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THE DEMAND FOR A RISKY ASSET WHEN ITS RETURNS ARE STOCHASTICALLY RELATED TO PRICES OF CONSUMPTION GOODS

by

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ABSTRACT

The presence of a stochastic relation between the returns on a risky asset and a price of a consumption good alters the effects of parametric changes on the demand for a risky asset and on the expected utility of a consumer-investor from what they are in the classical case in which such a relation does not exist. In particular the qualitative equivalence of the effects of risk and risk aversion on the demand for a risky asset breaks. The reason for this departure from the classical portfolio selection behavior is the existence of conflicting objectives. On the one hand the consumer-investor prefers a stable income over a random income with the same expected value. On the other, he prefers a lottery in the price of a consumption good over an assured price which equals the expected value of the lottery. These conflicting objectives come into play if the consumer-investor's income is stochastically related to a price of a consumption good. This is the case if the rate of return on the risky asset is stochastically related to a price of a consumption good.

1. Introduction

The theory of portfolio selection was developed in its early stage under the assumption of a complete independence of rates of return on risky assets and prices of consumption goods. A complete characterization of the demand for a risky asset by a small investor is available subject to the aforementioned independence assumption (see, for example, Arrow (1972).

The last decade witnessed a surge of interest in portfolio selection when rates of return on assets are related to prices of consumption goods, as indeed such relationships are evident. We do not intend to be comprehensive in our list of references. However, in order to appreciate the scope of that interest we note a few studies that testify to the diversity of fields.

The demand for Future's Contracts is analyzed by Britto (1984), the demand for commodity bonds is analyzed by O'Hara (1984). The demand for durables and common stocks is analyzed by Schwartz and Pines (1983). The prices of all of these assets are clearly related to prices of some corresponding consumption goods.

Within the context of International Trade, households can invest in stocks of foreign countries (including financial assets and foreign currencies) as well as in local stocks. The returns on foreign stocks are related to prices of imported goods consumed locally, see for example, Krugman (1983), Stulz (1983), and Branson and Henderson (1985).

Another recent branch of literature which recognizes the dependence of rates of return on assets and prices of consumption goods is the Consumption-based asset pricing macroeconomic model. See, for example, Shiller (1982), Hansen and Singleton (1983).

These models specify relationships of ratio of prices of consumption goods at different periods to prices of bonds of corresponding maturity (or to corresponding ratios of stock prices).

A majority of these studies characterize the effects of some parameters on the quantity demanded of the risky assets as the system which is analyzed shifts parametrically from one general-equilibrium point to another. The parameter of interest is mostly the measure of risk aversion. In some studies (see, for example, the international trade models) stylized facts are used to infer a plausible range for the measure of risk aversion. The consumption based asset pricing model uses the observed macrodata to estimate risk aversion and to test hypotheses of rational expectation.

All of these studies do not have a complete microeconomic characterization of the effects of the parameters of interest on the demand function (as is derived, for example, in Arrow (1972)). This is so because only displacements from one equilibrium point to another are considered.

It is the purpose of this paper to provide a general characterization of the demand for the risky asset in and out of equilibrium, similar to the analysis, available when a complete independence between returns or assets and prices of consumption goods is assumed. In particular we characterize the effects of parameters that affect the stochastic relationship of the price of a consumption good to the return on a risky asset. We also characterize the effects of the expectations and variances of the prices involved in this stochastic relationship and the effect of risk aversion.

The structure of the paper is as follows: In section 2 we set up the model. In section 3 we characterize the demand for the risky asset. In section 4 we characterize the associated welfare implicating and in section 5 we have some concluding remarks.

2. The Model

Consider a risk averse household that lives one period. It makes investment decisions at the beginning and consumes at the end of the period, subject to a budget constraint determined by the outcome of the investment. The household consumes two goods, bread to be denoted, H and an all-purpose good, A, of which it is endowed at birth with an amount I. The all-purpose good serves also as the numeraire. Its price is therefore one.

At the beginning of the period the household can purchase (or sell short) at a going market price, c, an amount, F, of "futures-contract" on a commodity, say wheat, which is harvested at the end of the period. A household which sells a "futures-contract" has to pay the holder of its contract $F \cdot p_w$, where p_w is the spot price of wheat after the harvest (at the end of the period). A household which purchases an amount F of "futures-contract" receives from the seller at the end of the period $F \cdot p_w$.

Before continuing with the 'story' and the formal setup we would like to emphasize the fact that although the investment in our 'story' is in futures market, our model is general and is applicable to any investment in risky asset.

Consumption takes place subsequent to these transactions and is thus subject to the following budget constraint:

(1) $A + Hq = y = I + F(p_w - c)$

where

A = consumption of the all-purpose good,

H = consumption of bread,

p_w = end-of-period spot price of the commodity F transacted in the Futures market (wheat),

c = futures price,

F = quantity of futures contract bought (negative, if sold),

q = the price of bread,

y = $I + F(p_w - c)$ = total income at the end of the period. It includes the initial endowment, I, and the net return from the futures contract $F(p_w - c)$.

