when disbursements are deterministic the solution can be derived using simple calculus and it is not necessary to resort to impulse control methods.

[21]Ignoring terms of third and higher order in the Maclaurin series of $e^{\alpha M/g}$ and second and higher order for $e^{\alpha\mu/g}$ since M is much larger than (the absolute value of) μ in this case.

$$(43) \qquad \mu = -\left(\frac{2gKr}{p^2}\right)^{1/2} - \frac{\alpha K}{p} \to 0$$

the Baumol-Tobin result.

Assume now that disbursements are deterministic and agents minimize expected cost per unit of time. This assumption of zero discount rate ($\alpha = 0$), the so-called "steady state approach", results in the following solution,[22]

$$(44) \qquad M = \left(\frac{2gKp}{r(p+r)}\right)^{1/2}$$

$$(45) \qquad \mu = -\frac{r}{p} M.$$

Once again, the Baumol-Tobin solution is derived for $p \to \infty$. Notice, however, the symmetry between μ and M in equations (44) and (45). For example, if $r = p$ then $\mu = -M$ and the average money holding of an individual is zero. This emphasizes the importance of the assumption $\mu = 0$, common to almost all studies of money demand, to get positive money demand with probability 1 at all times. We will elaborate on this point in the next section. Notice also that the deterministic steady state solution (44), (45) preserves the benchmark elasticity of 1/2 with

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[22] See Sulem (1986).

respect to expenditures g and the fixed cost K but not with respect to the interest rates p or r.

4.4. Numerical solutions

Comparative statics of the general solution for M and μ, given in equations (26) and (27), is straightforward but tedious. Most elasticities have, in general, ambiguous signs.[23] It is perhaps more illuminating to present numerical solutions for M and μ as a function of different parameters.[24] We concentrate on the effects of changes in the parameters p, r, and α, the most important variables which distinguish this analysis from previous ones. Figure 1 depicts the change of the target level M and the trigger μ vs. the overdraft rate p for discount rate $\alpha = 7\%$, cost of holding money $r = 2\%$ (corresponding to 5% interest paid on checking accounts), $c = 0$, $g = \sigma = 1$ and $K = .01$. Both M and μ increase with p at similar rates

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[23] An exception is the effect of an increase in the fixed cost on the target M. When $c = 0$ we have,

$$\frac{dM}{dK} = \left(\frac{\alpha}{r}\right)\frac{e^{-\lambda_1\mu}}{e^{-\lambda_1 M} - e^{-\lambda_1\mu}} \geq 0$$

and the elasticity varies with different parameters and is not fixed at the level 1/2.

[24] The applicability of this method is not as narrow as it sounds because of the homogeneity of M and μ as a function of (g, σ, K). We can thus interpret the numbers for M, μ, g, σ, and K as representing, say, thousands of dollars.

so that the difference $(M-\mu)$ is not very sensitive to p when p is not very small.[25] For example, when p increases from 20% to 40%, $(M-\mu)$ decreases from 1.141 to 1.094, which gives an arc elasticity of -.04. We shall make use of this observation later. Notice also that the convergence toward the limiting values of M (equation (32)) and μ (zero) for $p \to \infty$, the values which are closely related to the Frenkel-Jovanovic analysis, is fairly slow. When p = 50% then M = .865 and $\mu = -.219$; even when p rises to the outrageous rate of 200% then M = .936 and $\mu = .105$, compared to the values M = 1.017 and $\mu = 0$ which correspond to the $p \to \infty$ approximation. Hence, the generalization to finite values of the overdraft rate p does make a difference even if this rate is high.

