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GENERALIZED EXPECTED UTILITY ANALYSIS OF MULTIVARIATE RISK AVERSION

by

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1. Introduction

In expected utility theory preferences over risky prospects are assumed to be representable by a preference functional V such that $V(F) \equiv \int U(x)dF(x)$, where U is the decision maker's *von Neumann-Morgenstern utility function*, and F is the cumulative distribution function representing the prospect. This hypothesis, however, has been repeatedly contradicted by experimental evidence which indicates that preferences over risky prospects are in fact systematically ***nonlinear*** in the probabilities.[2] Motivated by this discrepancy between theory and evidence, Machina [1982a] presented an alternative model of preferences over *univariate* probability distributions which is consistent with some of the experimental evidence and, except for the specific property of global linearity, preserves most of the useful theoretical properties and behavioral implications of expected utility theory.[3]

In this paper we extend this approach to preferences over multivariate probability distributions and examine the robustness of results concerning multivariate risk aversion that have been obtained under the expected utility hypothesis. In particular, we are concerned with three well-known results:[4] (a) that interpersonal comparisons of attitudes toward multivariate risks require that the individuals being compared have the same *ordinal preferences*[5] over the commodity space; (b) that the properties of the individual's ordinal preferences affect the comparative statics effects of increasing risk aversion; and (c) that if the individual's ordinal preferences are homothetic then comparisons of his aversion to small multivariate risks at different wealth levels are possible. We shall find that each of these results extends when their respective assumptions are applied to what we shall term "local ordinal preferences."

2. Smooth Preferences over Multivariate Distributions

We take as our choice set the set $D_B(\mathbb{R}^n_+)$ of all multivariate cumulative distribution functions F over $\mathbb{R}^n_+ = \{(x_1,\ldots,x_n) \mid x_i \geq 0,\ i = 1,\ldots,n\}$ with bounded support, and assume that the individual's preference relation over $D_B(\mathbb{R}^n_+)$ is complete, transitive, and representable by a real-valued preference functional V. We assume that V is continuous with respect to the topology of weak convergence on $D_B(\mathbb{R}^n_+)$. As our "smoothness" condition we assume that V is continuously differentiable in the sense that at each $F \in D_B(\mathbb{R}^n_+)$ there exists a *local multivariate utility function* $U(\cdot,\cdots,\cdot;F)$, continuous in $X \equiv (x_1,\ldots,x_n)$ and F such that

$$(1)\quad V(F^*) - V(F) = \int U(X;F)[dF^*(X) - dF(X)] + o(\|F^* - F\|)$$

where $\|F^* - F\|$ is the L^1 norm $\int|F^*(x_1,\ldots,x_n) - F(x_1,\ldots,x_n)|dx_1\ldots dx_n$.[6]

As in Machina [1982a] we extend this approach to the analysis of nondifferential shifts by considering any path $\{F(\cdot,\cdots,\cdot;\alpha)|\alpha \in [0,1]\}$ in $D_B(\mathbb{R}^n_+)$ from F to F^* which is "smooth enough" so that the derivative $\frac{d}{d\alpha}\|F(\cdot,\cdots,\cdot;\alpha) - F(\cdot,\cdots,\cdot;\bar{\alpha})\|\Big|_{\alpha=\bar{\alpha}}$ exists for each $\bar{\alpha} \in [0,1]$, and noting that (1) implies

$$(2) \qquad \left.\frac{dV(F(\cdot,\cdots,\cdot;\alpha)}{d\alpha}\right|_{\bar{\alpha}} = \left.\frac{d}{d\alpha}[\int U(X;F(\cdot,\cdots,\cdot;\bar{\alpha}))dF(X;\alpha)]\right|_{\bar{\alpha}}.$$

Thus, by the fundamental theorem of integral calculus, we have,

$$(3) \qquad V(F^*) - V(F) = \int_0^1 \left[\left.\frac{d}{d\alpha}[\int U(X;F(\cdot,\cdots,\cdot;\bar{\alpha}))dF(X;\alpha)]\right|_{\bar{\alpha}}\right]d\bar{\alpha},$$

which shows how the individual's ranking of these two distributions depends upon the local utility functions along the path between them. We shall assume that all local utility functions are *strictly* increasing and differentiable (though not necessarily concave) in $(x_1,\cdots,x_n)$.

