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STAGGERED AND SYNCHRONIZED PRICE POLICIES BY MULTIPRODUCT MONOPOLIES

by

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1. Introduction

It is now widely accepted that the microeconomic background of an inflationary process is characterized by discrete jumps in individual prices. This observation has led to a number of studies dealing with the aggregation of discrete pricing policies into a smooth time path for the aggregate price level. The feasibility of such aggregation is necessary for the overall consistency of individual pricing policies (Caplin and Spulber [1988]). A crucial issue for such an analysis is the interaction among individual price policies. If all firms follow identical real price cycles which are uniformly spread over time, then consistent aggregation is feasible (Sheshinski and Weiss [1977]). There may, however, be important reasons why such uniformity may not emerge as an equilibrium outcome. In oligopolistic markets, where each firm takes into account the actions of its rivals, pricing policies will be interdependent. In multiproduct monopolies, there is a further source for interdependence, namely, increasing returns in the costs of price adjustment. Even under competitive conditions, bunching over

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* We would like to thank Avner Bar-Ilan for the reference to Bensoussan, Crouhy and Proth [1983] and Sulem [1986].

time may be caused by aggregate shocks, while idiosyncratic shocks are needed to maintain the spread.

Apart from the issue of consistent aggregation, the time pattern of individual price policies has important implications for the real costs of inflation. If individual price paths are staggered, then temporary shocks may be propagated over long periods. Synchronized price policies, on the other hand, may accelerate the adjustment process (see Blanchard [1983], Blanchard and Fischer [1986, Ch. 9], and Taylor [1980]). In addition, non-synchronized price policies lead to price variations across products and thereby to search costs incurred by consumers (Benabou [1987], Fishman [1988]).

A number of recent studies have analysed the dynamic interaction of pricing policies in oligopolistic markets (Maskin and Tirole [1988], and Gertner [1986]), [1987]). However, these studies take the time pattern as exogenous, and focus on the equilibrium price configuration.

The issue of staggering vs. synchronization has been taken up by Ball and Cecchetti [1987] and by McMillan and Zinde-Walsh [1988]. The approach of Ball and Cecchetti emphasizes the informational gains from staggered pricing policies. McMillan and Zinde-Walsh consider a closed-loop equilibrium in an oligopolistic market for a homogeneous good.

Our own objective is to analyse the dynamics of a Bertrand game with diffentiated goods. In the present paper we analyse only the cooperative outcome of a duopoly game, assuming a single profit maximizing decision maker, i.e. a multiproduct monopoly. We view the analysis as a first step in the investigation of various equilibria of the duopoly game.

The main objective of this paper is the determination of the conditions which lead to staggered or synchronized pricing policies, when the timing of price changes is endogenous. We emphasize two aspects of the multiproduct monopoly decision problem. First, the interaction in the profit function between the prices of the two goods. Second, the form of the price adjustment costs. In particular, we distinguish between menu costs and decision costs. Under menu costs, we consider an increasing return to scale technology of price adjustments, whereby costs are independent of the number of items in the price list. Under decision costs we consider a constant returns to scale technology, whereby each price change requires an adjustment cost. A similar distinction in the inventory adjustment context was made by Sulem [1986].

Our main results can be summarized as follows:

(1) In the menu costs case, with strictly quasi-concave profits, a synchronized steady-state is attained after the first price change. This policy is essentially the optimal policy in the one-good case (Sheshinski and Weiss [1977]). We use a counterexample to show that quasi-concavity is essential for this result;

(2) Under decision costs, we show that a synchronized pricing policy never occurs if goods are strategic complements;

(3) With positive interaction of real prices in the profit function (and some additional assumptions), there exists a unique synchronized steady state <u>and</u> a unique symmetric staggered steady state.

(4) Which of the above steady states is attained depends on initial conditions. There is a set of initial conditions which triggers an immediate switch to the synchronized steady state. However, under no circumstance

will a joint price change be followed by a staggered steady state. That is, a staggered steady state can only be reached asymptotically.

(5) We provide a complete characterization of the optimal policy for the case of additive profits. In this case, there is a multiplicity of steady states differing only in the timing of price adjustments, which can be fully ranked (in terms of the present value of net profits). The synchronized steady state is the most profitable. Thus, as stated above, it is the one to be chosen when both prices are changed simultaneously.

(6) We suggest an approximation method which can be used for stability analysis. This method is applied to a non-additive example, which suggests that the staggered steady state is not stable. A similar result in different context was derived by Ball and Cecchetti [1987].

We conclude the paper with a few remarks on the non-cooperative (duopoly) game. We intend to provide a detailed analysis of this case in a companion paper.

2. The Model

Consider an economy subject to an inflationary trend where the aggregate price level grows at a constant rate, g $(g > 0)$. We analyse a monopoly who sells two related products whose demands depend on the current real prices of the two goods. The monopoly controls the nominal price of each good and there is a fixed real cost of nominal price adjustments.

Let $z_i(t)$ denote the log of the real price of good i at time t, $t \in [0,\infty)$. The real profit function of the monopoly, denoted by $F(z_1,z_2)$, is assumed to be time invariant. We assume that $F(z_1,z_2)$ is symmetric, strictly quasi-concave, twice differentiable and that it has a maximum at some finite point, denoted (S^*,S^*). Furthermore, it is assumed that the set $\{z_1,z_2 | F(z_1,z_2) \geq 0\}$ is compact. The class of functions satisfying all of these conditions is denoted by $\mathcal{F}$.

The problem facing the monopoly is a choice of price paths, $(z_1^*(t),z_2^*(t))$, which maximize the present value of real profits over an infinite horizon, given some initial condition $(z_1(0),z_2(0))$.

The salient feature of our model is the discontinuous pattern of nominal price adjustments. This widely observable phenomenon is generated in our model by the presence of non-convex costs of price adjustment: any nominal price change, no matter how small, requires non-negligible costs of adjustment. Specifically, the real cost of any nominal price change is assumed to be a constant denoted by β $(\beta > 0)$.

The main question which the paper addresses is the following: will the monopoly adopt a <u>synchronized</u> policy of price adjustments, whereby both prices are changed simultaneously, or a <u>staggered</u> policy whereby the two nominal prices are changed at different points in time.

Special attention will be given to repetitive price paths. An optimal policy can be described by the sequences $\{S_\tau^i\}_{\tau=0}^{\tau=\infty}$ and $\{t_\tau^i\}_{\tau=0}^{\tau=\infty}$, $i = 1,2$, where S_τ^i is the real price of good i chosen at the beginning of the time interval $[t_\tau^i, t_{\tau+1}^i)$. A <u>steady-state</u> path is defined by

(1) $$S_{\tau+1}^i = S_\tau^i \quad \text{and} \quad t_{\tau+1}^i = t_\tau^i + \varepsilon^i , \qquad \tau = 0,1,2,\ldots$$

where ε^i $(\varepsilon^i > 0)$, $i = 1,2$, are constants denoting the time intervals between subsequent price changes. Thus, in steady-state, the real prices chosen at the beginning of each interval and the duration until the next price change remain constant. A symmetric steady-state is defined by the additional restriction

$$(2) \qquad S_\tau^1 = S_\tau^2 = S \quad \text{and} \quad \varepsilon^1 = \varepsilon^2 = \varepsilon , \qquad \tau = 0,1,2,...$$

where S $(S > 0)$ and ε $(\varepsilon > 0)$ are constants. Along such a path, the real price of each good follows the same cycle. Among the symmetric steady-states we can identify a synchronized steady-state by the added requirement that

$$(3) \qquad t_0^1 = t_0^2 ,$$

that is, the prices of both goods are always changed at the same time. Finally, a (symmetric) staggered steady-state is defined by

$$(4) \qquad |t_0^1 - t_0^2| = \frac{\varepsilon}{2} ,$$

that is, the prices of the two goods are changed alternately and the time distance between any two price changes is equal.

The time pattern of the monopolist's optimal price policy, in particular whether price changes will be synchronized, depends crucially on two features of the model. The first relates to the technology of price adjustments, and the second to the form of the profit function. One issue of concern is the degree of returns to scale when both prices are changed simultaneously. Under constant returns to scale in the costs of price adjustment the monopoly incurs a cost of 2β whenever prices are changed jointly. Under increasing returns to scale these costs will be less than 2β, possibly as low as β. The degree of returns to scale depends on the distinction between 'menu costs' and 'decision costs' of price adjustment. By 'menu costs' we refer to costs such as advertizing and updating of price lists. By 'decision costs' we refer to costs of acquiring information on the production and demand of different products and to organization and computation costs of coordinated price changes in multiproduct firms. If the costs of price adjustment are interpreted as '*menu costs*', one would expect these costs to be β independently of the number of items in the menu. If, however, these costs are interpreted as *decision costs*, one would expect that the complexity of the choice, and thus the costs, will depend on the number of items involved, suggesting that constant returns to scale is the more appropriate assumption.

The other issue of concern is the interaction in demand and possibly in the production of the two goods. One would expect that if the goods are *strategic complements*, i.e., raising z_i increases the marginal profits of z_j, $j \neq i$, then synchronization is more likely, and *vice versa*. We shall investigate these issues in due course.

3. Menu Costs

In this section we analyse the extreme case of increasing returns where the costs of nominal price adjustments are β irrespective of the number of items in the menu.

Let $V(z_1,z_2)$ be the value function associated with an optimal policy starting at real prices (z_1,z_2) at time 0. The existence of such a function is guaranteed by our assumption that $F(z_1,z_2)$ has a well-defined maximum and by assuming that the real interest rate, r, is positive. The value function can be defined recursively:

$$(5) \quad V(z_1,z_2) = \underset{t \geq 0}{\mathrm{Max}}\{[\int_0^t e^{-rx}F(z_1-gx,\ z_2-gx)dx + e^{-rt}[\underset{S_1,S_2}{\mathrm{Max}}\ V(S_1,S_2) - \beta]\},$$

where t is the time of the subsequent price change and (S_1,S_2) are the real prices chosen at that time (i.e., nominal prices are set so as to attain these real prices). If the optimal t is $t = 0$, then a price change occurs immediately; otherwise the current nominal prices will be kept unchanged, with real prices decreasing at the rate of inflation, g, over the interval $[0,t)$. It is asumed that for any initial (z_1,z_2), a price change is optimal after a finite lapse of time. Our assumptions on the profit function (specifically, that $F(z_1,z_2)$ has a well defined maximum and that it reaches zero at finite time from any initial (z_1,z_2)), ensure that V is positive for all (z_1,z_2), provided that β is sufficiently small relative to maximum profits.