Figure 2 presents the effect of the rate of money holding r on the trigger and target levels when the parameters are p = 20%, α = 7%, $g = \sigma = 1$, K = 0.1, and c = 0. As expected, both M, μ, and the difference $(M-\mu)$ fall when r rises. However, the elasticity of any of these three variables with respect to the holding rate r is, in general, less than one half (in absolute value), unlike the case when the overdraft rate $p \to \infty$. Notice that when $r \to 0$, $M \to \infty$ and $\mu \to 0$ and the money balance is always positive. On the other hand, when r

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[25]For very small values of p the trigger μ increases very rapidly. This is because $\mu \to -\infty$ when $p \to 0$, since it is optimal never to sell bonds in this case.

becomes large relative to p the target M is negative, resulting in negative money balances held at all times.[26]

The target and trigger levels are not very sensitive to changes in the market interest rate α. For a wide range of parameters M and μ are practically constant even when α changes from 1% to 17.5%. However, it is interesting to note that the unexpected result of a rise in the money demand when the interest rate α increases, found in the approximation for large p (equation (35)), still holds when p is finite: M increases slightly and μ decreases slightly when α rises such that the difference $(M-\mu)$ increases, although by a very small amount.

The effects on M and μ of changing the parameters of the Wiener process and the cost function are as follows. An increase in the mean-variance ratio g/σ^2 raises both M and μ but $(M-\mu)$ can rise or fall. An increase of the fixed cost K, by contrast, raises the target M and lowers the trigger μ by large amounts. Thus, the difference $(M-\mu)$ is quite sensitive to changes in the fixed cost. Variations in the proportional cost c have much milder effect: large increases in c produce slight decreases in M and μ. This numerical analysis thus extends existing comparative analysis results which are derived by approximate solutions.[27]

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[26] In figure 2 we allow $r > \alpha$ which implies negative interest on demand deposits.

[27] See, for example, Hadley and Whitin (1963) or Blinder (1981).

5. Aggregate Money Demand

The proposition that unused credit should count as money is an old point, going back at least to Keynes (1930) and Laffer (1970). In this section we compare the standard defnition of money stock with a new definition which takes overdrafts into account.

Begin the analysis by assuming that it is illegal, or prohibitively expensive, to hold negative account balances, and hence the trigger level μ is zero. Assume further, as we did previously, that agents can hold either bonds or money, which takes the form of demand deposits. Net disbursements flow at the constant rate of one dollar per period, taking place at the beginning of the period. The target stock M equals four dollars. Suppose that an agent begins a certain period with money stock which is equal to M. His money stock m is reduced right away by one dollar to give money stock $m = 3$ during the first period. In the second, third and fourth periods the money stock will be 2, 1, and 0, respectively. At the end of the fourth period the agent will sell \$4 worth of bonds to begin the fifth period with money stock of \$4 which is depleted immediately to $m = \$3$, and a second cycle begins. Summarize this as

period	1	2	3	4
money holding (\$)	3	2	1	0

The average money stock of an agent is \$6/4. Suppose now that the economy is comprised of four identical individuals with staggered money holdings. Each period a different person converts 4 dollars of bonds to money to match the four dollars of total disbursements. The result is that the average aggregate money stock equals \$6/4, the same as the individual's average. This can also be seen from the balance-sheet of the (only) bank in this economy:

Assets	Liabilities
Bonds: 6	Demand Deposits: 6

The story up to this point is a standard one. Suppose now that the overdraft rate p falls to some finite level. The result is that both the targe M and the trigger μ are reduced by the optimizing agents, probably by very similar amounts (see figure 1).

Suppose that the new optimal levels are $M = 3$ and $\mu = -1$. The money balances of the agents will now be

period	1	2	3	4
money holding (\$)	2	1	0	-1

in order to give \$2/4 as the average money stock held by an individual. However, if we look at the banking system for the staggered economy we see the following:

Assets		Liabilities
Bonds:	2	DD: 3
Loans:	1	

which gives a total money stock, as currency plus demand deposits, of \$3/4 per capita. Thus, the aggregage money stock is not identical to the average stock of a representative agent, even though all agents are basically alike. The source of the discrepancy is simple. From the bank's point of view, a negative balance is an asset which falls into the category of "loans". As such, the traditional money definition assigns weight zero to negative balances. However, from the optimizing agents' point of view, a negative balance is **not** identical to a loan. The central assumption in the transactions theory of money demand is that of minimizing transaction costs. Essential distinction between a loan and the use of an approved credit line is that there are transaction costs associated with the former, but not with the latter. Hence, negative balances **do** enter into the calculation of average money holdings of individuals.

