TEL-AVIV

WP 12-88

THE FOERDER INSTITUTE FOR ECONOMIC RESEARCH

TEL-AVIV UNIVERSITY

RAMAT AVIV ISRAEL

מכון למחקר כלכלי ע״ש ד״ר ישעיהו פורדר ז״ל

ע״י אוניברסיטת תל־אביב

Forthcoming in Economics Letters

FIXED RULES AND DECISION RULES:
TIME CONSISTENCY AND SUBGAME PERFECTION

by

Chaim Fershtman

Working Paper No.12-88

March, 1 9 8 8

I would like to thank Avinash Dixit for valuable comments and Ariel Rubinstein for many discussions on dynamic games

This research was supported by funds granted to the Foerder Institute for Economic Research by KEREN NUR MOSHE

FOERDER INSTITUTE FOR ECONOMIC RESEARCH
Faculty of Social Sciences
Tel-Aviv University, Ramat Aviv, Israel.

FIXED RULES AND DECISION RULES;
TIME CONSISTENCY AND SUBGAME PERFECTION

by
Chaim Fershtman

ABSTRACT

The paper investigates the relationship between time consistency and subgame perfection. We show that despite some suggestions in the literature, the two are not equivalent. Subgame perfection is a stronger refinement. The paper also discusses the classes of games in which time consistency and subgame perfection are equivalent.

Journal of Economic Literature Classification Nos.026, 113, 321.

Forthcoming in Economics Letters

FIXED RULES AND DECISION RULES: TIME CONSISTENCY AND SUBGAME PERFECTION

by

Chaim Fershtman

Working Paper No.12-88

March, 1 9 8 8

I would like to thank Avinash Dixit for valuable comments and Ariel Rubinstein for many discussions on dynamic games

This research was supported by funds granted to the Foerder Institute for Economic Research by KEREN NUR MOSHE

FOERDER INSTITUTE FOR ECONOMIC RESEARCH
Faculty of Social Sciences
Tel-Aviv University, Ramat Aviv, Israel.

FIXED RULES AND DECISION RULES;
TIME CONSISTENCY AND SUBGAME PERFECTION

by
Chaim Fershtman

ABSTRACT

The paper investigates the relationship between time consistency and subgame perfection. We show that despite some suggestions in the literature, the two are not equivalent. Subgame perfection is a stronger refinement. The paper also discusses the classes of games in which time consistency and subgame perfection are equivalent.

Journal of Economic Literature Classification Nos.026, 113, 321.

Fixed Rules and Decision Rules: Time Consistency and Subgame Perfection

1. Introduction

Since Simaan and Cruz (1973a,b), Kydland and Prescott (1977), and Calvo (1978), it has been recognized that government's optimal policy might be time inconsistent, i.e. the policy which is optimal at the outset ceases to be so at some later period, t, which implies that there are incentives for the policy maker to change his original plan.

In order to obtain time consistent policy, the standard procedure in the literature (see, for example, Kydland and Prescott (1977) and Cohen and Michel (1984)), is to use a backward induction technique to calculate an optimal feedback decision rule which is by construction subgame perfect. Indeed intuitively, the notion of time consistency is closely related to subgame perfection. However, despite some suggestions in the literature, the two are not equivalent. Subgame perfection is a stronger refinement. Thus, solving the time inconsistency problem by applying a technique that yields subgame perfection might be too restrictive. In this note we investigate the relationship between time consistency and subgame perfection and discuss the classes of games in which the two are equivalent.

2. Consistency and Subgame Perfection

The problem of optimal policy can be described as a two-level hierarchial dynamic game. Let (i) S^1 be the leader's set of admissible actions and S^2 the follower's set of admissible actions; (ii) U^1 and U^2 are the sets of all possible sequences of actions chosen by the leader and the follower respectively. A typical element in U^1 is $u_1 = (s_1^1, s_2^1, \ldots,)$ where $s_t^1 \in S^1$; (iii) For every $u_i \in U^i$ let $u_i(\tau)$ be the continuation of u_i from period τ on; (iv) $x(t) \in R^m$ is the value of the state variables at time t, $x(t+1) = f(x(t), s_t^1, s_t^2)$ is the law of motion of these state variables and $\tilde{X}$ is the set of all admissible sequences of state variables.