We begin our analysis by stating some properties of the value function which will be used subsequently:

(i) $V(z_1,z_2)$ is symmetric,

(ii) $V(z_1,z_2)$ is continuous,

(iii) $V(z_1,z_2)$ is differentiable, except possibly at some boundary points to be defined below.

Symmetry of $V(z_1,z_2)$ follows directly from the assumed symmetry of the profit function $F(z_1,z_2)$. Starting from $z_1 = a$ and $z_2 = a'$ or $z_1 = a'$ and $z_2 = a$, the monopoly can obtain the same present value of future profits simply by exchanging the optimal price sequences of the two products.

Continuity of $V(z_1,z_2)$ follows from the continuity of the integral of profits on the R.H.S. of (5) in (z_1,z_2), for any given choice of S_1, S_2 and t. Hence the maximum over these must also be continuous.

Differentiability may be determined as follows. Note first that the choice of S_1 and S_2 in (5) is <u>independent</u> of (z_1,z_2). Only the timing of the subsequent price change, t, depends on (z_1,z_2). By straightforward application of the envelope theorem, continuity of the optimal t is sufficient, in view of the differentiability of $F(z_1,z_2)$, to establish the differentiability of $V(z_1,z_2)$. Indeed,

(6) $$V_i(z_1,z_2) = \int_0^{t^*} e^{-rx} F_i(z_1-gx,\ z_2-gx)dx\ , \qquad i = 1,2\ ,$$

where $V_i = \partial V/\partial z_i$, $F_i = \partial F/\partial z_i$, $i = 1,2$, and $t^* = t^*(z_1,z_2)$ is the optimal choice of t.

We are now ready to characterize the optimal policy. Consider the following subsets of $\mathbb{R}^2$:

<u>Continuation Set</u> (C):

(7) $$C = \{z_1, z_2 \,|\, t^*(z_1, z_2) > 0\}$$

<u>Trigger Set</u> (T_0)

(8) $$T_0 = \{z_1, z_2 \,|\, t^*(z_1, z_2) = 0\} \ .$$

Clearly, these sets are mutually exclusive and $C \quad T_0 = R^2$ (Bensoussan, Crouhy and Proth [1983], and Sulem [1986]).

Let (z_1, z_2) be a point in the interior of C . Then the following restrictions on V must be satisfied:

(9) $$V(z_1, z_2) > V^* - \beta \ ,$$

where $V^* = \underset{S_1, S_2}{\text{Max}}\, V(S_1, S_2)$;

(10) $$F(z_1, z_2) = gV_1(z_1, z_2) + gV_2(z_1, z_2) + rV(z_1, z_2) \ .$$

Equation (10) is derived from (5) by the following consideration. For any $t^* > 0$, there exists an h , $0 < h < t^*$, such that:

(11) $$V(z_1, z_2) = \int_0^h e^{-rx} F(z_1 - gx,\ z_2 - gx)dx + e^{-rh} V(z_1 - gh, z_2 - gh) \ .$$

Expanding the R.H.S. of (11) in a Taylor series for h at (z_1, z_2) , one obtains

$$V(z_1,z_2) = F(z_1,z_2)h + V(z_1,z_2) - rV(z_1,z_2)h - \qquad (12)$$
$$- [gV_1(z_1,z_2) - gV_2(z_1,z_2)]h + O(h) \; .$$

Taking the limit as $h \to 0$, we obtain (10).

Equation (10) can be interpreted as an asset pricing formula. The imputed value of a state which does not generate a price change, $rV(z_1,z_2)$, is given by the current flow of profits, $F(z_1,z_2)$, less the depreciation caused by the inflationary erosion in real prices, $gV_1(z_1,z_2) + gV_2(z_1,z_2)$. In subsequent analysis we shall refer to equation (10) as the '<u>valuation formula</u>'.

Next consider a point in the interior of T_0 . In this case, $V(z_1,z_2)$ must satisfy the restriction

$$V(z_1,z_2) = V^* - \beta \qquad (13)$$

and

$$F(z_1,z_2) < gV_1(z_1,z_2) + gV_2(z_1,z_2) + rV(z_1,z_2) = rV(z_1,z_2) \; , \qquad (14)$$

since, by (13), $V_1(z_1,z_2) = V_2(z_1,z_2) = 0$. Equation (14) is obtained by noting that if a price change occurs then the L.H.S. must exceed the R.H.S. of (11).

We now wish to characterize the pair (S_1,S_2) chosen when (z_1,z_2) is in the trigger set.

<u>Proposition 1</u>. Assume that $F(z_1,z_2) \in \mathcal{F}$. Then, (a) for any $(z_1,z_2) \in T_0$, the optimal choice of (S_1,S_2) satisfies $S_1 = S_2 = S$, where $S > S^*$ is a unique singleton in the interior of C; (b) the continuation set C is defined by $F(z_1,z_2) > r(V^* - \beta)$.

<u>Proof</u>: (a) Any maximizer of V must be in C. Otherwise an additional cost β would be incurred to obtain the same value V^*. Observe that for any point in C, equation (6) applies and V is differentiable. Combining these facts, we obtain

$$rV(S_1,S_2) = F(S_1,S_2) \tag{15}$$

where $(S_1,S_2) \in \arg\max_{z_1,z_2} V(z_1,z_2)$.

Now suppose that $S_1 \neq S_2$. Then, by symmetry, the points (S_1,S_2) and (S_2,S_1) are both maximizers of $V(z_1,z_2)$, yielding the same value V^*.

For any $0 < \gamma < 1$, define (S_1^γ,S_2^γ) as:

$$\begin{aligned} S_1^\gamma &= \gamma S_1 + (1-\gamma)S_2 , \\ S_2^\gamma &= \gamma S_2 = (1-\gamma)S_1 . \end{aligned} \tag{16}$$

Using recursive equation (5), the value associated with $V(S_1,S_2)$ is

$$V(S_1,S_2) = \int_0^{t^*} e^{-rx}F(S_1-gx,S_2-gx)dx + e^{-rt^*}[V^*-\beta] \tag{17}$$

where $t^* = t^*(S_1,S_2)$ is the optimal time for the subsequent price change and V^* is the maximum value of V realized at t^*. Starting at (S_1^γ,S_2^γ), the same choices are still feasible. Hence,

$$(18) \qquad V(S_1^{\gamma},S_2^{\gamma}) \geq \int_0^{t^*} e^{-rx}F(S_1^{\gamma}-gx,S_2^{\gamma}-gx)dx + e^{-rt^*}[V^*-\beta] .$$

By strict quasi-concavity, $F(S_1-gx,S_2-gx)dx < F(S_1^{\gamma}-gx,S_2^{\gamma}-gx)dx$ for all x , $t^* \geq x \geq 0$. Thus, equations (17) and (18) imply that $V(S_1^{\gamma},S_2^{\gamma}) > V(S_1,S_2)$, which contradicts the assumption that (S_1,S_2) maximizes V .

This proves that $S_1 = S_2 = S$. To prove that S is unique we use again the quasi-concavity of F together with the valuation formula (10).

Suppose there are two values, (S^a,S^a) and (S^b,S^b) , $S^a \neq S^b$, that maximize V . Let $S^{\theta} = \theta S^a + (1-\theta)S^b$, $0 < \theta < 1$. Then,

$$(19) \qquad F(S^{\theta},S^{\theta}) > F(S^a,S^a) = F(S^b,S^b) = rV^* \geq rV(S^{\theta},S^{\theta}) .$$

Inequality (19) and the valuation formula (10) imply that for any $(S^{\theta},S^{\theta}) \in C$, we must have

$$(20) \qquad gV_1(S^{\theta},S^{\theta}) + gV_2(S^{\theta},S^{\theta}) > 0 .$$

Letting $\gamma \to 0$ or $\gamma \to 1$, (20) implies that V can be increased in the neighborhood of (S^a,S^a) or (S^b,S^b) , contrary to the assumption that these are local maxima.

To prove that (S,S) is in the interior of C , note that for (z,z) slightly above (S,S) , it is always worthwhile to wait until the price path reaches (S,S) , thus avoiding the cost of a nominal price change, β .

Finally, we prove that $S > S^*$, where F attains its unique maximum at $F(S^*,S^*)$. Suppose that $S < S^*$. Let $S^\gamma = \gamma S^* + (1-\gamma)S$, $0 < \gamma < 1$. Then, by (15) and the assumption that (S,S) maximizes V ,

$$F(S,S) = rV^* \geq rV(S^\gamma,S^\gamma) \ . \tag{21}$$

By strict quasi-concavity of $F(z_1,z_2)$,

$$F(S^\gamma,S^\gamma) > F(S,S) \ . \tag{22}$$

Hence,

$$F(S^\gamma,S^\gamma) > rV(S^\gamma,S^\gamma) \ , \tag{23}$$

which for (S^γ,S^γ) C implies, by (6), that $gV_1(S^\gamma,S^\gamma) + gV_2(S^\gamma,S^\gamma) > 0$. Since (S,S) is in the interior of C , by letting $\gamma \to 0$ we obtain a contradiction to the assumption that (S,S) maximizes V .

(b) To complete the proof we now describe the boundary points between T_0 and C . Consider any sequence $(z_1^n,z_2^n) \in C$ and $t^*(z_1^n,z_2^n) \to 0$ as $n \to \infty$. Any such limit point, (s_1,s_2) , is in T_0 and therefore must satisfy equations (13) and (14).

Furthermore, equation (14) must hold with equality. Hence,

$$rV(s_1,s_2) = rV^* - r\beta = F(s_1,s_2) \qquad \| . \tag{24}$$

Condition (24) implies that the boundary between T_0 and C is given by an iso-profit curve in the (z_1,z_2) space. The critical level of profits is the imputed flow of profits associated with the optimal choice, (S_1,S_2), net of the imputed flow of the costs of price changes. In other words, the gain from a delay, $F(s_1,s_2)$, is equated to the opportunity costs, $rV^* - r\beta$.