As already noted by Keynes, these two definitions of money are not proper. As far as total availability of means of payment is concerned, there is no difference between a case of target $M = 4$ and trigger $\mu = 0$ or $M = 3$ and $\mu = -1$. The volume of transactions that the agent can finance is $(M-\mu)$, irrespective of whether μ is positive or negative; there is nothing inherently special about the level 0 which requires defining money up to this point. Instead the proper way of measuring money is up to the trigger level, whatever its magnitude. Using the new definition, the quantity of money for the four agents in the staggered economy with $\mu = -1$ is (3, 2, 1, 0) to give aggregate quantity of 6, even though account balances are now (2, 1, 0, -1).

Thus, in spite of the change in the individual money holdings and in the conventionally defined quantity of money, if we correctly focus on the total availability of means of payment, there is no change in money stock when the economy adjusts to the new trigger and target levels (3, -1). Before we elaborate on some of the implications of the new definition, an interesting point about the banking system of this economy should be emphasized. Although the quantity of money according to the proposed definition does not change, there is a real change in the financial position of banks, a change which takes on an unexpected form: when banks extend credit, their volume of assets falls. In the example of the staggered economy banks' assets dropped from \$6 to \$3 when they introduced the overdraft option. We realize again that automatic use of credit lines is fundamentally different from bank loans in the way it affects the economy.[28] The drop in their assets implies that banks do not have strong incentives to introduce automatic use of credit lines.

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[28] I would like to reemphasize the reason for that. Availability of loans does not change the parameters of money demand. Credit lines, which unlike loans do not incur transaction costs, reduce not only the trigger level, but the target money level as well.

Assuming zero profits, banks will have to find compensation for the decrease in the volume of interest-bearing assets by charging high interest when customers use their credit. This might be a reason for the exceptionally high and downwards rigid interest rates charged on credit card balances.

Returning to the issue of aggregate money demand, we have found that there are three possible ways of measuring this variable when the trigger level μ is different from zero:

(1) average money stock of a representative agent ($2/4 in the staggered economy example when $\mu = -1$). If we denote by $\phi(m,t)$ the probability density function for having a money balance m at time t,[29] then average holding, denoted by $E_1(m)$, will be

$$(46) \qquad E_1(m) = \int_{\mu}^{\infty} m\phi(m,t)dm$$

(ii) average positive money stock ($3/4 in the above example). This definition is the closest to the current definition of money and is given by

[29] $\phi(m,t)$ is a function of the initial money stock. I omit this as an explicit parameter for simplicity of exposition. Similarly, the time subscript is omitted in $E_1(m)$.

(47) $$E_2(m) = \int_0^\infty m\phi(m,t)\,dm$$

(iii) average money stock measured by its distance from the trigger level μ (\$6/4 in the example). This redefinition of "zero level" of money stock yields

(48) $$E_3(m) = \int_\mu^\infty (m-\mu)\phi(m,t)\,dm = E_1(m) - \mu.$$

As long as $\mu \leq 0$, as will always be the case in our framework, the relationship between the three measures will be

(49) $$E_3(m) \geq E_2(m) \geq E_1(m)$$

when the equalities hold for $\mu = 0$ ($p \to \infty$), the standard case in the literature.

The steady state distribution of the money stock is defined as[30]

(50) $$\phi(m) = \lim_{t\to\infty} \phi(m,t)$$

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[30] The steady-state distribution is not a function of the initial money stock.