Bread is produced from wheat and another all purpose factor.

The household perceives the price of bread, q, to depend linearly on the price of wheat, p_w, and on the price of the other (all purpose) factor, p_z as follows:

(2) $q = \mu_1 p_w + \mu_2 p_z$

where p_z = end-of-period spot price of an (all purpose) factor which is used in the production of bread; μ_1 and μ_2 are the perceived parameters of the price relation. p_w and p_z are not known at the beginning of the period. The household perceives their distribution to be governed by the following mechanisms:

(3) $p_w = \mu_w + bx$

(4) $p_z = \mu_z + du$

where x and u are independent random variables with known distributions which are symmetric around zero. $\underline{x}$ is the lower bound and $\bar{x}$ is the upper bound of x, likewise $\underline{u}$ and $\bar{u}$ are the lower and upper bound of u. These bounds, along with the parameters, μ_w, b, μ_z and d are such that (3) and (4) are nonnegative. μ_w and μ_z are then respectively the expected values of p_w and p_z.

Substituting (4) and (3) in (2), we get:

(2′) $q = (\mu_1\mu_w + \mu_2\mu_z) + \mu_1 bx + \mu_2 du$

The information set available to the household at the beginning of the period (when the investment decision takes place) is composed of the price of futures-contract c, the initial endowment I, the perceived Mechanisms (2), (3) and (4) and the distributions of x and u.

The utility function is restricted in this analysis to the family of homogeneous concave utility functions. The associated particular indirect utility is

(5) $$V(q,y) = (y\theta(q))^{\gamma} = [(I+F(\mu_w + bx-c)\theta(\mu_1\mu_w+\mu_2\mu_z + \mu_1 bx + \mu_2 du)]^{\gamma} \equiv$$
$$\equiv [y(x)\theta(q(x,u))]^{\gamma} \equiv y(x)^{\gamma}\lambda(q(x,u)) \quad 0 < \gamma < 1.$$

The first equality follows from the homogeneity assumption. The second is obtained by substituting (3) in $y = I + F(p_w - c)$ and substituting (2') for q. The third is a short-hand notation that explicitly recognizes that the income depends on the random variable x (through its dependence on the price of wheat) and the price, q, depends on both random variables x and u. In the last equality we denote $\lambda(q(x,u)) = \theta(q(x,u))^{\gamma}$. γ denotes the degree of the homogeneity.

At the beginning of the period the household chooses the level of futures contract, F, which maximizes its expected indirect utility. Its formal problem is thus:

(6) $$\max_F \iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u)$$

where

$G(x,u) = G^1(x)G^2(u)$ is the distribution function of x and u which are assumed to be independent.

The first-order condition for this maximization problem is:

(7) $$\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w + bx - c)dG(x,u) = 0$$

The second-order condition is:

(8) $$(\gamma-1)\gamma\iint y(x)^{\gamma-2}\lambda(q(x,u))(\mu_w + bx - c)^2 dG(x,u) < 0.$$

The inequality follows because $\gamma - 1 < 0$.

The explicit solution of equation (6), expressed as a function of the parameters which are given to the household is the beginning of the period demand function for F:

$$(9) \qquad F = f(c, I, G^1(x), G^2(u), \gamma, \mu_1, \mu_2, \mu_w, \mu_z, b, d).$$

In the subsequent analysis the function $\lambda(q)$ is assumed to satisfy the following condition for all possible realizations of the price q:

$$(10) \qquad \lambda > 0; \quad \lambda' < 0; \quad \lambda'' > 0; \quad \lambda''' < 0 \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\lambda'(q(x,u))}{\lambda(q(x,u))}x\right) < 0.$$

$\lambda > 0$; $\lambda' < 0$ are satisfied by all proper utility functions. The other conditions are satisfied by a wide variety of utility functions, but not all. One example is the family of homogeneous functions whose $\lambda(q)$ is: $\lambda(q) = ((\alpha + \mu q)^{-\epsilon})^{\gamma}$. This function satisfies all the conditions in equation (10) for all the positive values of the parameters α, μ, ϵ and $\gamma < 1$. Another example is the most widely used family of functions which exhibits constant elasticity of substitution. In this case the $\lambda(q)$ function takes the form:

$$\lambda(q) = ((\delta + \beta q^{1-\sigma})^{-\frac{\gamma}{1-\sigma}}$$

$\lambda'' > 0$ is satisfied for all positive values of δ, β and the elasticity of substitution σ. $\lambda''' < 0$ is satisfied at least as long as the parameter of the elasticity of substitution, σ, does not exceed 5. The condition $\frac{\partial}{\partial x}(\frac{\lambda'(q(x,u))}{\lambda(q(x,u))}x) < 0$ is satisfied at least as long as the elasticity of substitution, σ, does not exceed 2. These are lower bounds but not the highest lower bounds. Moreover, these more stringent conditions are sufficient, not necessary, for the propositions derived in this paper. These propositions will hold even if the conditions are not satisfied on the complete range of possible realizations of the price, q, but on a substantial portion of that range.