The optimal policy can now be described with the aid of Figure 1.

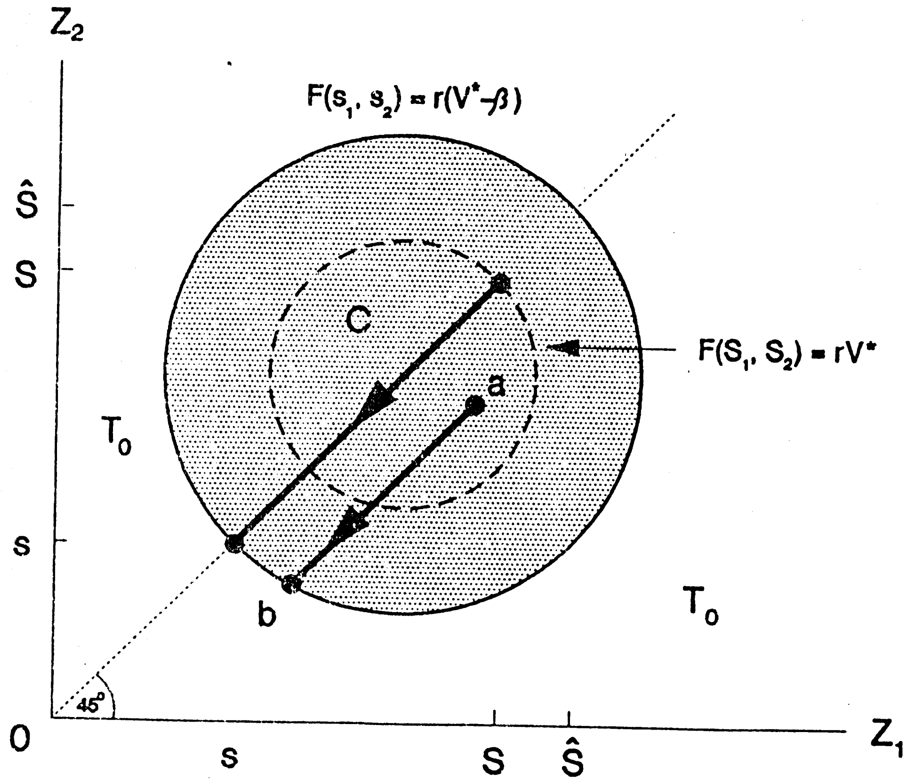

Figure 1

Starting at any initial $(z_1, z_2) \in T_0$, the optimal policy is to jump immediately to (S,S). At any initial $(z_1, z_2) \in C$ (point a, in Figure 1), the optimal policy is to wait until the trajectory reaches the boundary (point b) and then, again, (S,S) is chosen. After the first price change the policy is fully repetitive. The price path starts at (S,S). Then, after a fixed interval, ε, allowing real prices to erode to (s,s), where $s = S - g\varepsilon$, nominal prices are adjusted to attain again the level (S,S). Along this path, prices are always equal. We have referred to this solution as the Synchronized Steady-State.

The optimal policy of the multi-product monopoly is seen to be identical to the case of a monopolist selling one good analysed in Sheshinski-Weiss [1977]. The actual values of (S,S) and (s,s) can be calculated as in the single product case. Specifically, equations (5), (15) and (24) can be reduced to:

$$F(S,S) - F(s,s) - r\beta = 0 , \tag{25}$$

$$F(S,S) = \frac{r}{1 - e^{-r\varepsilon}} \left[\int_0^{\varepsilon} e^{-rx} F(S-gx, S-gx)dx - \beta e^{-r\varepsilon}\right] , \tag{26}$$

where $\varepsilon = \frac{1}{g}(S-s)$. As in Sheshinski-Weiss [1977], strict quasi-concavity of $F(z_1, z_2)$ is sufficient to guarantee that the solution, (S,s), is unique.

At this point we can comment on the differentiability of $V(z_1, z_2)$ on the boundary between T_0 and C. Consider first the boundary point (s,s). Recall that by (24), $rV(s,s) = F(s,s)$. If V is differentiable at this point then, by (10), the gradient $gV_1(s,s) + gV_2(s,s)$ must be zero, which under (6) is consistent with the differentiability of $V(z_1,z_2)$ at (s,s). Consider, however, the point $(\hat{S},\hat{S})$ which is also a solution to equation (24), satisfying $s_1 = s_2$. At this point, there is a discontinuity in the choice of t. At a point slightly above on the diagonal, $t^*(z_1,z_2) = 0$, but at points on the diagonal slightly below, $t^*(z_1,z_2) = \hat{\varepsilon} > \varepsilon > 0$. This discontinuity in the action leads to non-differentiability of $V(z_1,z_2)$ at $(\hat{S},\hat{S})$. On the diagonal above $(\hat{S},\hat{S})$, by (13), $V_1(z_1,z_2) = V_2(z_1,z_2) = 0$. However, on the diagonal below $(\hat{S},\hat{S})$, the gradient must be strictly negative. This follows from (6) using the strict quasi-concavity of $F(z_1,z_2)$ and the fact that $V_1(S,S) = V_2(S,S) = 0$. It should be noted that in the related literature on inventories (see Sulem [1986] and Costantinides and Richard [1978]), $V(z_1,z_2)$ <u>is</u> differentiable at the boundary. The difference in our model is caused by the fact that we allow the price to be raised <u>and</u> to be lowered. In contrast, the inventory literature always assumes positive orders.

Our results so far suggest that, under the menu costs hypothesis, a fully synchronized price policy is optimal for a wide class of profit functions. The crucial assumption behind this result is that profits are quasi-concave in real prices. While this assumption is quite natural in a context where prices are the choice variables, there are plausible circumstances where it fails to hold. To illustrate this point we consider, in the next section, an example where quasi-concavity fails.

4. An Example Without Quasi-Concavity

Consider a given number of customers, N, each demanding one unit of a product at all prices below a reservation price e^b, and nothing at prices exceeding b. The monopoly operates two stores, generally charging different prices for the same product. Given the announced prices, customers are divided between the stores according to a symmetric probability distribution function defined on the price ratio of the two stores.

Let z_i be the log of the real price at store i, $i = 1,2$ and $p(z_2-z_1)$ the probability of purchase at store one when $z_1 \leqslant b$ and $z_2 \leqslant b$. Under symmetry, $p(z_2-z_1) = 1 - p(z_1-z_2)$. We further assume that if either store announces a price above b then no customer arrives at this store.

Total real profits are

$$
(27) \qquad F(z_1,z_2) = \begin{cases} N[p(z_2-z_1)e^{z_1} + p(z_1-z_2)e^{z_2}] & z_1 \leqslant b\,, \quad z_2 \leqslant b \\ Ne^{z_1} & z_1 \leqslant b\,, \quad z_2 > b \\ Ne^{z_2} & z_1 > b\,, \quad z_2 \leqslant b \\ 0 & z_1 > b\,, \quad z_2 > b \end{cases}
$$

The iso-profit curves corresponding to these assumptions are presented in Figure 2. Note the discontinuities at $z_1 = b$ and $z_2 = b$. Selling at the same price at both stores or selling at that price at one store, yields the same profits. However, as long as both prices are below b it is always worthwhile to increase either one of the two.

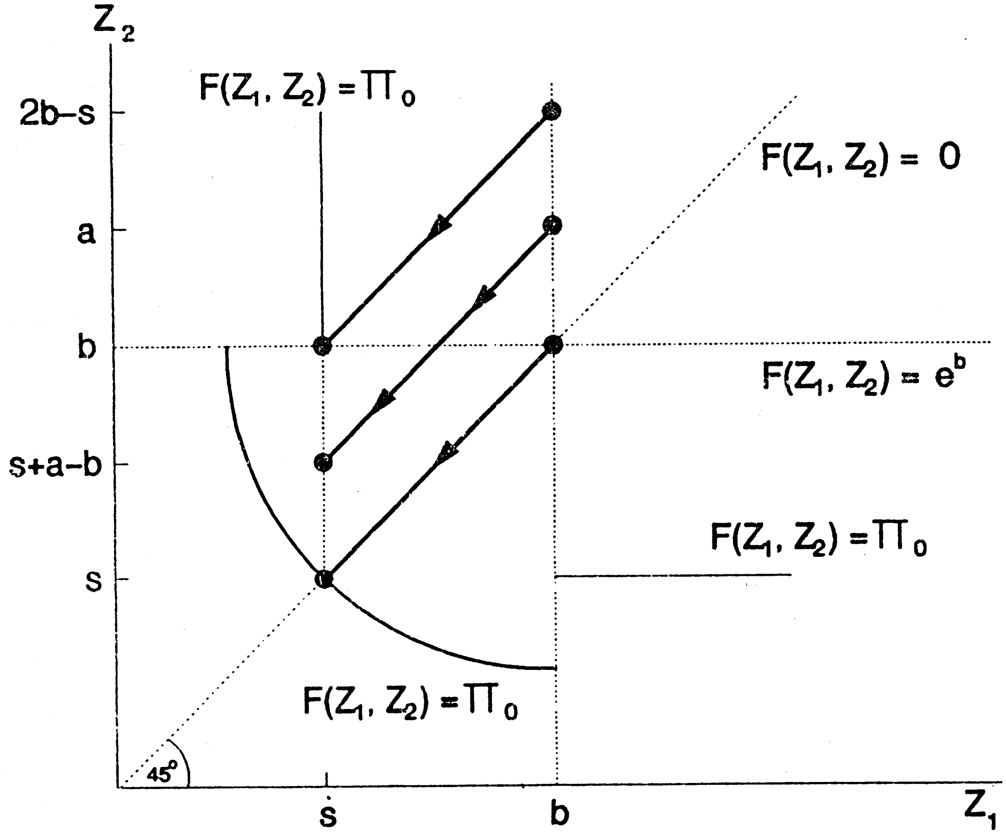

Figure 2

For this example a synchronized policy can never be optimal. Referring again to Figure 2, consider the synchronized steady-state indicated by the points (b,b) and (s,s) . Along this path, sales are divided equally between the two stores. Now consider the path alternating between (b,2b-s) and (s,b) . Along this path, $z_2 > b$ at all times and hence all the customers purchase at the first store. The time between price adjusmtents is the same as in the synchronized path and real profits at any point along the cycle are also the same. Finally, note that any path with the same periodicity but associated with positive sales at both stores over part of the interval, such as (b,a) and (s,s+a-b) , is superior to both. Therefore the synchronized steady state can never be optimal. The main feature of this example is that, starting with equal prices for the two goods, the monopoly can never lose, and generally gains, from increasing either price. This situation can never arise when $F(z_1,z_2)$ is strictly quasi-concave and reaches a maximum at a finite (z_1,z_2). In that case, deviations from the diagonal yield lower profits at prices above the maximum profits.