3. Interpersonal Comparisons of Risk Aversion

The main result of Kihlstrom and Mirman [1974] concerning interpersonal comparisons of risk aversion is that the partial ordering of multivariate von Neumann-Morgenstern utility functions representing *identical ordinal preferences* by concavity is equivalent to the partial ordering by their certainty equivalents for risky multivariate distributions. Essentially the same result obtains in the generalized expected utility case provided the individual's local ordinal preferences, i.e., the rankings over $\mathbb{R}^n_+$ induced by their local utility functions $U(\cdot,\cdots,\cdot;F)$, are identical for each distribution $F \in D_B(\mathbb{R}^n_+)$. Notice, however, that the ordinal preferences induced by the local utility functions do not necessarily represent the individual's preferences between certain bundles. The latter are represented by preferences over the distributions δ_X defined as $\delta_X(x_1',\cdots,x_n') = 1$ if $x_1' \geq x_i$ for all i and $\delta_X(x_1',\cdots,x_n') = 0$ otherwise.

The formal statement of this result requires the following definitions: For any F, $F^* \in D_B(\mathbb{R}^n_+)$ and $p \in (0,1]$, we define the individual's ***conditional certainty equivalent set*** of F given p and F^* by $C(F;F^*,p) = \{X \in \mathbb{R}^n_+ \mid V((1 - p)F^* + pF) = V((1 - p)F^* + p\delta_X)\}$. The set $C(F;F^*,p)$ will be said to be *at* *least as great as* $C^*(F;F^*,p)$ if for each $(x_1,\cdots,x_n) \in C^*(F;F^*,p)$ there exists an $(x_1',\cdots,x_n') \in C(F;F^*,p)$ such that $x_i' \geq x_i$ for all i. Given this, we have:

<u>**THEOREM 1:**</u> *The following conditions on a pair of smooth preference functionals* V *and* V^* *over* $D_B(\mathbb{R}_+^n)$ *with local utility functions* $U(X;F)$ *and* $U^*(X;F)$ *are equivalent:*

(i) *For each* F^*, $F \in D_B(\mathbb{R}_+^n)$ *and* $p \in (0,1]$, $C(F;F^*,p)$ *is at least as great as* $C^*(F;F^*,p)$;

(ii) *For each* $F \in D_B(\mathbb{R}_+^n)$ *there exists an increasing concave function* $T_F[\cdot]$ *such that* $U^*(\cdot,\cdots,\cdot;F) = T_F[U(\cdot,\cdots,\cdot;F)]$.

In this case we say that V is ***at least as risk averse*** as V^*. (For proofs of this theorem and subsequent results see section 6).

4. <u>Application and Comparative Statics Effects: Consumption-Saving Decisions</u>

Consider a two-period consumption model. Let (y_1,y_2) denote the decision maker's earnings in the two periods and let $\tilde{r}$ be a random variable in $[-1, \infty)$ representing the rate of interest on the first period saving, s. Let $C(s,\tilde{r}) \equiv (c_1(s),c_2(s,\tilde{r})) \equiv (y_1-s,\ y_2 + s(1+\tilde{r}))$ denote the two period consumption stream corresponding to s. Thus, $C(s,\tilde{r})$ is a random variable in $\mathbb{R}_+^2$. Let $F \in D_B(\mathbb{R}_+)$ be the distribution of $(1+\tilde{r})$ and denote by $F(s)$ the distribution of $C(s,\tilde{r})$ induced by F, i.e., $F(s)(c_1(s),c_2(s,r)) = F(r)$.