Note that:

(a) ceteris paribus, the individual prefers a stable income over a random income because $y(x)^{\gamma}$ is a concave function;

(b) ceteris paribus, the individual prefers lotteries in the price, q, (of the consumption good, bread) over a stable price, because $\lambda(q)$ is convex (see equation (10)).

(c) The income of the consumer-investor in this analysis, y, is correlated to the price of bread, q (see equations (1) and (2)).

(a) needs no elaboration - it is the behavioral characteristic of risk aversion in income. The intuition behind (b) is the following. As a price of a good goes down ceteris paribus, we have an income effect and a substitution effect. The income effect by itself raises utility concavely. The substitution away from the good that is now relatively more expensive raises the utility further. For the class of utility functions characterized by (10), this additional increase in the utility more than compensates for the concavity in income leading to a convex increase in the utility.

The objective of this paper is to provide a complete characterization of the demand function for the risky asset F (i.e., equation (9)) within the framework that was set up above. It is a theory of behavior of a "small" investor-consumer who believes that his own actions cannot affect market prices by themselves. However, incomplete information and uncertainty leads him to choose his portfolio on the basis of his subjective perceptions, represented by equations (2), (3) and (4).

We do not impose an apriori market equilibrium condition. Thus, the individual in this study may perceive, for example, that there exists a large gap between the price of Future's contract on wheat and the expected price of wheat (after the harvest), although in equilibrium that gap may have to be small but not necessarily zero. Market equilibrium should be characterized only after a complete characterization of the demand and supply are available.

3. The Demand for the Risky Asset

3.1. The classical case

We start our analysis with a summary of the widely known results from the classical portfolio selection theory. This will serve as a reference set for the general results derived in this paper.

The classical case of portfolio selection is characterized by a stochastic independence of the rates of return on risky assets and the prices of consumption goods. In this paper's notation it requires that μ_1 be set equal to zero.

In the classical case, the demand for a risky asset shifts upward as its expected rate of return increases, ceteris paribus. It shifts downward in the positive quadrant and upward in the negative quadrant if there is either an increase in the risk of the asset or an increase in the consumer-investor's risk aversion. This is demonstrated graphically in figures 1a, 2a, and 3. The price of a risky asset, c, is measured along the vertical axis and the quantity, F, along the horizontal axis. In this paper's notation, a higher b implies a higher risk and a higher γ implies a lower risk aversion. Note that if the price of the asset, c, equals its expected end-of-period unit return, μ_w, then the asset, F, is neither purchased nor sold short for any level of risk aversion (measured by γ) and for any level of risk (measured by b). Only when there exists a perceived gap between c and μ_w will a risk averse individual invest in a risky asset. He will invest by selling short (in which case $F < 0$) if $c > \mu_w$ and by purchasing (in which case $F > 0$) if $c < \mu_w$. Note also (see figures 1a and 2a) that as either the risk or the risk aversion increases the consumer-investor invests less in the risky asset for any perceived gap between c and μ_w. He sells short less (i.e. $|F|$ decreases) if $c > \mu_w$ and he purchases less if $c < \mu_w$.

Note that if the asset cannot be sold short (a possibility that is indeed ignored in the classical case), then the positive quadrant is the only relevant quadrant in these figures.

3.2. **The general case**

In the classical case (i.e., $\mu_1 = 0$, $F \geq 0$), the effects of an increased risk (increased b) and an increased risk aversion (decreased γ) on the demand for F are qualitatively equivalent. Also, in the expanded classical case where F can be sold short (i.e., F can be negative), there exists a qualitative symmetry (of the effects of an increased risk and of an increased risk aversion) between the positive quadrant (where $F > 0$) and the negative quadrant (where $F < 0$) as is demonstrated in figures 1a and 2a.

This qualitative equivalence breaks if $\mu_1 > 0$. Recall that the consumer-investor prefers a stable income over a random income with the same expected value, but that he prefers a lottery in the price of bread, q, to an assured price which equals the expected value of the lottery.

Note now that $\mu_1 > 0$ leads to a covariation of the individual's income, y, and the price of bread, q, that he faces. The covariance of y and q (which is derived from (1), (2), (3), (4) and under the independence of u and x is:

$$\text{(11)} \qquad \text{covariance } (y,q) = F\mu_1 b^2 \text{Varx}$$

This covariance increases with F, it is negative for $F < 0$, zero for $F = 0$ and positive for $F > 0$.

Any parametric change which increases the expected utility gain from a lotetry in the price, q, causes the investor-consumer to desire a lower covariance between y and q. This he can achieve by reducing F, the only choice variable at his disposal (see equation (11)). That is, if $F < 0$, the consumer-investor wishes to increase $|F|$. Any parametric change that increases the advantage of a more stable income leads him to desire a lower investment in the risky asset $|F|$, (i.e., to increase F if $F < 0$ and to decrease F if $F > 0$). These two (and sometimes opposing) effects govern the total effect of parametric changes on the demand for F.