5. Decision Costs

In this section we analyse the case of constant returns to scale in the costs for nominal price adjustments. That is, each price change requires a cost of β, and if both prices are changed simultaneously the costs are 2β. Our justification for this assumption is that with more prices to be changed, the decision problem facing the monopoly increases in complexity. A typical organizational solution to this problem is decentralization, whereby separate divisions are allowed to follow separate pricing policies, maximizing objective functions set by the center. The overall outcome of this process is that adjustment costs for the monopoly are the sum of the costs incurred by the separate 'price centers'.

The value function now satisfies the following recursive equation:

$$(28) \quad V(z_1,z_2) = \max_{t \geq 0}\{(\int_0^t e^{-rx}F(z_1-gx,\ z_2-gx)dx + e^{-rt}\max[\max_{S_1,S_2} V(S_1,S_2) - 2\beta,\ \max_{S_1} V(S_1,z_2-gt) - \beta,\ \max_{S_2} V(z_1-gt,S_2) - \beta]\}.$$

The difference between equations (5) and (28) stems from the fact that it may now be optimal for the firm to change only one of the two prices in order to save the adjustment costs. Using the same arguments as in Section 3, $V(z_1,z_2)$ can be shown to be symmetric, continuous and differentiable except at some boundary points.

Let $t_i^*(z_1,z_2)$ be the optimal time for a change in the price of good i, and S_i, be the optimal choice when the price of good i is changed, $i = 1,2$. Then, whenever $V(z_1,z_2)$ is differentiable, its partial derivatives are given by

$$
\begin{aligned}
&V_1(z_1,z_2) = \int_0^{t_1^*} e^{-rx} F_1(z_1-gx, z_2-gx)dx \\
(29)\quad & \\
&V_2(z_1,z_2) = \int_0^{t_1^*} e^{-rx} F_2(z_1-gx, z_2-gx)dx + \int_{t_1^*}^{t_2^*} e^{-rx} F_2(S_1-g(x-t_1^*), z_2-gx)dx
\end{aligned}
$$

for $t_1^*(z_1,z_2) \leq t_2^*(z_1,z_2)$; and

$$
\begin{aligned}
&V_1(z_1,z_2) = \int_0^{t_2^*} e^{-rx} F_1(z_1-gx, z_2-gx)dx + \int_{t_2^*}^{t_1^*} e^{-rx} F_1(z_1-gx, S_2-g(x-t_2^*))dx \\
(30)\quad & \\
&V_2(z_1,z_2) = \int_0^{t_2^*} e^{-rx} F_2(z_1-gx, z_2-gx)dx
\end{aligned}
$$

for $t_2^*(z_1,z_2) \leq t_1^*(z_1,z_2)$.

The solution is now described with the aid of four distinct sets:

$$
(31)\quad
\begin{aligned}
C &= \{z_1, z_2 \mid t_1^*(z_1,z_2) > 0\ ,\ \ t_2^*(z_1,z_2) > 0\} \\
T_0 &= \{z_1, z_2 \mid t_1^*(z_1,z_2) = t_2^*(z_1,z_2) = 0\} \\
T_1 &= \{z_1, z_2 \mid t_1^*(z_1,z_2) = 0\ ,\ \ t_2^*(z_1,z_2) > 0\} \\
T_2 &= \{z_1, z_2 \mid t_1^*(z_1,z_2) > 0\ ,\ \ t_2^*(z_1,z_2) = 0\}
\end{aligned}
$$

The set of initial conditions C is the continuation set where no price change occurs. The set T_0 triggers a change in both prices, while T_i , $i = 1,2$, is the set which triggers a change in the price of good i only. Clearly, these sets are mutually exclusive and $C \cup T_0 \cup T_1 \cup T_2 = R^2$.

The restrictions on $V(z_1,z_2)$ in the sets C and T_0 are the same as in Section 3, except that β is replaced by 2β when both prices are changed simultaneously.

The restrictions on $V(z_1,z_2)$ in T_1 are:

$$(32)\qquad V(z_1,z_2) = \underset{S_1}{\text{Max}}\ V(S_1,z_2) - \beta\ ,$$

$$(33)\qquad V(z_1,z_2) \geq \text{Max}[\underset{S_1,S_2}{\text{Max}}\ V(S_1,S_2) - 2\beta,\ \underset{S_2}{\text{Max}}\ V(z_1,S_2-\beta]\ ,$$

and

$$(34)\qquad gV_2(z_1,z_2) + rV(z_1,z_2) \geq F(z_1,z_2)\ ,$$

where (34) is derived from equation (14), using (32) (which implies that $V_1(z_1,z_2) = 0$). These conditions restate the requirement that the four sets defined above, (31), must be distinct. That is, if $(z_1,z_2) \in T_1$, then changing the price of the first good is superior to any alternative. Analogous restrictions on $V(z_1,z_2)$ apply when $(z_1,z_2) \in T_2$.

We are now ready to investigate the properties of the optimal price path for the decision costs case.

The restrictions on the value function, (32)-(34), impose restrictions on the real price changes along the optimal path. At a boundary point between C and T_0 , (s_1,s_2) , where a change in both prices is triggered, we must have

(35) $$F(S_1,S_2) - 2r\beta = F(s_1,s_2) .$$

Similarly, at a boundary point between T_1 and C , (s_1,z_2) , we have

(36) $$F(S_1,z_2) - r\beta = F(s_1,z_2) .$$

Finally, at a boundary point between T_2 and C , (z_1,s_2) , we have

(37) $$F(z_1,S_2) - r\beta = F(z_1,s_2) .$$

Whenever V is differentiable at the boundary of C , one can use the valuation formula (10) to derive these equations. Otherwise, we use the continuity of V at the boundary. We shall sketch the derivation of equation (36). Differentiating (32), using the envelope relation, we obtain

(38) $$V_1(s_1,z_2) = V_1(S_1,z_2) = 0$$

(39) $$V_2(s_1,z_2) = V_2(S_1,z_2) .$$

By the valuation formula (10),

$$F(s_1,z_2) = rV(s_1,z_2) + gV_2(s_1,z_2) \tag{40}$$

$$F(S_1,z_2) = rV(S_1,z_2) + gV_2(S_1,z_2) \tag{41}$$

Thus, using equations (32) and (39), equation (36) follows.

If $t^*(z_1,z_2)$ is not continuous at the boundary between T_1 and C, then $V_1(s_1,z_2)$ does not exist, but equation (40) still holds, since $V_2(s_1,z_2)$ is well defined. Equations (35) and (37) are derived similarly.

The economic interpretation of these equations is clear. The L.H.S. of each equation is the cost of the delay in a price change consisting of foregone profits at the new real prices net of adjustment costs, while the R.H.S. is the benefit consisting of profits at the old prices.

We are now ready to investigate the properties of the optimal price path for the decision costs case.

It is clear that in contrast to the menu costs case, decision costs imply that the class of profit functions which entails synchronization is narrower. There is no savings in adjustment costs when prices are changed simultaneously. This observation is highlighted by the following proposition.

Proposition 2. Assume that $F(z_1,z_2) \in \mathcal{F}$. Assume further that $F_{12}(z_1,z_2) < 0$ for all (z_1,z_2). Then there is no sequence $(z_1^n,z_2^n) \in C$, whose limit as $n \to \infty$ is in T_0. In particular, a simultaneous price change can occur at most once.

Proof. Suppose there is a sequence $(z_1^n, z_2^n) \in C$ whose limit point as $n \to \infty$, (s_1, s_2), is in T_0 and let (S_1, S_2) be the choice of prices triggered at (s_1, s_2). Clearly, $(S_1, S_2) \in C$, and

(42) $$V(s_1, s_2) = V(S_1, S_2) - 2\beta ,$$

(43) $$V(S_1, S_2) - 2\beta \geq V(S_1, s_2) - \beta ,$$

(44) $$V(S_1, S_2) - 2\beta \geq V(s_1, S_2) - \beta .$$

Conditions (43) and (44) imply, respectively, that $(S_1, s_2) \in T_2$ and $(s_1, S_2) \in T_1$. Hence, by (15), (32) and (34),

(45) $$F(S_1, S_2) - r\beta \geq F(S_1, s_2) ,$$

and

(46) $$F(S_1, S_2) - r\beta \geq F(s_1, S_2) .$$

Condition (42) together with the valuation formula (6) imply that

(47) $$F(S_1, S_2) - 2r\beta = F(s_1, s_2) .$$

Substituting (47) into (46) and using (45) we obtain

(48) $$F(S_1, S_2) - F(S_1, s_2) \geq r\beta \geq F(s_1, S_2) - F(s_1, s_2) .$$

Clearly, the inequalities in (48) cannot hold simultaneously if $F_{12}(z_1, z_2) < 0$ uniformly ||.

To illustrate the implication of a negative interaction between the two prices ($F_{12} < 0$), consider the following example. Let $F(z_1, z_2) = G(z_1+z_2)$, where $G(\cdot)$ is strictly concave.[1] Clearly, $F_{12} = G'' < 0$.

For this example it is easy to see that a synchronized price policy cannot be optimal. Consider the set of fully synchronized price paths. In this class, the choice of an optimal path is equivalent to the one good case. Thus, applying the results in Sheshinski and Weiss [1977], the optimal path converges after one price change to a steady-state (S,s) policy. The profit levels associated with these (S,s) values are indicated in Figure 3.