In expected utility theory the comparative statics effects of an increasing risk aversion on the level of saving are analyzed in Kihlstrom and Mirman [1974]. To extend this analysis to the generalized expected utility case we adopt the the following notation: Let ρ be an increasing index of risk aversion. Let $\{V_\rho\}$ be a family of preference functionals ordered by their risk aversion and denote by U_ρ, the local utility functions corresponding to V_ρ. Consider the problem of choosing the level of saving so as to maximize a preference functional V. Then, the comparative statics effect of an increase in risk aversion on the optimal level of saving is given by:

<u>THEOREM 2:</u> *Given* $\{V_\rho\}$, *let* $s(\rho) = \text{argmax}\ V_\rho(F(s))$. *Then* $s(\rho)$ *increases (decreases) with* ρ *if there exist values* $r^*(F(\beta(\rho)))$ *such that:* $\text{sgn}[\partial U_\rho(C(s,r);F(s(\rho)))/\partial s] = \text{sgn}[r - r^*(F(s(\rho)))]$ $(= -\text{sgn}[r - r^*(F(s(\rho)))])$ *for all* ρ *and* $s(\rho)$.

In the framework of expected utility theory this single crossing property of the marginal utility of the control variable s obtains if the optimal level of s is a monotonic function of r when r is certain. In the generalized expected utility framework the ordinal preferences induced by the local utility functions may vary. Thus, it is impossible to infer the relevant single crossing property from the behavior of an individual under certainty.

5. Decreasing Risk Aversion

In the expected utility framework the attitudes toward small risks of a given individual at different wealth levels are comparable if his ordinal preferences are homothetic. (See Kihlstrom and Mirman [1981].)

Although in our generalized expected utility framework we may drop the (linearity) assumption that the local utility functions are all identical, we still require that they induce the same homothetic ***ordinal*** preferences. In other words, we must have $U(x_1,\cdots,x_n;F) \geq U(x_1',\cdots,x_n';F)$ if and only if $U(\lambda x_1,\cdots,\lambda x_n;F^*) \geq U(\lambda x_1',\cdots,\lambda x_n';F^*)$ for all $(x_1,\cdots,x_n)$, $(x_1',\cdots,x_n')$, F, F^* and $\lambda > 0$. This is equivalent to the condition that $U(x_1,\cdots,x_n;F) \equiv h(u(x_1,\cdots,x_n);F)$ for some linear homogenous u and function h which is increasing in its first argument.

The identity of the ordinal preferences induced by each local utility function implies that any two distributions in $D_B(\mathbb{R}^n_+)$ which induce the same probability distribution over the range of u must be equally preferred. To see this, note that between any two distributions F^* and F in $D_B(\mathbb{R}^n_+)$ that induce the same probability distribution over the range of u we may construct a smooth path $\{F(\cdot,\cdots,\cdot;\alpha)|\alpha \in [0,1]\}$ consisting of distributions that also induce the same probability over the range of u so that the derivative in equation (3) above is zero for all $\bar{\alpha}$, which implies $V(F) = V(F^*)$. Defining $H_F(\cdot)$ as the cumulative distribution function of $u(\tilde{X})$ when $\tilde{X}$ has the distribution $F \in D_B(\mathbb{R}^n_+)$ and $W(H_F) \equiv V(F)$, we may apply the univariate, generalized expected utility characterization of decreasing risk aversion developed in Machina [1982b]. In particular, defining the risk premium π in terms of the level of u by $V(F_{\tilde{u}-\pi}) = V(F_{\tilde{u}+\tilde{\epsilon}})$, where $F_{\tilde{u}-\pi}$ and $F_{\tilde{u}+\tilde{\epsilon}}$ denote the cumulative distribution functions of $\tilde{u} - \pi$ and $\tilde{u} + \tilde{\epsilon}$, respectively, and following Machina [1982b, Theorem 1] we have:

__THEOREM 3:__ *Let V be a smooth preference functional with twice continuously differentiable local utility functions $U(\cdot,\cdots,\cdot;F)$ which induce the same homothetic ordinal preferences for all $F \in D_B(\mathbb{R}^n_+)$. Then the following properties are equivalent:*