The derivation of $\phi(m)$ for the one-target-one-threshold (μ,M) policy is presented in the appendix.[1] The result is

$$(51)\qquad \phi(m) = \begin{cases} (M-\mu)^{-1}\,[1-e^{-\pi(m-\mu)}] & \text{for } \mu \le m \le M \\ (M-\mu)^{-1}[e^{-\pi(m-M)}-e^{-\pi(m-\mu)}] & \text{for } M < m \end{cases}$$

where $\pi \equiv \frac{2g}{\sigma^2}$.

Using (51) we get,

$$(52)\qquad E_1(m) = \frac{1}{2}(M + \mu + \frac{\sigma^2}{g})$$

$$(53)\qquad E_2(m) = (M-\mu)^{-1}[\frac{M^2}{2} + \frac{M}{\pi} + \pi^{-2}(1-e^{\pi\mu})]$$

$$(54)\qquad E_3(m) = \frac{1}{2}\,(M-\mu + \frac{\sigma^2}{g}).$$

Comparing the definition of the money stock $E_3(m)$ with the conventional definition $E_2(m)$ we see that the quantity of medium of exchange available to agents $E_3(m)$, depends only on the difference $(M-\mu)$. However, if both M and μ fall, but $(M-\mu)$ does not change, the false conclusion from equation (53) is that the money stock $E_2(m)$ is lower. Hence our intuition, based on the simple staggered economy, is verified by the more general, stochastic analysis.

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[1] Actually, the appendix presents the derivation for the more general (μ_1, M, μ_2) rule where μ_2 is a second, upper threshold which triggers a reduction in the money stock to the target M. Only then is the condition $\mu_2 \to \infty$ applied. Notice also that when $M = 0$, equation (51) reduces to equation (26) in the Frenkel and Jovanovic (1980) paper.

The surprising prediction that extension of credit by banks might lead to a reduction of the conventional measure of the money stock carries strong empirical appeal. Increased popularity of credit cards and availability of credit lines, represented by lower p in our model, induces people to lower both the trigger μ and the target M. The availability of credit does not mean that consumers have lower purchasing power, but allows some to hold, on average, lower money balances. The official money stock, $E_2(m)$, which treats the trigger level μ as fixed at zero, is therefore erroneously perceived to fall. A potentially useful insight into the puzzle of the "missing money" (Goldfeld (1976)) is to look at the aggregate $E_3(m)$ and not the false measure $E_2(m) \leq E_3(m)$. The difference between the two tend to be quite significant. For instance, when $\alpha = 7\%$, $r = 2\%$, $p = 20\%$, $\mu = \sigma = 1$, $K = .01$ and $c = 0$, we get $M = .780$ and $\mu = -.362$ to give $E_2(m) = .721$ but $E_3(m) = 1.071$. A similar percentage of discrepancy arises for a wide range of parameters.

The model also suggests a similar explanation to the seemingly excess variability of the velocity of money. Availability of credit will lower both the trigger μ and the target M by similar amounts without changing the velocity of money. If the quantity of money is measured properly, by $E_3(m)$, this will be the conclusion. However, when the standard definition $E_2(m)$ is used, the velocity seems to rise.

To conclude, our model suggests that proper definition of the money stock must include some measure of approved credit lines available to the public. Otherwise, empirical estimations of money might be significantly different from the quantity of the medium of exchange, which might lead to erroneous conclusions of "missing money","unstable money demand", and "excess variability" of the money velocity. The standard definition of money, which assumes that the trigger point μ is constant at level zero, or equivalently that credit is ignored while computing money stock, might deliver an outcome which is quite remote from the true one when so much credit is available to the public.