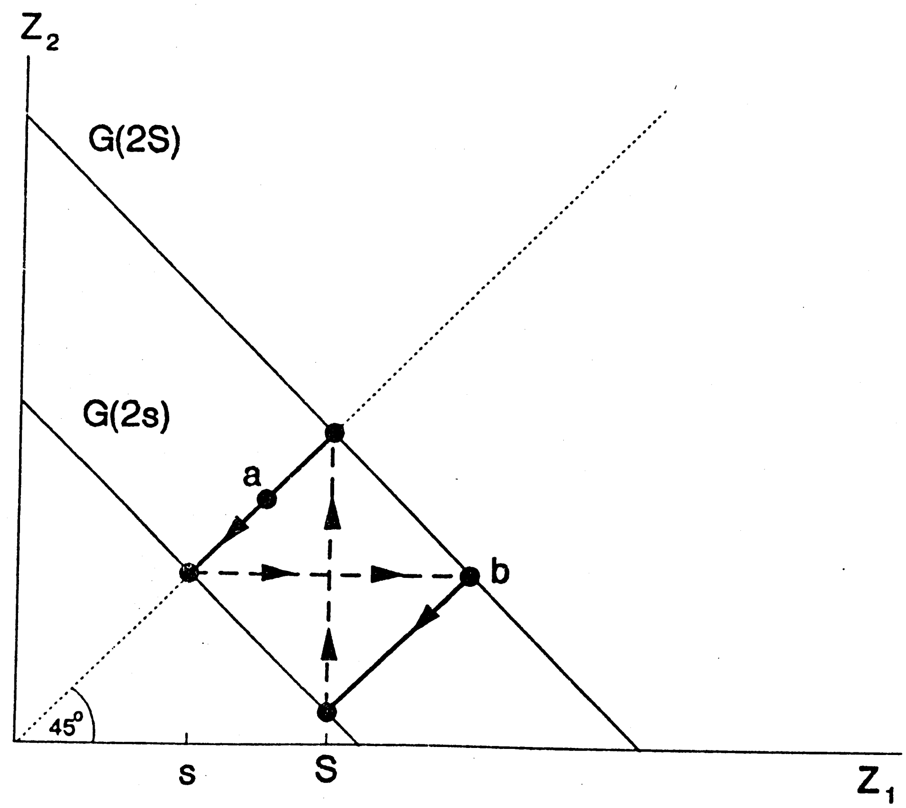

Figure 3

Starting at any part of this supposedly optimal path, it is optimal to diverge from it after the first price change. Thus, starting at point a on the diagonal, reaching the point (s,s), then changing one price only, say the first, to $2S - s$ (point b), then allowing real prices to drift down to $(S, 2s-S)$, and changing the second price to S, is superior to a simultaneous change from (s,s) to (S,S). The non-synchronized policy, while providing the same profits, requires half the adjustment costs, since at any change only one price is adjusted while the same sum of prices is maintained. In this example, since prices are perfect-substitutes, any level of profits can be attained by changing only one price, thus avoiding the additional adjustment costs that would be incurred if both prices were changed. Hence, a synchronized policy cannot be optimal.

The previous example suggests that further restrictions have to be imposed on profit functions to permit both synchronized and staggered price policies. In particular, in the decision costs case, some complementarity between prices is needed to overcome the higher costs of simultaneous price changes.

In the subsequent analysis, we shall therefore adopt two additional assumptions:

A1. Complementarity

For any (z_1,z_2),

$$F_{12}(z_1,z_2) \geq 0 . \tag{49}$$

A2. Non-Reversibility

For any z_1, z_2 and x $(x > 0)$,

(50) $$F_i(z_1,z_2) > 0 \Rightarrow F_i(z_1-x,z_2-x) > 0 , \qquad i = 1,2 .$$

Assumption A1 requires that any increase in the price of good j also increases marginal profits from an increase in the price of good i, $i \neq j$, thereby enhancing an accompanying increase in the price of good i.

Assumption A2 imposes the natural requirement that if a price increase is profitable at (z_1,z_2), then it is also profitable after these real prices are eroded by inflation to (z_1-x, z_2-x). Clearly, monotonicity of F_i w.r.t. equal changes in z_1 and z_2, $F_{11}(z_1,z_2) + F_{12}(z_1,z_2) < 0$ for all (z_1,z_2) implies this assumption but not vice-versa.

We now turn to an analysis of the optimal price policy, applying assumptions A1 and A2.

6. Positive Interactions

In this section we consider a monopoly whose profit function displays positive interactions between the prices of the two goods. As specified in A1, an increase in z_2 which increases profits also increases the marginal profits from raising z_1, and vice-versa. We continue to assume that the costs of price change are decision costs and thus that a simultaneous price change requires a cost of 2β.

We first prove two lemmas which will be used in the sequel. The first concerns a property of symmetric, quasi-concave functions and the second concerns the patern of marginal profits along an optimal path.

<u>Lemma 1</u>. If $F(z_1,z_2)$ is quasi-concave, symmetric and differentiable, then for any (z_1,z_2),

(51) $$(z_1-z_2)(F_2(z_1,z_2) - F_1(z_1,z_2)) \geq 0 .$$

<u>Proof</u>: By symmetry, $F(z_1,z_2) = F(z_2,z_1)$. By quasi-concavity, $F(\gamma z_1+(1-\gamma)z_2) \geq F(z_1,z_2)$, for all $0 \leq \gamma \leq 1$. Expanding around $\gamma = 0$,

(52) $$F(z_1,z_2) + (F_1(z_2,z_1)(z_1-z_2) + $$
$$+ F_2(z_2,z_1)(z_2-z_1))\gamma + 0(\gamma) \geq F(z_1,z_2) .$$

Cancelling, dividing by γ and taking the limit, we otain (51) ||.

<u>Lemma 2</u>. Under assumptions A1 and A2, on any interval with a fixed nominal price for good i, $i = 1,2$, we have $F_i < 0$ at the intital point and $F_i > 0$ at the end point of such an interval.

<u>Proof</u>: Without loss of generality, suppose that at time $t = 0$, the price of the first good has been raised to S_1. Then,

(53) $$V_1(S_1,z_2) = 0 .$$

Let $t^*(S_1,z_2)$ be the optimal time of the subsequent price change. There are two cases to consider.

Case 1: the subsequent price change involves a change in z_1 (possibly together with z_2). Differentiating (28),

$$V_1(S_1,z_2) = \int_0^{t^*(S_1,z_2)} e^{-rx} F_1(S_1-gx,z_2-gx)dx \quad (54)$$

By the non-reversibility assumption A2,

$$F_1(S_1,z_2) > 0 \Rightarrow F_1(S_1-gx,z_2-gx) > 0 \quad (55)$$

for all $t^* \geq x \geq 0$. Hence, $V_1(S_1,z_2) > 0$, contrary to (53). Therefore, $F_1(S_1,z_2) < 0$ and $F_1(S_1-gt^*,z_2-gt^*) > 0$.

Case 2: the subsequent price change is an increase in the second price. In this case, applying equation (6) again,

$$V_1(S_1,z_2) = \int_0^{t^*(S_1,z_2)} e^{-rx} F_1(S_1-gx,z_2-gx)dx + \int_{t^*(S_1,z_2)}^{t^{**}} e^{-rx} F_1(S_1-gx,S_2^*-g(x-t^{**}))dx \quad (56)$$

where $t^{**} = t^*(S_1-gt^*(S_1,z_2),\ z_2-gt^*(S_1,z_2))$ and $S_2^* = S_2^*(S_1-gt^*(S_1,z_2))$. Under A1, at $t^*(S_1,z_2)$, as a consequence of the increase in z_2 , F_1 is raised. Thus, using monotonicity (A2), equation (53) implies that $F_1(S_1,z_2) < 0$ and $F_1(S_1-gt^{**},\ S_2^*-g(t^{**}-t^*)) > 0$.

These two cases exhaust all possibilities, since a reduction in price can only occur at the first change. ||

We now begin the analysis by proving some properties of the actions triggered by the sets T_1 , T_2 , and T_0 .

Proposition 3. Under assumtions A1 and A2, for any $(z_1,z_2) \in T_i$, $i = 1,2$, there is a unique real price chosen for good i , $S_i^*(z_j) \in C$, whose value depends only on z_j , $j \neq i$.

Proof: Suppose that $(z_1,z_2) \in T_1$ and hence a change in z_1 is triggered. Since the costs of price adjustment are independent of the size of the price change, the optimal policy satisfies $S_1 \in \underset{x}{\operatorname{argmax}}\, V(x,z_2)$. Clearly, the set of maximizers depends on z_2 but is indpendent of z_1 . This reflects our Markovian assumptions, whereby profits depend on current prices only.

Now assume that there are two maximizers S_1 and S_1' . Without loss of generality, let $S_1 > S_1'$. By equation (32),

(57) $$V(S_1,z_2) = V(S_1',z_2) = V(z_1,z_2) + \beta > V^*-\beta .$$

Clearly, S_1 and S_1' are in the interior of C , for otherwise an additional price change is optimal contradicting (57). We can thus differentiate (51), using the envelope conditions, to obtain

(58) $$V_2(S_1,z_2) = V_2(S_1',z_2) .$$

In addition, since S_1 and S_1' are maximizers,

(59) $$V_1(S_1,z_2) = V_1(S_1',z_2) = 0 .$$

Applying the valuation formula (10), we obtain from (57), (58) and (59)

(60) $$F(S_1,z_2) = F(S_1',z_2) .$$

Then, by the strict quasi-concavity of F , $F_1(S',z_2) > 0$, contrary to Lemma 2 ||.

To complete the characterization of the optimal policy we need to specify the optimal choices triggered by $(z_1,z_2) \in T_0$. In view of the symmetry imposed by our assumptions, it is clear that if (S_1,S_2) is an optimal choice, so is (S_2,S_1) . Hence, in general, uniqueness cannot be expected. However, we can prove the following:

<u>Proposition 4</u>. The Synchronized steady state is unique. Under A1 and A2, the symmetric staggered steady state is also unique.