(i) The term $-h_{11}(u;F)/\int h_1(\omega;F)dF(\omega)$ is everywhere nonincreasing in u and $F(\cdot)$ (i.e., $-h_{11}(u^*;F^*)/\int h_1(\omega;F^*)dF^*(\omega) \le -h_{11}(u;F)/\int h_1(\omega;F)dF(\omega)$ whenever $u^* \ge u$ and F^* equals or stochastically dominates F);

and

(ii) if $\tilde{u}$, $\Delta\tilde{u} \geq 0$, $E[\tilde{\epsilon}|u] = E[\tilde{\epsilon}|u + \Delta u] = 0$, *and* π *and* π^* *satisfy* $V(F_{\tilde{u}+\pi}) = V(F_{\tilde{u}+\tilde{\epsilon}})$ *and* $V(F_{\tilde{u}+\Delta\tilde{u}-\pi^*}) = V(F_{\tilde{u} + \Delta\tilde{u} + \tilde{\pi}})$, *then* $\pi \geq \pi^*$.

The proof is as that of Theorem 1 in Machina [1982b]. In the case where $\tilde{u}$ and $\Delta\tilde{u}$ are degenerate, Theorem 3 may be regarded as a statement of equivalent definitions of decreasing risk aversion analogous to those of Kihlstrom and Mirman [1981].

6. Proofs

Proof of Theorem 1: (i) => (ii): Given $F \in D_B(\mathbb{R}^n_+)$ suppose there were no increasing $T_F[\cdot]$ for which $U^*(\cdot,\cdots,\cdot;F) = T_F[U(\cdot,\cdots,\cdot;F)]$, so that $U(X';F) < U(X;F)$ and $U^*(X';F) > U^*(X;F)$ for some X, and X'. From equation (2) it follows that $V((1 - 2p)F + p\delta_X + p\delta_{X'}) < V((1 - 2p)F + 2p\delta_X)$ and $V^*((1 - 2p)F + p\delta_X + p\delta_{X'}) > V^*((1 - 2p)F + 2p\delta_X)$ for some small positive p. This implies that no element of $C(0.5\delta_X + 0.5\delta_{X'};F,2p)$ strictly vector dominates X, but that some element of $C^*(0.5\delta_X + 0.5\delta_{X'};F,2p)$ does strictly dominate X, contradicting (i).

Now suppose that the function $T_F[\cdot]$ is not concave, so that $[U^*(X'';F)-U^*(X';F)]/[U^*(X';F)-U^*(X;F)]>1>[U(X'';F)-U(X';F)]/[U(X';F)-U(X;F)]$ for some X, X', and X" for which $U(X'';F) > U(X';F) > U(X;F)$. From (2) we have that $V^*((1 - 2p)F + p\delta_X + p\delta_{X''}) > V^*((1 - 2p)F + 2p\delta_{X'})$ and $V((1 - 2p)F + p\delta_X + p\delta_{X''}) < V((1 - 2p)F + 2p\delta_{X'})$ for some small positive p. This implies that there exists some element of $C^*(0.5\delta_X + 0.5\delta_{X''};F,2p)$ which dominates X', but that no element of $C(0.5\delta_X + 0.5\delta_{X''};F,2p)$ dominates X', contradicting (i).

(ii) => (i): Let F^*, $F \in D_B(\mathbb{R}^n_+)$ and $p \in (0,1]$ be given. Consider first the case in which there exist $\bar{X} \in C(F;F^*,p)$ such that $\bar{X} < X'$ for some $X' \in$ Supp F, where $\bar{X} < X'$ means that $\bar{x}_i \leq x'_i$ for all i with strict inequality for some i. Let $A_h = \{X \in \mathbb{R}^n_+ \mid X \geqq \bar{X}\}$ and let A_ℓ be the complement of A_h in $\mathbb{R}^n_+$. Define $\phi_h : \mathbb{R}^n_+ \to [0,1]$ as $\phi_h(X) = F(X) - \delta_{\bar{X}}(X) = F(X) - 1$ for X in A_h and $\phi_h(X) = 0$ otherwise, and $\phi_\ell : \mathbb{R}^n_+ \to [0,1]$ as $\phi_\ell(X) = F(X) - \delta_{\bar{X}}(X) = F(X)$ for X in A_ℓ and $\phi_\ell(X) = 0$ otherwise. Let $F(X;\alpha,\beta) = \delta_{\bar{X}}(X) + \alpha\phi_h(X) + \beta\phi_\ell(X)$. For $\alpha \in [0,1]$ let $\beta(\alpha)$ be defined by the equation:

(4) $V((1 - p)F^* + pF(.;\alpha,\beta(\alpha))) = V((1 - p)F^* + pF) = V((1 - p)F^* + p\delta_{\bar{X}})$.

The existence of β follows from the assumption that $A_\ell \cap \text{Supp } F \neq \phi$ and $A_h \cap \text{Supp } F \neq \phi$. By strict monotonicity of V, $\beta(\cdot)$ is unique, increasing, and $\beta(0) = 0$, $\beta(1) = 1$. The smoothness of V implies that $\beta(\cdot)$ is differentiable. Hence, for all $\alpha^* \in [0,1]$ we have:

(5)
$$0 = \frac{d}{d\alpha} V((1 - p)F^* + pF(.;\alpha,\beta(\alpha)))\bigg|_{\alpha^*}$$
$$= p\int U(X;(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*)))[d\phi_h(X) + \beta'(\alpha^*)d\phi_\ell(X)].$$

Thus, $d\phi_h + \beta'(\alpha^*)d\phi_\ell$ represents a mean utility preserving shift from the point of view of $U(X;(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*)))$. Next we show that this shift represents mean utility preserving increase in risk. Let $I(u;\alpha^*) = \{X \in \mathbb{R}^n_+ \mid U(X;(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*))) = u\}$ and denote by $\bar{u}$ the value of $U(\bar{X};(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*)))$. Then, by definition, $d\phi_h + \beta'(\alpha^*)d\phi_\ell$ represents a decline in the probability measure of $I(\bar{u};\alpha^*)$ and an increase in that of $I(u,\alpha^*)$ for all $u \neq \bar{u}$.

Let $S(u,\alpha^*) = \{X \in \mathbb{R}^n_+ \mid U(X;(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*))) \leqq u\}$ and let $H(\cdot;\alpha^*) : \mathbb{R} \to [0,1]$ be given by:

$$H(u,\alpha^*) = \int_{S(u,\alpha^*)} [(1 - p)dF^*(X) + pdF(X;\alpha^*,\beta(\alpha^*))]$$

Then,

(6) $$\frac{d}{d\alpha} H(u;\alpha)\Big|_{\alpha^*} = p\int_{S(u,\alpha^*)} d\phi_h(X) + \beta'(\alpha^*)d\phi_\ell(X).$$

Hence, by the above argument,

(7) $$\frac{d}{d\alpha} H(u;\alpha^*)\Big|_{\alpha^*} \begin{cases} \geqq 0 & \text{if} \quad u < \bar{u} \\ < 0 & \text{if} \quad u \geqq \bar{u}\ . \end{cases}$$

Consequently, $\frac{d}{d\alpha} H(u;\alpha)\Big|_{\alpha^*}$ represents a mean utility preserving increase in risk from the point of view of $U(X;(1 - p)F^*(.;\alpha^*,\beta(\alpha^*)))$.

By hypothesis, for all $\alpha^* \in [0,1]$, $U^*(.;(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*)))$ is a concave monotonic transformation of $U(.;(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*)))$. Hence, by Diamond and Stiglitz [1974, Theorem 3],

(8) $$\int U^*(X;(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*)))[d\phi_h + \beta'(\alpha^*)d\phi_\ell] \leqq 0$$

for all $\alpha^* \in [0,1]$. Thus,

(9) $$\begin{aligned} &V^*((1 - p)F^* + pF) - V^*((1 - p)F^* + p\delta_{\bar{X}}) \\ &= \int_0^1 \{\int U^*(X;(1 - p)F^* + pF(.;\alpha^*,\beta(\alpha^*)))[d\phi_h(X) + \beta'(\alpha^*)d\phi_\ell(X)]\}d\alpha^* \leqq 0. \end{aligned}$$

By monotonicity of V^* this implies that $\bar{X}^* \leqq \bar{X}$, where $\bar{X}^* \in C^*(F;F^*,p)$, for all $\bar{X}$ such that there exist $X' \in \text{Supp } F$ and $\bar{X} < X'$.