6. Summary

This paper presents a general analysis of money demand when net disbursements follow a Wiener process and overdrafting is allowed at some penalty rate. Using recent developments in optimal control theory, the "impulse control" method, a solution for both the target and trigger money levels is presented. This relaxes the assumption that overdrafts are excluded and the trigger level is exogenously fixed at level zero and thus extends the works of Frenkel and Jovanovic (1980) and others. By allowing for a variety of interest rates, the model generates rich dynamics of the money stock. It is shown, for instance, that the short-run interest elasticity of money demand is probably large (in absolute value) and negative, but in the long run this elasticity is much smaller or even positive. It is also argued that inappropriate

current definitions of the monetary aggregates, which exclude unused credit, may spuriously generate instability of the money demand. An alternative definition of money stock is suggested which seems to be conceptually more satisfying.

APPENDIX

In this appendix we derive equation (51) which gives the steady-state distribution of a Brownian motion with a drift when there are two thresholds, μ_1 and μ_2, and one target, M. The derivation is based on chapter 15 of Karlin and Taylor (1981) (hereafter known as K-T). They show (section 8.E) that (μ_1, M, μ_2) diffusion process with mean g and variance σ^2 converges to the following limiting distribution:

$$\text{(A.1)} \qquad \phi(m) = \lim_{t\to\infty} \phi(m,t) = G(M,m)/\int_{\mu_1}^{\mu_2} G(M,y)dy$$

where G(x,y) is the Green function of the diffusion proceess defined by

$$\text{(A.2)} \qquad G(x,y) = \begin{cases} 2[S(x) - s(\mu_1)][S(\mu_2) - S(y)]/\{\sigma^2 s(x)[S(\mu_2)-S(\mu_1)]\} & \text{for } \mu_1 \le x \le y \le \mu_2 \\ 2[S(\mu_2) - S(x)][S(y) - S(\mu_1)]/\{\sigma^2 s(x)[S(\mu_2)-S(\mu_1)]\} & \text{for } \mu_1 \le y \le x \le \mu_2 \end{cases}$$

and where S( ) and s( ), for a Brownian motion with a drift, can be expresssed as follows (K-T, p.205),

$$s(x) = \exp(-2\mu x/\sigma^2)$$

(A.3)

$$s(x) = As(x) + B \quad (A \text{ and } B \text{ are constants}).$$

The integration required in the denominator of equation (A.1) can be performed straightforwardly to get

$$\text{(A.4)} \qquad \int_{\mu_1}^{\mu_2} G(M,y)dy = \int_{\mu_1}^{M} G(M,y)dy + \int_{M}^{\mu_2} G(M,y)dy$$

$$= 2AN/\{[s(\mu_2) - s(\mu_1)]\sigma^2 s(M)\}$$

where N is defined by

$$\text{(A.5)} \qquad N \equiv (\mu_1 - \mu_2)s(\mu_1)\, s(\mu_2) + s(M)[s(\mu_1)(M-\mu_1) + s(\mu_2)(\mu_2 - x)].$$

The division of equation (A.2) by (A.4) yields the following solution for the steady-state distribution,

$$\text{(A.6)} \qquad \phi(m) = \begin{cases} (1/N)[s(M) - s(\mu_2)][s(\mu_1) - s(m)] & \mu_1 \le m \le M \\ (1/N)[s(\mu_1) - s(M)]\,[s(m) - s(\mu_2)] & M \le m \le \mu_2 \end{cases}$$

Since we are interested in the case of no upper boundary, $\mu_2 \to \infty$, we get from equation (A.5)

$$\lim_{\mu_2 \to \infty} N = s(\mu_1)\, s(M)\, (M - \mu_1) \tag{A.7}$$

And, from equation (A.6)

$$\lim_{\mu_2 \to \infty} \phi(m) = \begin{cases} [s(\mu_1) - s(m)]/[s(\mu_1)(M-\mu_1)] & \mu_1 \le m \le M \\ s(m)[s(\mu_1) - s(M)]/[s(\mu_1)s(M)\,(M-\mu_1)] & M \le m \end{cases} \tag{A.8}$$

which is equation (51), section 5.