For every possible initial condition $y = (x(t_0), t_0)$ let J_y^i be a real value functional, J_y^i: $U^1 \times U^2 \to R$ such that $J_y^i(u_1, u_2)$ is the payoff to player i when the players choose the sesquences (u_1, u_2) and the game starts at y.

Definition 1: A **fixed rule** or an **open loop** strategy is a sequence of actions $u_i \in U^i$ chosen by the i'th player at the outset of the game.

Definition 2: A function B_y^i: $U^j \to P(U^i)$ is defined as a **best response set** (or reaction set) such that if $\hat{u}_j \in U^j$

$$B_y^i(\hat{u}_j) = \{\hat{u}_i \in U^i \mid J_y^i(\hat{u}_i, \hat{u}_j) \geq J_y^i(u_i, \hat{u}_j) \text{ for all } u_i \in U^i\}.$$

For convenience we assume that for every $u_j \in U^j$, B_y^i is a singleton so that B_y^i is a mapping from U^j to U^i.

Definition 3: (u_1^*, u_2^*) is an open loop Stackelberg equilibrium for the game that starts at $y = (x_0, t_0)$ if (i) $u_1^* \in \underset{u_1 \in U^1}{\text{Argmax}}\, J_y^1(u_1, B_y^2(u_1))$ and (ii) $u_2^* = B_y^2(u_1^*)$.

The problem of optimal (fixed rule) policy in a rational expectations model is structually identical to the problem of finding an open loop Stackelberg equilibrium in a two-level hierarchic game. In discussing such a problem, Kydland and Prescott (1977) pointed out that an optimal open loop policy in subsequent periods is not the continuation of the optimal open loop plan formulated at the outset. They denoted this phenomenon as time inconsistency.

Definition 4: An open loop Stackelberg equilibrium (u_1^*, u_2^*) is time consistent if for every intermediate period τ, $(u_1^*(\tau), u_2^*(\tau))$ constitute an open loop Stackelberg equilibrium for the game starting at $y^*(\tau) = (x^*(\tau), \tau)$ where $x^*(\tau)$ is the value of the state variable that is reached at period τ when the players play (u_1^*, u_2^*).

Since the open loop Stackelberg equilibrium is, in most cases, inconsistent, attention was shifted to Stackelberg equilibrium with feedback strategies and in particular to the subgame perfect feedback Stackelberg equilibrium. A feedback strategy (or a closed loop no-memory strategy) is a Markovian decision rule $d_i(x,t)$ that prescribes an action in S^i for every state variables and time. Let D^i be the set of all possible such rules and let $g_y : D^1 \times D^2 \rightarrow U^1 \times U^2$, such that $g_y(d_1, d_2)$ is the player's actual time sequence of actions when the game starts at the initial condition y and the two players play (d_1, d_2). Note that g_y is well defined only if there are some restrictions on the set of feedback strategies.

Definition 6: (d_1^*, d_2^*) is a Stackelberg equilibrium with decision rule strategies if $d_1^* \in D^1$, $d_2^* \in D^2$ and

$$d_1^* \in \underset{d_1 \in D^1}{\text{Argmax}}\ J_y^1(g_y(d_1, R_y^2(d_1)))$$

$$d_2^* \in R_y^2(d_1^*)$$

where $R_y^2 : D^1 \rightarrow D^2$ is the best response function of the follower which is defined similar to Definition 2 but for decision rules.

The above Stackelberg equilibrium (d_1^*, d_2^*) is **time consistent** if for every intermediate period t, (d_1^*, d_2^*) constitutes a Stackelberg equilibrium for the game that starts at $y^*(t) = (x^*(t), t)$ where $x^*(t)$ is the value of the state variables that is reached at time t when the players play (d_1^*, d_2^*). This Stackelberg quilibrium is **subgame perfect** if (d_1^*, d_2^*) constitutes a Stackelberg equilibrium for **every** possible initial condition y.