Next consider the case where $F \in D_B(\mathbb{R}^n_+)$ is such there is no $X \in C(F;F^*,p)$ such that $X < X'$ for some $X' \in \text{Supp } F$. Augment the support of F by including the point $Z \equiv (z_1,\ldots,z_n)$ such that $Z > X'$ in the sense that $z_i \geq x_i'$, i=1, ...,n for all $X' \in \text{Supp } F$. Let $\bar{X} \in C(F;F^*,p)$ and $X' \in \text{Supp } F$ such that $X' < \bar{X} < Z$. (That such $\bar{X}$ exists follows from the monotonicity of

V with respect to first order stochastic dominance.) Let A_h be defined as before, let $A_\ell = \{X \in \mathbb{R}^n_+ \mid X \leq \bar{X}\}$, and let A_s be the complement of $A_h \cup A_\ell$ in $\mathbb{R}^n_+$. For j = h, ℓ, s, let $\phi_j : A_j \to [0, 1]$ be defined by $\phi_j(X) = F(X) - \delta_{\bar{X}}(X)$. Let $F(X; \alpha, \beta, \gamma) = \delta_{\bar{X}}(X) + \alpha\phi_s(X) + \beta\phi_\ell(X) + \gamma\phi_h(X)$, where, for all $\alpha \in [0,1]$, β and γ are nonnegative and are defined by the conditions:

$$(10)\quad V((1-p)F^* + pF(.;\alpha, \beta(\alpha), \gamma(\alpha)) = V((1 - p)F^* + pF) = V((1 - p)F^* + p\delta_{\bar{X}}).$$

and

$$(11)\quad \begin{aligned} &\gamma'(\alpha) > 0 \Rightarrow \beta'(\alpha) = 0, \\ &\beta'(\alpha) > 0 \Rightarrow \begin{cases} \gamma'(\alpha) = 0 \text{ if } \gamma(\alpha) = 0 \\ \gamma'(\alpha) = -\beta'(\alpha) \text{ if } \gamma(\alpha) > 0. \end{cases} \end{aligned}$$

By strict monotonicity of V, $\beta(.)$ and $\gamma(.)$ are unique, $\beta(.)$ is monotonic increasing (but not necessarily strictly monotonic increasing), and $\beta(0) = \gamma(0) = 0$, $\beta(1) = 1$, $\gamma(1) = 0$. The smoothness of V implies that β and γ are differentiable. Thus, the operation described by (α, β, γ) shifts probability mass from $\bar{X}$ to points in the augmented support of F in a way that preserves the value of V. In the process some probability mass may be shifted to Z, i.e., when $\gamma'(\alpha) > 0$, but as α tend to 1 the entire mass is distributed on the support of F with zero mass at Z. Following the steps of the proof from equation (5) to (9) will establish that $\bar{X}^* \leq \bar{X}$.

To extend this result to every $X \in C(F;F^*,p)$, suppose that there exists $\hat{X} \in C(F;F^*,p)$ and $\hat{X}^* \in C^*(F;F^*,p)$ such that $\hat{X} < \hat{X}^*$. Let $Y \in \mathbb{R}^n_+$ be such that $\hat{X} < Y < \hat{X}^*$. Since $\bar{X}^* \leqq \bar{X}$ and $C(F;F^*p)$ and $C^*(F;F^*,p)$ are connected sets, there exist $X^o \in C(F;F^*,p) \cap C^*(F;F^*,p)$. Let $\hat{X}$, $\hat{X}^*$ and Y be in a neighborhood of X^o, and define $J(X;\nu) = \delta_{X^o}(X) + \nu[\delta_Y(X) - \delta_{X^o}(X)]$. Obviously $J(.;\nu) \in D_B(\mathbb{R}^n_+)$. Hence,