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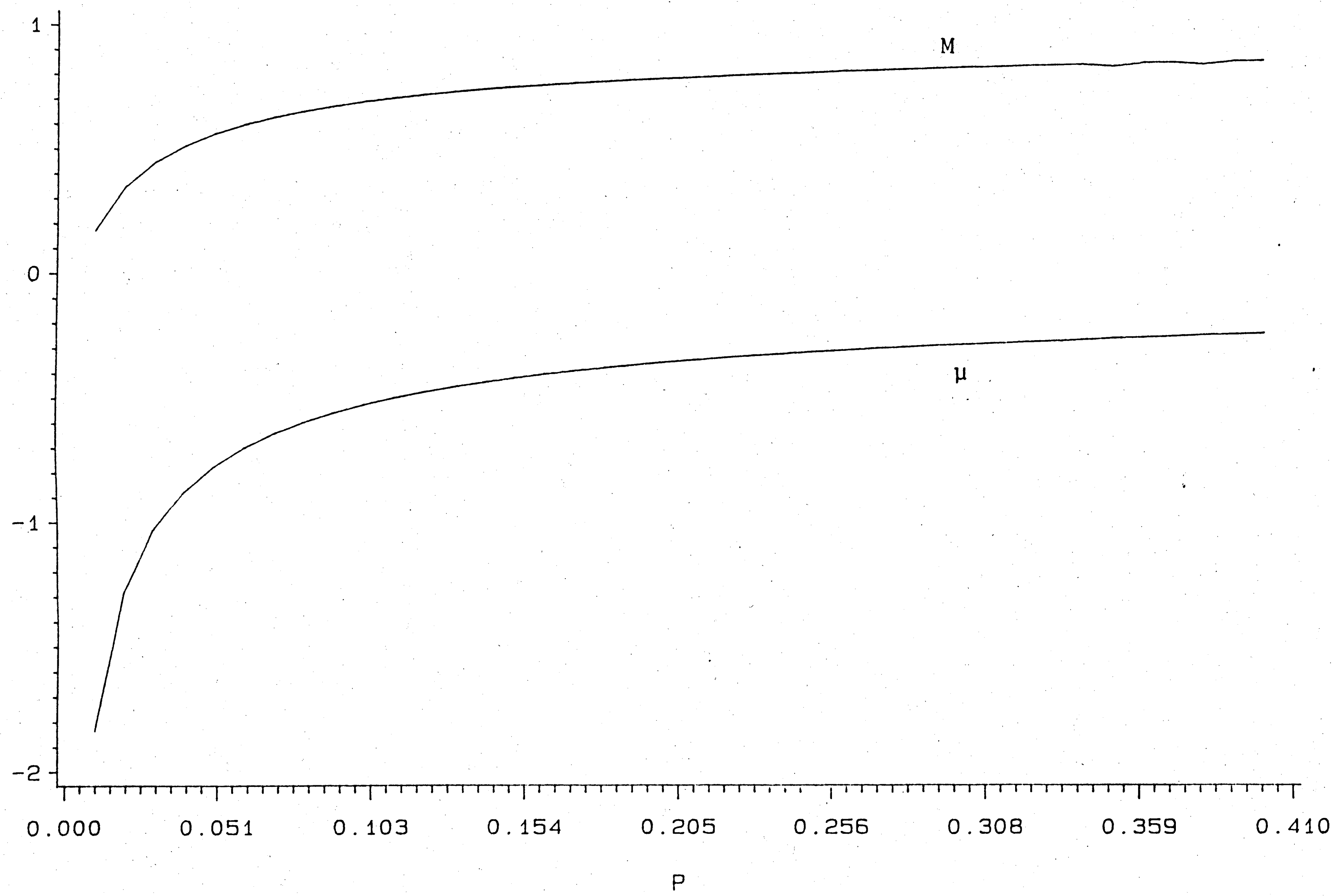