3.2.1. <u>The effect of risk aversion (γ)</u>

An increase in the risk aversion (a decrease in γ in our notation) increases the desire for a more stable income implying a desired reduction of $|F|$ which is the quantity invested in the risky asset. In addition, an increased risk aversion decreases the convexity of the indirect utility in the price, q, (i.e., the convexity of $\lambda(q)$ see Lemma A3 in Appendix A and the discussion therein). This reduces the expected utility gain from a lottery in the price, q, which in turn implies a desired increase in F in order to increase the covariance of y and q (see equation (11)).

The two effects (the desire for a more stable income and the decreased desire for a lottery in q as γ decreases) enhance each other on the negative quadrant (where F is negative), both leading to an increased F in real terms if γ decreases. They exert opposite forces on the positive quadrant. However, for large values of F the desire for a more stable income when γ decreases dominates the associated reduced desire for a lottery in q so that beyond some boundary, say $F^*(\mu_1)$ the total effect on F of a decreased γ is negative. This is demonstrated in figure 1 b and summarized in Proposition 1.

Proposition 1: There exists a boundary $F^*(\mu_1) \geq 0$ which equals zero if $\mu_1 = 0$ and increases with μ_1. The demand for F decreases with γ for F $< F^*(\mu_1)$ and increases for $F > F^*(\mu_1)$.

Proof.

See Lemma B10 in Appendix B.

3.2.2. The Effect of the Risk (b)

An increased b increases the variance of p_w thus making the investment in F more risky. However, an increased b raises the expected utility gain from a lottery in the price, q. The first effect leads to a desired decrease in $|F|$ whereas the second effect leads to a desired decrease in F in order to decrease the covariance of y and q (see equation (11)). The two effects enhance each other on the positive quadrant, but have opposing directions on the negative quadrant. For $F < 0$, but sufficiently small (i.e. large in absolute value) the risk aversion effect dominates the desired lottery effect. Thus there exists an $F(\mu_1) < 0$ such that F decreases for $F > \tilde{F}(\mu_1)$. This is demonstrated in figure 2b and summarized in Proposition 2.

Proposition 2. There exists a boundary $\tilde{F}(\mu_1) \leq 0$ which equals zero for $\mu_1 = 0$ and decreases with μ_1. The demand for F decreases with b for $F > \tilde{F}(\mu_1)$ and increases for $F < \tilde{F}(\mu_1)$.

Proof.

See Lemma B6 in Appendix B.

The effects of an increased risk and an increased risk aversion on the desire for a stable income are qualitatively the same. These effects differ in their impact on the expected utility gain from a lottery in q. An increased risk aversion (decreased γ) reduces this gain whereas an increased risk (increased b) increases this gain. Thus the consumer-investor desires to increase the covariance between the income, y and the price q, if risk aversion increases and to decrease it if the risk increases. This implies a desire to increase F if the risk aversion increases and to decrease F if the risk increases (see equation (11).

Consequently, an increased risk aversion will shift the demand for F so that the demand associated with the higher risk aversion will intersect the demand (associated with the lower risk aversion) from above on the positive quadrant, whereas for an increased risk the intersection will also be from above, but on the negative quadrant, thus breaking the qualitative equivalence of these two effects. The qualitative equivalence is restored only if $\mu_1 = 0$. This is demonstrated in figures 1 and 2, by comparing 1a and 2a to 1b and 2b.

Finally note also that if $\mu_1 > 0$, the existence of a possible gain in the expected utility from a lottery in the price, q, shifts the demand function so that the price, c, at which F is neither bought nor sold short is strictly less than the expected unit return, μ_w (on the investment in F). The gap between μ_w and c(F = 0) (i.e., the cost at which F = 0)) can be termed the desired lottery premium.

3.2.3. The Effect of the Variance of p_z (the effect of d).

An increase in d increases the variance of the price, q, but does not affect the variance of the risky asset, F. Thus it increases the desire for a lottery in the price, q, but does not affect the desire for a stable income. The argument discussed in the previous subsection leads to the conclusion that if $\mu_1 > 0$, the demand function for F will shift downwards as d increases in spite of the fact that p_z affects neither the expected rate of return of F nor its variance. However, If $\mu_1 = 0$ the demand function for F will not be affected by d. This is summarized in Proposition 3.

Proposition 3. (a1) The demand for F is unaffected by changes in d if $\mu_1 = 0$; (a2) The demand for F decreases with d if $\mu_1 > 0$.

Proof.

See Lemma B7 in Appendix B.