Before elaborating on the difference between time consistency and subgame perfection, the above definition of subgame perfection needs some clarification. The standard definition of subgame perfection (Selten, 1975) is that the equilibrium strategies constitute an equilibrium for every subgame of the original game where a subgame is a game that starts after a particular history has been played. In our above definition, however, we require that (d_1^*, d_2^*) will be an equilibrium for every possible initial condition $y = (x,t)$, which is a weaker requirement. In presenting this paper we had two alternatives. One, to define subgame perfection in the standard way and then to show, as a proposition, that when we restrict our attention to Markovian decision rules, our definition implies subgame perfection.[1] The alternative chosen by us was to

---

[1] In order to prove that (d_1^*, d_2^*) is a subgame perfect Stackelberg equilibrium we need to show that for every t and every possible history h_t, (d_1^*, d_2^*) is a Stackelberg equilibrium for the game that starts after h_t has been played. Every h_t uniquely defines the state variables $x(t) = \xi(h_t)$. Since the conditions of the above definition hold for $y = (\xi(h_t), t)$, $(d_1^*,$

adopt the standard procedure in differential games and to define subgame perfect equilibrium as a pair of Markovian strategies constituting an equilibrium when viewed from any possible (state, date) pair. For more details see Reinganum and Stokey (1985) and Fershtman and Kamien (1987). It is important to note that the above equivalence is correct **only** when we discuss Markovian strategies.

Comparing the above definitions we can conclude the following:

Corollary 1: Subgame perfection implies time consistency.

Corollary 2: When the optimal policy problem does not have a state space, then, if the strategies are assumed to be Markovian decision rules, a time consistent Stackelberg equilibrium is also a subgame perfect Stackelberg equilibrium. Note that this corollary does not hold if we assume a general history dependent strategies.

Corollary 2 indicates that it is the existence of state variables that imply the difference between subgame perfection and time consistency. The intuition of this conclusion is clear from the above discussion. When we consider a game with state variables and Markovian strategy space, a subgame is defined by (x,t). When the problem does not have state variables but still Markovian strategies are used, a subgame can be defined by t, the period the subgame starts. The definition of time consistency implies that for every t the equilibrium strategies induce an equilibrium for the continuation of the game and therefore implies subgame perfection for games with no state variables.

$d_2^*)$ is a feedback Stackelberg equilibrium for the game that starts at $(\xi(h_t),t)$ **regardless** of the history leading to it.

However, when there are state variables, time consistency implies that the equilibrium strategies constitute an equilibrium only for subgames (x,t) **along** the equilibrium path and not for all possible (x,t) as the definition of subgame perfection requires.

In Fershtman (1987) I define a necessary and sufficient condition that the game has to satisfy so that the open loop Nash equilibrium is also a (subgame perfect) feedback Nash equilibrium. It is straightforward to verify that for the same classes of games a consistent fixed rule (open loop) Stackelberg equilibrium is subgame perfect.

As an example, we can consider a differential game in which the players' payoff functions are exponential in the state variable, i.e. $J^i = \int_0^T \phi_i(s,t)e^{-\lambda_i x(t)} dt$ where $\lambda_i \in R^m$ and $\phi_i(s,t) \in R^1$ and the state variables evolve according to $\dot{x} = f(s,t)$. In such a game, we can use the semi-group property of the exponential function and the separability in x and s to prove that for this class of problems time consistency and subgame perfection are equivalent.

References

Calvo, G. (1978) "On the Consistency of Optimal Policy in a Monetary Economy" **Econometrica** 46, 1411-28.

Cohen, D. and P.Michel (1984), "Toward a Theory of Optimal Pre-Commitment I: An Analysis of the Time-Consistent Equilibria," CEPREMAP, Paris, Working Paper NO.8412.

Fershtman, C. (1987), "Identification of Classes of Differential Games for which the Open Loop is a Degenerated Feedback Nash Equilibrium," **Journal of Optimization Theory and Applications**, 55.

Fershtman, C. and M.I.Kamien (1987), "Dynamic Duopolistic Competition with Sticky Prices," **Econometrica**, Vol.55, 1151-1164.

Kydland, F.E. and E.C.Prescott (1977), "Rules rather than Discretion: The Inconsistency of Optimal Plans," **Journal of Political Economy** 85, 473-491.

Reinganum, J.F. and N.I.Stokey (1985), "Oligopoly Extraction of a Common Property Natural Resource: The Importance of the Period of Commitment in Dynamic Games," **International Journal of Economics**, 161-173.

Selten, R. (1975), "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," **International Journal of Game Theory** 4, 25-55.

Simaan, M. and J.B.Cruz (1973a), "On the Stackelberg Strategy in Non Zero-Sum Games," **Journal of Optimization Theory and Applications** 11, 5, 533-55.

Simaan, M. and J.B.Cruz (1973b), "Additional Aspects of the Stackelberg Strategy in Nonzero-Sum Games," **Journal of Optimization Theory and Applications** 11, 6, 613-626.