Figure 1. PLOT OF TRIGGER AND TARGET POINTS VS. OVERDRAFT RATE

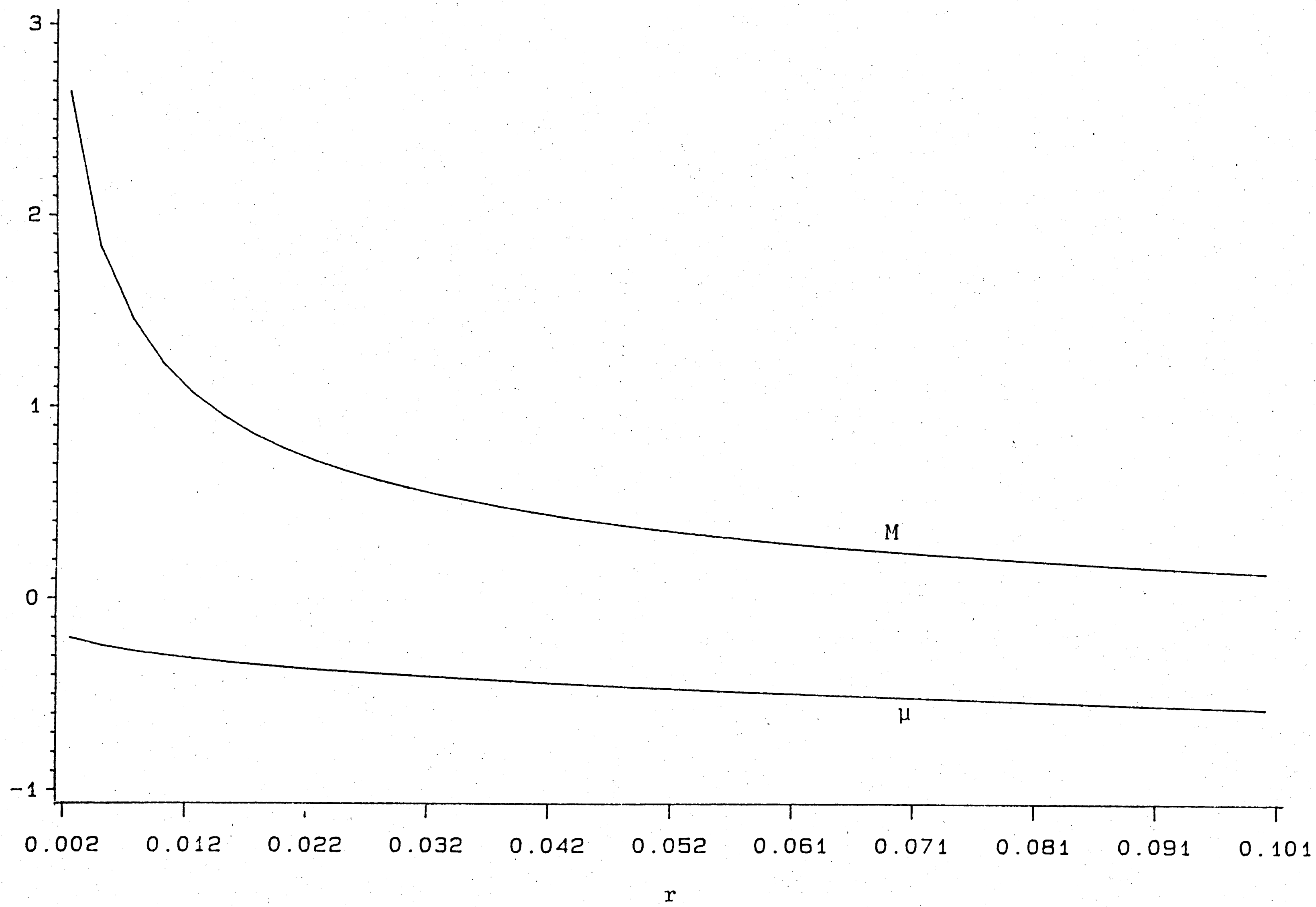

Figure 2. PLOT OF TRIGGER AND TARGET POINTS VS. HOLDING RATE