3.2.4. The Hedging Motive (the Effects of μ_w and μ_z)

An increase in the mean of the price of the risky asset, μ_w, increases the expected rate of return if $F > 0$. It decreases the expected rate of return if $F < 0$. Thus, even if $\mu_1 = 0$ the demand for F will shift upward (implying a reduction in the investment in the negative quadrant). The fact that $\mu_1 > 0$ also introduces the hedging motive . If one believes that on average the price of a good will increase in the future he will rush and buy more of it now. The consumption good (bread) in this model cannot be purchased in the beginning of the period, but a futures contract on an input (wheat) which participates in the production of that good serves the same purpose.

An increase in the expected price of the all purpose factor, μ_z does not affect the random returns on F. However, if $\mu_1 > 0$, the hedging motive exists. Therefore an increased μ_z will shift the demand function for F upward., This is summarized in Proposition 4.

Proposition 4. (a) The demand for F increases with μ_w; (b) the demand for F increases with μ_z.

Proof.

See lemma B3 in Appendix B

3.2.5. **The effects of the parameters of the price mechanism** (μ_1, μ_2)

The Effect of μ_1

An increase in μ_1 increases the contribution of P_w (the price of wheat) to q (the price of bread). It increases both the contribution of P_w to q's expected value via the term $\mu_1 \cdot \mu_w$ and to its variance, via the term $(\mu_1 \beta)^2 \text{Var } x$. It does not affect the expected return and the variance of the risky asset (and thus of the income for any given F), but it affects the covariance of the income, y, and the price, q. If $F > 0$, this covariance increases and if $F < 0$ it decreases (see equation (11)).

The effect of μ_1 on the demand for F through its effect on the mean of q is qualitatively equivalent to the effect which was termed "the hedging motive" in the discussion leading to Proposition 4. This effect is positive as is argued therein (see third term on the r.h.s. of equation (B8) in Appendix B).

We now turn to the effect of μ_1 on the demand for F, through its effect on the variance of q and its covariance with y.

An increase in the variance of q increases the gain to the expected utility from a lottery in q. This in itself (holding the covariance of q and y unchanged) leads to a desire to reduce the covariance between y and q by reducing F.

On the positive quadrant (where $F > 0$) an increased μ_1 raises the covariance of q and y (see equation (11)) which leads to a desire to partially restore it by adjusting F downward. Thus, on the positive quadrant, the two effects enhance each other, so that an increased μ_1 leads, unambiguously, to a desired decrease in F.

On the negative quadrant (where $F < 0$) an increased μ_1 decreases the covariance of q and y (because $F < 0$, see equation (11)), thus generating two opposite forces. The increased variance of q by itself leads to a desire to decrease the covariance between q and y by reducing F, whereas the decreased covariance by itself leads to a desire to partially restore it by increasing F. For very large $|F|$ the second effect may dominate the first (see Lemma B4).

We conclude that the effect of μ_1 through its effect on the variance of q and its covariance with y is negative except, possibly, on the left tail of the negative quadrant. The effect of μ_1 through its effect on the mean of q is positive anywhere. Thus, except for the left tail these are opposite forces. If μ_w is sufficiently larger than b, the effect on the mean denominators and the demand for F increases with μ_1. If, on the other hand, b is sufficiently larger than μ_w the effect of the variance of q and its covariance with y dominate and the demand for F decreases with μ_1 except possibly for the left tail of the negative quadrant.

The effect of μ_2

An increase in μ_2 increases the contribution of P_z (the price of the all-purpose factor) to q. It increases both the contribution of P_z to q's expected value and to its variance. It affects neither the expected returns and the variance of the risky asset nor the covariance of q and y.

The effect of μ_2 on the demand for F through its effect on the mean of q is qualitatively identical to the "hedging motive" of the effect of μ_z which is summarized in Proposition 4. The effect of μ_2 on the demand for F through its effect on the variance of q is qualitatively identical to the effect of d which is summarized in Proposition 3.

These two effects are of opposite signs and therefore the effect of μ_2 on the demand for F depends on which of the two is larger in absolute value. Clearly, if μ_w is sufficiently larger than d the demand for F will increase with μ_2 and vice versa. This is summarized in Proposition 5.

Proposition 5.

(a) **The effect of μ_1**

(a1) If μ_w/b is sufficiently large, then the demand for F increases with μ_1.

(a2) If μ_w/b is sufficiently small, then the demand for F decreases with μ_1 on the range $F > \tilde{F}(\mu_1)$, where $\tilde{F}$ is negative.

(b) **The effect of μ_2**

(b1) If μ_z/d is sufficiently large, then the demand for F increases with μ_2.

(b2) If μ_z/d is sufficiently small, then the demand for F decreases with μ_2.

Proof.

See Lemmas B8 and B9 in Appendix B.

4. Welfare Implications

The expected utility is a U-shaped function in the price of the risky asset, c. It is minimized where c attains a value $\tilde{c}$ at which it is optimal to choose $F = 0$ (see Lemma C7 in Appendix C). As c departs from $\tilde{c}$ the consumer-investor gains from an investment in the risky asset. He purchases F if $c < \tilde{c}$ and sells it short if $c > \tilde{c}$ which explains the U-shape behavior. This is demonstrated in figure 4 (see graph marked μ_w^0). The expected utility is measured along the vertical axis and c along the horizontal axis.