(12) $$\frac{d}{d\nu} V(J(.;\nu))\Big|_{\nu=0} = U((Y;\delta_{X^o}) - U(X^o;\delta_{X^o}) > 0.$$

and

(13) $$\frac{d}{d\nu} V^*(J(\cdot;\nu))\Big|_{\nu=0} = U^*(Y;\delta_{X^o}) - U^*(X^o;\delta_{X^o}) < 0.$$

But $U^*(X;G_{X^o})$ is a monotonic transformation of $U(X;G_{X^o})$, a contradiction. Thus, $C(F;F^*,p)$ is at least as great as $C^*(F;F^*p)$.

Q.E.D.

__Proof of Theorem 2__: By theorem 1 and the hypothesis there exist functions $u(.;F)$ and $T_F[.,\rho]$ such that $U_\rho(.;F(s)) \equiv T_{F(s)}[u(.;F(s)),\rho]$ for all s and ρ, where $T_F[.,\rho']$ is concave transformation of $T_F[.,\rho]$ whenever $\rho' > \rho$. Since $s(\rho)$ maximizes $V_\rho(F(s))$ we have:

$$0 = \frac{d}{ds}[V_\rho(F(s))]\Big|_{s(\rho)} = \frac{d}{ds}[\int T_{F(s)}[u(C;F(s)),\rho]dF(s)(C)]\Big|_{s(\rho)} =$$

$$\int\left[\frac{\partial}{\partial u}T_{F(s(\rho))}[u(C(s(\rho),r);F(s(\rho))),\rho]\frac{\partial}{\partial s}u(C(s(\rho),r);F(s(\rho)))\right]dF(r).$$

By implicit differentiation we get,

$$\frac{ds(\rho)}{d\rho} = \frac{-\int\frac{\partial^2}{\partial u\partial\rho}T_{F(s(\rho))}[.,\rho]\ \frac{\partial}{\partial s}u(C(s(\rho),r);F(s(\rho)))dF(r)}{\int\{\frac{\partial^2}{\partial u^2}T_{F(s(\rho))}[.,\rho](\frac{\partial}{\partial s}u(.))^2 + \frac{\partial}{\partial u}T_{F(s(\rho))}[.,\rho]\frac{\partial^2}{\partial s^2}u(.,F(s(\rho)))\}dF(r)}.$$

Assuming that the second order optimality condition is satisfied, the denomonator is negative. Hence, the sign of $ds(\rho)/d\rho$ is the same as that of the numerator. The remainder of the proof follows from Diamond and Stiglitz [1974, theorem 4].

Q.E.D

FOOTNOTES

[1]This paper was originally conceived jointly with Mark Machina whose contribution was a necessary condition for its successful completion. I am grateful to Mark for his invaluable help. I am also indebted to Peter Wakker, for many helpful comments and suggestions and to Beth Allen, Chew Soo Hong, and Joel Sobel for helpful discussions. Any remainning errors are my sole responsibility.

[2]See MacCrimmon and Larsson [1979] and Machina [1987] for surveys of this literature.

[3]For alternative approaches see Chew and MacCrimmon [1979], Chew [1983, 1984], Fishburn [1983, 1984], Quiggin [1982], and Yaari [1987].

[4]These results are collectively due to Kihlstrom and Mirman [1974], [1981], and Diamond and Stiglitz [1974].

[5]By this term we mean the ordinal preference ranking over outcomes induced by the individual's (cardinal) von Neumann-Morgenstern utility function.

[6]We impose our continuity and smoothness assumptions on V rather than on the underlying preference relation for expositional convenience. See Allen [1987] for conditions on preferences over multivariate and more general distributions which imply a local expected utility representation similar to equation (1).

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