<u>Proof</u>: Appendix A.

Proposition 4 proves that the synchronized and (under certain conditions) the symmetric staggered steady states are both unique. However, we shall now show that only the synchronized steady state can be chosen following a simultaneous price change

<u>Proposition 5</u>. A staggered steady state will never be attained immediately following a joint price change. Furthermore, under assumptions A1 and A2, the only steady state following a joint price change is a synchronized steady state.

Proof: Any non-synchronized steady-state satisfies the following necessary conditions:

$$(61) \qquad V_1(S_1, S_2-gt_1) = \int_0^{t_2} e^{-rx}F_1(S_1-gx, S_2-gt_1-gx)dx + $$

$$+ \int_{t_2}^{t_2+t_1} e^{-rx}F_1(S_1-gx, S_2-g(x-t_2))dx = 0$$

and

$$(62) \qquad V_2(S_1-gt_2, S_2) = \int_0^{t_1} e^{-rx}F_2(S_1-gt_2-gx, S_2-gx)dx + $$

$$+ \int_{t_1}^{t_1+t_2} e^{-rx}F_2(S_1-g(x-t_1), S_2-gx)dx = 0$$

where t_1 and t_2 denote the time between alternating price changes (if the price of, say, good one is changed more than once, then t_2 is the time elapsed from the <u>last</u> price change of the first good).

If a non-synchronized steady state is triggered, then the choice at time $t = 0$ must be either (S_1, S_2-gt_1) or (S_1-gt_2, S_2) .

Suppose the choice is (S_1, S_2-gt_1) . Since this choice must be optimal at time $t = 0$, we must have in addition to (61) that

$$(63) \qquad V_2(S_1, S_2-gt_1) = \int_0^{t_2} e^{-rx}F_2(S_1-gx, S_2-gt_1-gx)dx = 0 .$$

Conditions (62) and (63) imply that

$$(64) \qquad \int_0^{t_1} e^{-rx}F_2(S_1-gt_2-gx, S_2-gx)dx = 0 .$$

Under A2, (64) implies that $F_2(S_1-g(t_1+t_2), S_2-gt_1) > 0$. It follows, under A1, that $F_2(S_1,S_2-gt_1) > 0$. Furthermore, applying A2 again, F_2 remains positive for all x , $t_1 \leqslant x \leqslant t_1+t_2$. Hence, the second integral in (62) cannot be equal to zero, contrary to (63).

For staggered symmetric steady states, $S_1 = S_2$ and $t_2 = t_1 > 0$. Condition (63) implies that the second integral in (61) is equal to zero. Hence,

(65) $$\int_0^{t_2} e^{-rx} F_1(S_1-gx, S_2-gt_1-gx)dx = 0 .$$

Comparing (63) and (65), and using Lemma 1, we see that both can hold if and only if $S_1 = S_2 - gt_1$, which contradicts the characterization $S_1 = S_2$ and $t_1 = t_2 > 0$ ||.

Proposition 4 excludes an immediate switch to a non-synchronized steady-state. Thus, for a broad class of profit functions, a staggered steady-state can be attained only as a limit of a non-stationary path. In contrast, a switch to a synchronized steady-state can occur immediately. A rather puzzling aspect of the model is that when the initial conditions are subject to choice, the outcome is not necessarily a steady-state.

Since a steady-state is, in general, not attained immediately, it remains to analyse the conditions for convergence to synchronized and to staggered steady states, respectively. This requires a detailed analysis of the boundaries between the continuation and each of the trigger sets.

7. Stability Analysis

To examine the behavior of the price path out of steady state we shall use a special approximation method. To motivate our approach, let us consider again the single good case (Sheshinski-Weiss [1977]).

In the continuation set, this model is characterized by the valuation equation

$$rV(z) + gV'(z) = F(z) \ . \tag{66}$$

Since $V(\cdot)$ is a continuous function and constant outside the continuation set, we can characterize it completely by finding a solution of the differential equation (66) and the appropriate boundary condition. The general solution to (66) is of the form

$$V(z) = \underset{t}{\mathrm{Max}}[\int_0^t e^{-rx}F(z-gx)dx + qe^{-rt}] \ , \tag{67}$$

where q is a constant to be determined. In particular, if $V(z)$ solves (66) then $U(z) = V(z) + qe^{-\frac{r}{g}z}$ is also a solution. Thus, to fully characterize the solution we need to determine the constant q. This constant is easily determined from the steady-state solution. Specifically,

$$V(S) = \frac{1}{1 - e^{-r\varepsilon}} [\int^{\varepsilon} e^{-rx}F(S-gx)dx - \beta e^{-r\varepsilon}] \ , \tag{68}$$

$$F(S) = - F(S-g\varepsilon) - r\beta = 0 \ , \tag{69}$$

and

(70) $$\int_0^{\varepsilon} e^{-rx} F'(S-gx)dx = 0 \ .$$

Equations (69) and (70) determine S and ε which, in turn, by (68), determine $V(S)$. since $S \in C$, we can also use (67) to evaluate $V(S)$. Equating the two equations we obtain

(71) $$q = V(S) - \beta \ .$$

Let us now consider the two-goods case. Recall the valuation formula (10),

(10') $$gV_1(z_1,z_2) + gV_2(z_1,z_2) + rV(z_1,z_2) = F(z_1,z_2)$$

which holds for any (z_1,z_2) C . Let $V(z_1,z_2)$ be a solution to (10'). Then $U(z_1,z_2) = V(z_1,z_2) + e^{-\frac{r}{2g}(z_1+z_2)}\psi(z_2-z_1)$, for $\psi(\cdot)$ some symmetric function, $\psi(0) = 0$, is also a solution. Hence, if we can find a <u>specific</u> solution to (10') and the function $\psi(\cdot)$, then $V(z_1,z_2)$ is fully determined. A specific solution to (10') which is always available is

(72) $$V(z_1,z_2) = \underset{t}{\text{Max}}[\int_0^t e^{-rx}F(z_1-gx,\ z_2-gx)dx + qe^{-rt}] \ ,$$

where q is determined from the <u>synchronized</u> steady-state solution, in analogous fashion to the one-good case.

The function $\psi(\cdot)$ is not so easily deetermined. We therefore use a second-order Taylor approximation for $\psi(\cdot)$:

(73) $$\psi(z_2-z_1) = \psi(0) + \psi'(0)(z_2-z_1) + \frac{1}{2}\psi''(0)(z_2-z_1)^2 .$$

The constraints on $\psi(\cdot)$ imply that $\psi(0) = 0$ and $\psi'(0) = 0$ (symmetry). Hence, we only need to find one additional parameter, $\psi''(0)$. This parameter can be found from the staggered steady state.

To illustrate the application of this method, we shall provide an analysis of the dynamics of the optimal path for a class of profit functions which can be represented by a weighted sum:

(74) $$F(z_1,z_2) = p(z_2-z_1)f(z_1) + (1-p(z_2-z_1))f(z_2)$$

for strictly quasi-concave functions $f(\cdot)$ and a symmetric distribution function $p(z_2-z_1)$, where $0 < p(z_2-z_1) \leq 1$, $p'(z_2-z_1) > 0$ and $p(z_2-z_1) = 1 - p(z_1-z_2)$. The profits of the monopolist can be viewed as generated by a distribution of customers between two distinct stores. This distribution depends on the ratio of prices charged in these stores. This class includes the example in Section 4, the additive case, $p(z_2-z_1) = \frac{1}{2}$ and the single-good case, $p(z_2-z_1) = 1$ (Sheshinski and Weiss [1977]) as special cases.

8. Weighted-Average Profit Functions

Assume that the profit function has the form (74) and that it is strictly quasi-concave (note that $f(\cdot)$ strictly quasi-concave is necessary but not sufficient). It is easy to verify that for this class of profit functions, the specific solution given by (72) inherits the weighted average property. That is,

(75) $$V(z_1,z_2) = p(z_2-z_1)U(z_1) + (1-p(z_2-z_1))U(z_2)$$

for some function $U(\cdot)$ which satisfies condition (66). Thus, the function $U(\cdot)$ can be interpreted as the value function associated with a one-good optimization problem.

To illustrate, let us first consider the additive case $p(z_2-z_1) = \frac{1}{2}$.[2] Define $U(z)$,

(76) $$U(z) = \max_{t \geq 0}\left[\int_0^t e^{-rx} f(z-gx)dx + qe^{-rt}\right] ,$$

where q, $q > 0$, is a constant to be determined. As already noted, it is easy to verify that (76) solves (66). To find q, consider the synchronized steady state characterized by the following conditions:

(77) $$U(S) = \frac{1}{1 - e^{-r\varepsilon}}\left[\int_0^\varepsilon e^{-rx} f(S-gx)dx - 2\beta e^{-r\varepsilon}\right] ,$$

(78) $$U'(S) = \int_0^\varepsilon e^{-rx} f'(S-gx)dx = 0 , \qquad i = 1,2$$

and

(79) $$f(S) - 2r\beta = f(S-g\varepsilon) .$$

Equations (78) and (79) determine S and ε uniquely. Thus, (77) implies a unique value for $U(S)$ and, in turn, by (76), the value of q,

(80) $$q = \frac{1}{1 - e^{-r\varepsilon}} \left[\int_0^{\varepsilon} e^{-rx} f(S-gx)dx - \beta\right] .$$

To complete the characterization we need to find $\psi''(0)$. Consider the symmetric staggered steady-state conditions for this case:

(81) $$\frac{1}{2} U(S) + \frac{1}{2} U(S-gt) + \psi(-gt) =$$

$$= \frac{1}{1 - e^{rt}} \left[\frac{1}{2} \int_0^t e^{-rx} f(S-gx)dx + \frac{1}{2} \int_t^t e^{-rx} f(S-gt-gx)dx - \beta e^{-rt}\right] ,$$

(82) $$\frac{1}{2} \int_0^t e^{-rx} f'(S-gx)dx + \frac{1}{2} \int_t^{2t} e^{-rx} f(S-gx)dx = 0$$

and

(83) $$\frac{1}{2} f(S) + \frac{1}{2} f(S-gt) - r\beta = \frac{1}{2} f(S-2gt) + \frac{1}{2} f(S-gt) .$$

Using definition (76) to eliminate $U(S-gt)$ in (81), we see that the system (77)-(79) is identical with the system (81)-(83), for $t = \frac{\varepsilon}{2}$ and $\psi(-gt) = 0$. We have thus found that $\psi''(0) = 0$. Indeed, it can be easily shown that $\psi(\cdot) = 0$ everywhere is the exact solution for the additive case. Not surprisingly, the additive case yields two independent price policies for the two goods, each being of the (S,s) type.