4.1. **The Effects of the Means** (μ_w, μ_z)

If $\mu_1 > 0$ an increased μ_w increases the expected price of q, which leads to a decreased expected utility. However, an increased μ_w increases also the rate of return on the risky asset if $F > 0$ but decreases it if $F < 0$. Thus for $c > \tilde{c}$ both effects enhance each other and the expected utility decreases with μ_w. For $c < \tilde{c}$ the two effects are of opposite signs. However, the positive effect of the increased rate of return dominates the negative effect of the increased expected q for sufficiently large values of F. The smaller is μ_1 the smaller is the boundary (on F) at which the two effects cancel each other. When $\mu_1 = 0$, this boundary occurs at $\tilde{c}$ (at which point $F = 0$). This is demonstrated in figure 4.

An increase in μ_z has no effect on the rate of return, thus it decreases the expected utility anywhere. This is summarized in Proposition 6.

Proposition 6. (a) There exists a boundary $\bar{c}(\mu_1)$ which decreases with μ_1 $\bar{c}(\mu_1 = 0) = \tilde{c}$ (where $\tilde{c}$ is the price of the risky asset at which it is optimal to choose $F = 0$).

The expected utility increases with μ_w for $c < \bar{c}(\mu_1)$ and decreases with μ_w for $c > c\,(\mu_1)$.

(b) The expected utility decreases with μ_z anywhere.

Proof.

See Lemmas C1 and C2 in Appendix C.

2. The Effects of the Variances (b,d)

An increased d increases the variance of q but leaves the returns on the risky asset unaffected. Since the consumer-investor prefers lotteries in q, an increased variance increases his expected utility.

An increased b increases the variance of the returns on the risky asset as well as the variance of q. An increase in the variance of q leads to an increase in the expected utility but an increase in the variance of the return on the risky asset leads the expected utility to fall. The variance of the risky asset is $(Fb)^2 Varx$. This variance decreases as the absolute value of F(i.e., $|F|$) decreases. On the other hand, the size of F has no effect on the variance of q. Therefore, for sufficiently small $|F|$ the effect of an increased b on the variance of q dominates the effect it exerts on the variance of the returns on the risky asset, causing the expected utility to rise. For $|F|$ large enough, the effect of b on the variance of the returns to the risky asset dominates its effect on the variance of q causing the expected utility to fall.

Note that as μ_1 goes to zero, the gain to the expected utility because of the lottery in q decreases. It is zero if $\mu_1 = 0$ (the classical case). In this case an increased b leads to a decrease in the expected utility anywhere except at $F = 0$ because it affects only the returns on the risky asset. This is demonstrated in figure 5, and summarized in Proposition 7.

Proposition 7. (a) The expected utility increases with d.

(b) There exist two boundaries $c^*(\mu_1)$ and $c^{**}(\mu_1)$ such that $c^*(\mu_1)$ increases with μ_1 and $c^{**}(\mu_1)$ decreases with μ_1. $c^*(\mu_1) = c^{**}(\mu_1) = \tilde{c}$ for $\mu_1 = 0$ (where $\tilde{c}$ is the price that makes $F = 0$ optimal).

The expected utility increases with b if c is in the range $c^{**}(\mu_1) \leq c \leq c^*(\mu_1)$. The expected utility decreases with b otherwise.

Proof.

See Lemmas C3 and C4 in Appendix C.

4.3. The Effects of the Parameters of the Price Mechanism (μ_1, μ_2).

The Effect of μ_1

An increase in μ_1 increases both the expected price of bread, q, via the term $\mu_1\mu_w$, and its variance via the term $\mu_1 b$ (which multiplies X). It does not affect the expected returns and the variance of the risky asset, F. However, the covariance between the consumer-investor's income, y, and the price of bread, q, is also affected by μ_1. The covariance decreases with μ_1 if $F < 0$, and increases with μ_1 if $F > 0$ (see equation 11)).

An increase in the variance of q in conjunction with a decrease in the covariance between y and q, increases the expected utility. Thus, for $F < 0$ (for which the covariance decreases with μ_1) the expected utility increases. For $F > 0$ the increased variance of q and the increased covariance of y and q have opposite effects. For large values of F the negative effect of the increased covariance more than offsets the positive effect of the increased variance, causing the expected utility to fall.

An increased μ_1 also increases the expected value of q which leads, by itself, to a decrease in the expected utility for any F.

Thus, for large values of F the expected utility unambiguously diminishes with μ_1. For small values of F (including also the negative values) we have opposing effects. However, if b is sufficiently large relative to μ_w, the effect of μ_1 on the expected utility, via its effect on the variance of q and its covariance with y dominates the effect of μ_1 via its effect on the expected value of q leading the expected utility to increase. If b is sufficiently small relative to μ_w, the effect of μ_1 via its effect on the expected value of q will dominate the other and the expected utility will fall. This is summarized in Proposition 8.

Proposition 8. (a) There exists a boundary N, on the price of the risky asset, c, such that the expected utility decreases with μ_1 if $c < N$;

(b) There exists two boundaries S and T, $T >> S$ such that:

(b1) The expected utility decreases with μ_1 for every c if $\frac{\mu_w}{b} > T$.

(b2) The expected utility increases with μ_1 if $c > N$ and $\frac{\mu_w}{b} < S$.

Proof.

See Lemma C5 in Appendix C.

The Effect of μ_2

An increase in μ_2 increases both the expected price of bread, q, via the term $\mu_1\mu_z$, and its variance via the term $\mu_1 d$ (which multiplies u). It affects neither the expected returns on the asset, F, nor its variance. It also does not affect the covariance of the income y and the price of bread q.

An increase in $\mu_1\mu_z$ is equivalent to an increase in μ_z which decreases the expected utility. An increase in $\mu_1 d$ is equivalent to an increase in d which increases the expected utility. If d is sufficiently large, relative to μ_z, then the effect of μ_1 on the expected utility via its effect on the variance of q will dominate its effect via the expected value of q and vice versa. This is summarized in Proposition 9.

Proposition 9. There exist two boundaries S and T, T >> S such that:

(1) The expected utility decreases with μ_2 if $\frac{\mu_z}{d} > T$

(2) The expected utility increases with μ_2 if $\frac{\mu_z}{d} < S$.

Proof.

See Lemma C6 in Appendix C.

Concluding Remarks

The presence of a stochastic relation between the returns on a risky asset and a price of a consumption good alters the effects of parametric changes on the demand for a risky asset and on the expeded utility (of the investor-consumer) from what they are in the classical case in which there is no such relation.

In the classical case the effect of an increased risk on the demand for a risky asset is qualitatively equivalent to the effect of an increased risk aversion. Both decrease the investment. This qualitative equivalence breaks when the returns on the risky asset are stochastically related to a price of a consumed good. The gap between the two effects is larger the stronger is the relation between the returns on the risky asset and the consumer good's price (the larger is μ_1 in our notation).

The stronger is this stochastic relation the more to the right will the demand shift if risk aversion increases, and the more to the left will it shift if the risk increases. However, in both cases, the shifted demand curve (associated with either the increased risk aversion or the increased risk) will intersect the initial demand curve from above.

The reason for this departure from the classical portfolio selection behavior is the existence of conflicting objectives. On the one hand the consumer-investor prefers a stable income over a random income with the same expected value. On the other, he prefers a lottery in the price of a consumptin good over an assured price equalling the expected value of the lottery. These conflicting objectives come into play if the consumer-investor's income is stochastically related to a price of a consumption good which is hereafter termed the stochastic dependence case.. Such a relation exists if the income depends in part on an investment made in a risky asset whose returns are related to a price of a consumption good. We now turn to some other comparisons.

An increase in the expected returns of the risky asset increases the demand for the risky asset in both the classical and the stochastic dependence cases. The expected utility, on the other hand, increases anywhere with the expected returns in the classical case, whereas in the case of stochastic dependence it decreases if the price of the risky asset (c, in this paper's notation) is high, but increases if this price is low.

In the classical case, an increase in the variance of a price of a consumption good does not affect the demand for the risky asset, but raises the expected utility. In the case of stochastic dependence, the effect depends on whether the increased variance of the price of the consumption good is caused by the same economic variable that increases also the variance of the returns on the risky asset, or whether it is caused by an economic variable that does not affect the returns on the risky asset. In the first case the demand decreases in the positive quadrant, but increased on the left tail of the negative quadrant, whereas in the second case it decreases anywhere! The expected utility in the first case (where the increased variance of the price causes also an increase in the variance of the returns on the risky asset) increases for some intermediate range of prices of the risky asset (c, in the paper's notation), but decreases outside of this range whereas in the second case the expected utility increases anywhere!

In the classical case the optimal investment level $|F|$ in the risky asset is zero if the expected unit return equals the unit cost (in this paper's notation $|F| = 0$ if $c = \mu_w$). In the presence of a stochastic relation (between the returns on the asset and a price of consumption good), the optimal investment level is zero at a unit cost which is strictly below the expected unit return. The gap between the expected unit return and the unit cost which makes zero investment optimal, increases with the strength of the stochastic relation between the returns on the risky asset, and the price of the consumption good (this strength is measured by μ_1 in this paper's notation). This is a reflection of the consumer-investor's desire for a lottery in the price of the consumption good leading the consumer-investor to reduce the covariance between his income and the price of the consumption good which he attains by reducing the investment in the risky asset. This gap can be termed the desired price lottery premium.