The four sets C, T_0, T_1 and T_2 are described in Figure 4. The regions designated by T_0 trigger a simultaneous change in both prices to

(S,S) . Following this change, the optimal policy is fully repetitive (a synchronized steady state). Both prices are allowed to erode to (s,s) and then are again changed simultaneously to (S,S) . The regions designated by T_1 trigger a change in the price of the first good only. At point g , for example, z_1 is reduced instantaneously to S . Following this change, the price path is e b c d e , which is also fully repetitive (a non-synchronized steady state). The regions designated by T_2 have similar implications. Finally, if the initial point is in the continuation set C , e.g. point a , then the nominal prices will remain unchanged for awhile, until point b is reached, at which time the price of the second good is

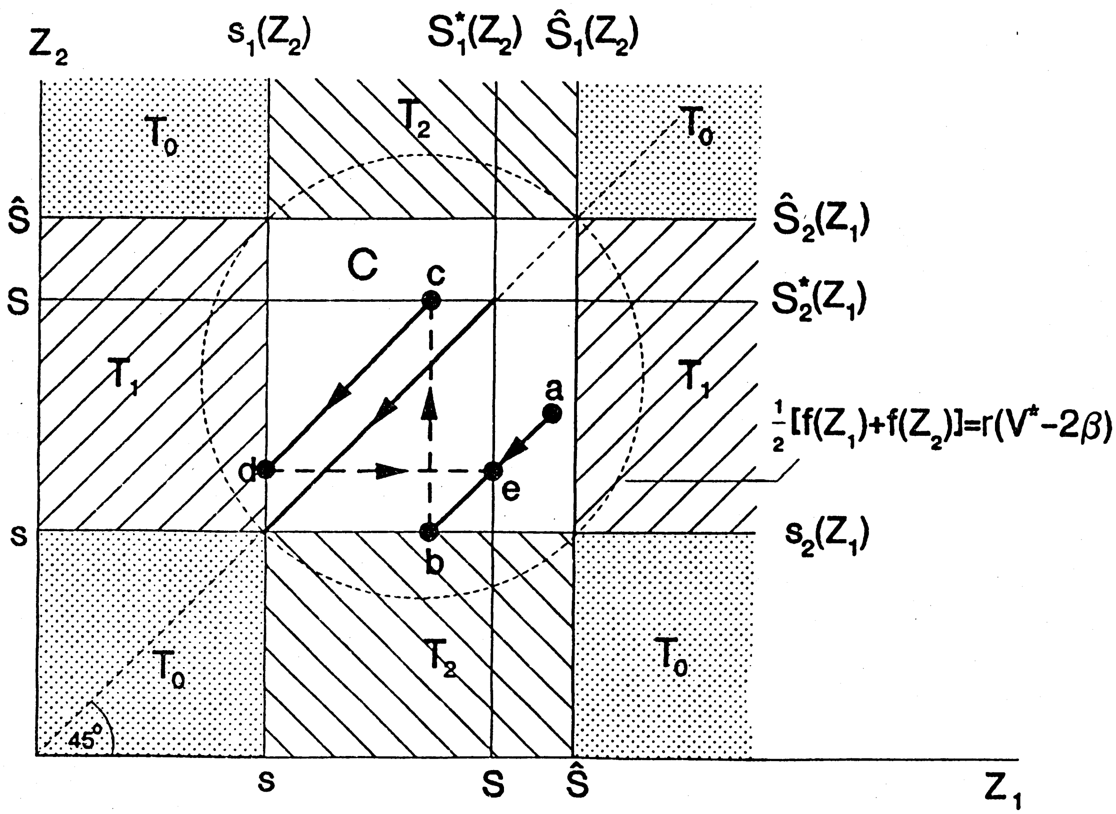

Figure 4

changed to S . Following this change the price path is again in a non-synchronized steady state.

The main features of the additive case can now be summarized: (1) The choice functions $S_i^*(z_j)$, $j \neq i$ $i = 1,2$, are both constants, reflecting the lack of interaction; (2) the optimal policy is to jump to a steady state immediately after the first price change. This reflects the irrelevance of history following any price change; (3) there is a unique synchronized steady state and a multiplicity of non-synchronized steady states, one of which is a symmetric staggered steady state.

While each steady state can be optimal for some initial conditions, they can nevertheless be ranked by their present value.

Proposition 5. Under additive profits, the value of each steady state increases as the difference between the time of price adjustment of different goods decreases.

Proof: For any steady state, let us define

$$(84) \qquad V(S, S-gt) = \frac{1}{1 - e^{-r\varepsilon}} \Big[\frac{1}{2} \int_0^{\varepsilon} e^{-rx} f(S-gx)dx + \\ + \frac{1}{2} \Big(\int_0^{\varepsilon-t} e^{-rx} f(S-gt-gx)dx + \frac{1}{2} \int_{\varepsilon-t}^{\varepsilon} e^{-rx} f(S-g(x-\varepsilon+t)dx\Big) \\ - \beta e^{-r\varepsilon} - \beta e^{-r(\varepsilon-t)}\Big] ,$$

where $\varepsilon = \frac{1}{g}(S-s)$ is the time span between consecutive changes in the price of each good and t is the time span between price changes of the first and the second goods.

Steady states vary only in t, where $t = 0$ indicates the synchronized steady state.

Differentiating (75) w.r.t. t, using (78) and (79),

(85)
$$\frac{dV}{dt} = \frac{-g}{2(1-e^{-r\varepsilon})} [\int_0^{\varepsilon-t} e^{-rx} f'(S-gt-gx)dx + \int_{\varepsilon-t}^{\varepsilon} e^{-rx} f'(S-g(x-\varepsilon+t)dx =$$

$$= \frac{-g}{2(1-e^{-r\varepsilon})} e^{rt}(1-e^{-r\varepsilon}) \int_t^{\varepsilon} e^{-rx} f'(S-gx)dx .$$

Since $f(\cdot)$ is strictly quasi-concave and (78) holds,

(86)
$$\int_t^{\varepsilon} e^{-rx} f'(S-gx)dx > 0 \quad \text{for all} \quad 0 < t < \varepsilon .$$

It follows that $\frac{dV}{dt} < 0$, for all $0 < t < \varepsilon$, implying a strict ranking of the steady states with a maximum at the synchronized steady state, $t = 0$ ‖.

Since the additive case is essentially the same as the one-good case, a more meaningful application of our method is to the case where there is positive interaction between the two prices. To this end, we set

(87)
$$f(z) = \begin{cases} ae^z & \text{if } 0 \leq z \leq b \\ 2ae^b - ae^z & \text{if } z > b \end{cases} ,$$

and

(88)
$$p(x) = \frac{e^x}{1 + e^x} .$$

The specification of $f(z)$ assumes a 'triangular' profit function, which is widely used in the related literature on optimal inventory policy (see, for example, Sulem [1986] and Constantinides and Richard [1978]). The specification of $p(\cdot)$ corresponds to a logistic probability distribution.

The value function, $V(z_1,z_2)$, for any $(z_1,z_2) \in C$, is in this case:

$$
(89) \quad V(z_1,z_2) = \begin{cases}
p\left(\dfrac{ae^{z_1}}{r+g} + q_1 e^{-\frac{r}{g}z_1} \right) + (1-p)\left(\dfrac{ae^{z_2}}{r+g} + q_1 e^{-\frac{r}{g}z_2} \right) + \\
\qquad + e^{-\frac{r}{2g}(z_1+z_2)} \psi(z_1,z_2), \quad \text{if } z_1,z_2 \leq b \\[2ex]
p\left(\dfrac{2ae^{b}}{r} - \dfrac{ae^{z_1}}{r+g} + q_2 e^{-\frac{r}{g}z_1} \right) + (1-p)\left(\dfrac{2ae^{b}}{r} - \dfrac{ae^{z_2}}{r+g} + q_2 e^{-\frac{r}{g}z_2} \right) \\
\qquad + e^{-\frac{r}{2g}(z_1+z_2)} \psi(z_2-z_1), \quad \text{if } z_1,z_2 > b \\[2ex]
p\left(\dfrac{ae^{z_1}}{r+g} + q_1 e^{-\frac{r}{g}z_1} \right) + (1-p)\left(\dfrac{2ae^{b}}{r} - \dfrac{ae^{z_2}}{r+g} + q_2 e^{-\frac{r}{g}z_2} \right) \\
\qquad + e^{-\frac{r}{2g}(z_1+z_2)} \psi(z_1,z_2), \quad \text{if } z_1 \leq b \text{ and } z_2 > b \\[2ex]
p\left(\dfrac{2ae^{b}}{r} - \dfrac{ae^{z_1}}{r+g} + q_2 e^{-\frac{r}{g}z_1} \right) + (1-p)\left(\dfrac{ae^{z_2}}{r+g} + q_1 e^{-\frac{r}{g}z_2} \right) + \\
\qquad + e^{-\frac{r}{2g}(z_1+z_2)} \psi(z_2-z_1), \quad \text{if } z_1 > b \text{ and } z_2 \leq b
\end{cases}
$$

where q_1 and q_2 are constants.

The continuity of $V(z_1,z_2)$ implies that

$$q_1 - q_2 = \frac{2ag}{(r+g)r} e^{\frac{b(r+g)}{g}} . \quad (90)$$

Both constants can be determined from the synchronized steady state. Approximating $\psi(\cdot)$ by $\frac{\psi''(0)}{2}(z_2-z_1)^2$, the coefficient $\frac{\psi''(0)}{2}$ can be found from the staggered steady state. Assuming known values for these coefficients, we can now describe the trigger sets that determine the optimal path (Figure 5).

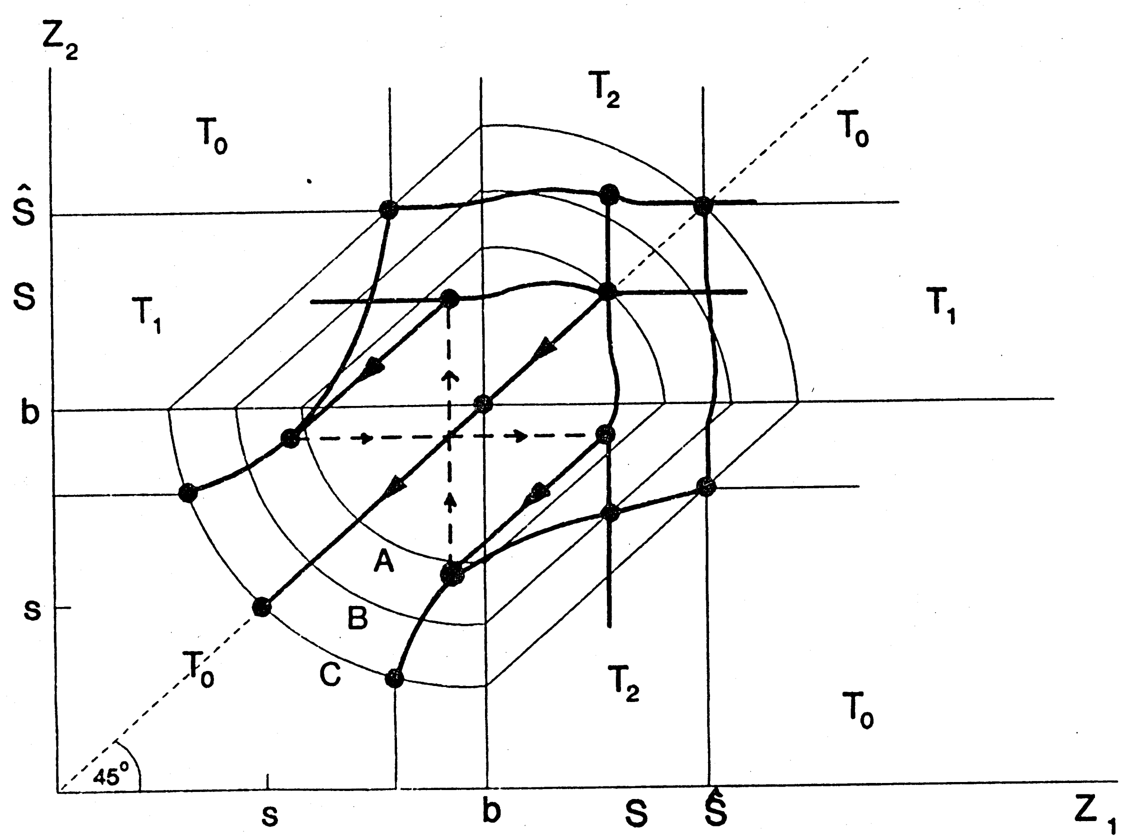

Figure 5

A: $F(z_1,z_2) = rV^*$; B: $F(z_1,z_2) = r(V^*-\beta)$; C: $F(z_1,z_2) = r(V^*-2\beta)$

The main difference of this from the additive case is that the range of initial conditions which leads to synchrcnization is enlarged. This is a reflection of the positive interaction imposed by the distribution function $p(\cdot)$. As z_2 increases, the profits generated by the first 'store' always increase for a given z_1 . Furthermore, marginal profits also increase whenever they are positive (and vice-versa). This creates an inducement for synchronization. It also appears from Figure 5 that the symmetric staggered steady state is the only steady state (except the syncrhonized) and that it is ;unstable. however, we did not investigate this issue in detail. Another important difference from the additive case is that, in general, a steady state is not attained after the first price change. With positive interaction, unless both prices are changed simultaneously, history always remains relevant. Thus, the impact of initial conditions is propagated over time.

9. Some Remarks on the Duopoly Problem

The analysis so far focused on the case of a single decision maker. One may also regard this problem as arising from cooperation among different firms, each controlling it's own nominal price. An analysis of the non-cooperative duopoly solution is the subject of on-going research, which will be reported separately. However, we wish to make some remarks on the difference between the duopoly and the monopoly cases.

In contrast to the monopoly case, each duopolist controls only his own nominal price. The concept of equilibrium depends on the type of interaction perceived by each decision maker. In the context it is common to distinguish two types of equilibrium: open-loop and closed-loop equilibrium. In the open-loop equilibrium, the optimal price paths viewed as functions of time, are mutually consistent. In the closed-loop equilibrium, the optimal strategies, viewed as functions of current real prices, are mutually consistent.

In some special cases, however, this distinction desappears. In particular, with additive profits, the open and closed loops solutions coincide. However, even in this case, they may both differ from the cooperative (or monopoly) solution. This situation occurs if the price of one firm affects the level of its rival's profits, but not the marginal profits. The monopoly always internalizes this external effect, but none of the duopolists takes it into account.

To illustrate this observation, let us write the profits of each duopolist, $F^i(z_1,z_2)$ in the additive form

(91) $$F^i(z_1,z_2) = f(z_i) + g(z_j) , \quad j \neq i , \quad i,j = 1,2 .$$

We shall say that the profits of the two firms are independent if $g(\cdot)$ = constant. Otherwise, each firm imposes an 'externality' on the other. However, due to additivity, this has no effect on any firm's actions. The optimal policy is thus the same as in the single good case, i.e., an (S,s) policy characterized by:

(92) $$f(S) - f(S-g\varepsilon) - r\beta = 0 ,$$

and

(93) $$\int_0^{\varepsilon} e^{-rx} f'(S-gx)dx = 0 .$$

In the decision costs case, the monopoly satisfies the same conditions, where the function $f(z)$ is replaced by $f(z) + g(z)$. It is easy to show that if $g'(z) > 0$, i.e., the two goods are substitutes, then the monopoly will select a higher initial price, S. However, the frequency of price changes, ε, under monopoly may be higher or lower than in the duopoly case (see Rotemberg and Saloner [1986]).

With non-additive profits, it is still easy to compare the open-loop with the monopoly solution, and the same prinicple applies. That is, the monopoly internalizes all interactions. Thus, with positive interaction, $F_{12} > 0$, we would expect a stronger tendency towards synchronization by the monopoly.

Another important distinction between the monopoly and duopoly solutions is the potential for increasing returns in the price adjustment costs. The monopoly may coordinate the timing of price changes and inform consumers jointly, thereby saving adjustment costs. As we have seen in Section 1, in the menu costs case, where the costs of price adjustment are independent of the number of items in the price list, the monopoly will always follow a synchronized price path. With this extreme assumption, staggering may arise only under duopoly.

Appendix A

Proof of Proposition 4:

The proof that the synchronized steady state is unique is the same as in the case of menu costs.

To prove the uniqueness of the symmetric staggered steady state, consider the point $(S,\hat{z})$ C , where the price of the first good has just been changed, and let t_2 be the timing of the subsequent price change. The F.O.C. satisfied at that point are:

(A.1) $$V_2(S-gt_2,S_2) = 0$$

and

A.2. $$F(S-gt_2, \hat{z}-gt_2) - r(V(S-gt_2, S_2)-\beta) - gV_1(S-gt_2, S_2) = 0 .$$

At a symmetric steady state, $S_2 = S$, $t_2 = t$ and $\hat{z} = S - 2gt$. Evaluating the second-order conditions at this point, we have the requirement that the matrix A,

(A.3) $$A = \begin{bmatrix} V_{22}(S-gt, S) & -gV_{12}(S-gt, S) \\ -gV_{12}(S-gt, S) & -g(F_1(S-gt, S-2gt) + F_2(S-gt, S-2gt) + \\ & + rV_1(S-gt, S) + gV_1(S-gt, S)) \end{bmatrix}$$

be negative definite. To prove the uniqueness of the symmetric staggered steady state, consider the system

$$V_2(S-gt, S) = 0 \tag{A.4}$$

and

$$F(S-gt, S-tgt) - r(V(S-gt, S)-\beta) - gV_1(S-gt, S) = 0 \tag{A.5}$$

as two equations in the unknowns S and t (in contrast to (A.1)-(A.2), where S_2 and t_2 are undnons and $\hat{z}$ and S were given).

To prove uniqueness, we shall show that the Jacobian, B,

$$B = \begin{bmatrix} V_{12}(S-gt, S) + & -gV_{12}(S-gt, S) \\ + V_2(S-gt, S) & \\ & \\ F_1(S-gt, S-2gt)+ & -g[F_1(S-gt, S-2gt) + 2F_2(S-gt, S-2gt) + \\ + F_2(S-gt, S-2gt) - & + rV_1(S-gt, S) + gV_{11}(S-gt, S-2gt)] \\ - rV_1(S-gt, S) + & \\ + gV_1(S-gt, S) & \end{bmatrix} \tag{A.6}$$

is negative definite. The first diagonal term is, under A1 and A2,

$$b_{11} = V_{12} + V_{22} = F_2(S-gt, S) = 0 . \tag{A.7}$$

The other diagonal term, b_{22}, is equal to the lower diagonal term in (A.3), a_{22} minus $gF_2(S-gt, S-2gt)$. By A2, F_2 just prior to a price change has to be positive. Thus, the whole term is negative. The determinant condition can now be written in the form

$$b_{11}(-2gF_2(S-gt, S-2gt)) + a_{22} V_{22} > 0 \tag{A.8}$$ ||.

Footnotes

[1] This is the case, for instance, if the demand for good i, $i = 1,2$, is given by $e^{z_j} - ae^{z_i+2z_j}$, $j \neq i$.

[2] Example: let demand for the i-th good, q_i, be: $q_i = 1 + e^{z_j - z_i} - e^{2z_i}$, $j \neq i$, $i = 1,2$. With zero costs, $F^i = e^{z_i} + e^{z_j} - e^{2z_i}$.

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