APPENDIX A - PRELIMINARY LEMMAS

Let $h(x,u)$ be a function that multiplies the integrand of the first-order condition $y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)$ and consider the integral $\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)h(x,y)dG(x,u)$. We state without proof:

Lemma A1 If

(a) $\frac{\partial h}{\partial x} > 0$ and $\frac{\partial h}{\partial y} > 0$ anywhere then this integral is positive and if

(b) $\frac{\partial h}{\partial x} < 0$ and $\frac{\partial h}{\partial u} < 0$ anywhere then this integral is negative.

Let $f(x)$ be a function that satisfies:

(a) $f(x) \gtrless 0$ if $x \gtrless c$; c is a constant

(b) $\int f(x)dG^1(x) = 0$

and let $n(x)$ be a monotone function. We state without proof:

Lemma A2

$$\int f(x)n(x)dG^1(x) \gtrless 0 \quad \text{if} \quad \frac{dn}{dx} \gtrless 0$$

Lemma A3 (characteristic of the function $\theta(q)$)

If $\lambda(q) = (\theta(q))^\gamma$ satisfies (10) for any γ such that $0 < \gamma < 1$ then $\theta(q)$ satisfies:

$$\theta(q) > 0 \quad \theta'(q) < 0 \quad \theta''(q) > 0 \quad \text{and} \quad \theta(q)\cdot\theta''(q) > (\theta'(q))^2$$

Proof.

Differentiate $\lambda(q)$ to obtain

$$\frac{\partial}{\partial q}\lambda(q) = \gamma\theta(q)^{\gamma-1}\theta'(q)$$

$$\frac{\partial^2}{\partial q^2}\lambda(q) = (\gamma-1)\gamma\theta(q)^{\gamma-2}(\theta'(q))^2 + \gamma\theta(q)^{\gamma-1}\theta''(q)$$

$$= \gamma\theta(q))^{\gamma-2}[(\gamma-1)(\theta'(q))^2 + \theta(q)\theta''(q)]$$

Equation (10) implies $\theta'(q) < 0$ and $\theta''(q) > 0$. Moreover, the r.h.s. has to be positive for any γ such that $0 < \gamma < 1$. This implies that $\theta(q)\theta''(q) > (\theta'(q))^2$. Note further that the convexity increases as γ increases (see the term in the square brackets).

Lemma A4

If $\lambda(q)$ satisfies equation (10) then $\frac{\partial}{\partial q}\frac{\lambda'(q)}{\lambda(q)} > 0$

Proof

$$\frac{\partial}{\partial q}\left[\frac{\lambda'(q)}{\lambda(q)}\right] = \frac{[(\gamma-1)\gamma\theta(q)^{\gamma-2}(\theta'(q))^2+\gamma\theta(q)^{\gamma-1}\theta''(q)]\theta(q)^{\gamma}-\gamma^2\theta(q)^{2(\gamma-1)}(\theta'(q))^2}{(\theta(q))^{2\gamma}}$$

$$= \frac{\gamma\theta(q)^{2(\gamma-1)}[\theta(q)\theta''(q)-(\theta'(q))^2]}{\theta(q)^{2\gamma}} > 0$$

The last inequality follows from Lemma A3.

APPENDIX B: THE DEMAND FUNCTION - FORMAL PROOFS

Define:

(B1) $$\delta(x) = \int \lambda(q(x,u))dG^2(u)$$

Lemma B1

$\delta(x) > 0, \quad \delta'(x) < 0, \quad \delta''(x) > 0.$

Proof

(B2) $\delta(x) = \int \lambda(q(x,u))dG^2(u) > 0$ since $\lambda > 0$ anywhere

(B3a) $\delta'(x) = \int \lambda_x(q(x,u)dG^2(u)) = \mu_1 b \int \lambda'(q(x,u))dG^2(u) < 0$ since $\lambda' < 0$ anywhere

(B3b) $\delta''(x) = (\mu_1 b)^2 \int \lambda''(q(x,u))dG^2(u) > 0$ since $\lambda'' > 0$ anywhere

Define

(B4) $$\alpha(x) = \int \lambda'(q(x,u))dG^2(u)$$

Lemma B2

$\alpha(x) < 0; \quad \alpha'(x) > 0$

Proof

(B5) $\alpha(x) = \int \lambda'(q(x,u))dG^2(u) < 0$ since $\lambda' < 0$ anywhere

(B6) $\alpha'(x) = \mu_1 b \int \lambda''(q(x,u))dG^2(u) > 0$ since $\lambda'' > 0$ anywhere

Note that $\text{Sign} \frac{\partial F}{\partial \theta} = \text{Sign} \frac{\partial^2 EV}{\partial \theta \partial F} = \text{Sign} \frac{\partial}{\partial \theta} \iint y(x)^{\gamma-1} \lambda(q(x,u))(\mu_w + bx - c)dG(x,u)$ where θ is a parameter given to the investor-consumer. In what follows we sign the derivatives $\frac{\partial}{\partial \theta} \iint y(x)^{\gamma-1} \lambda(q(x,u))(\mu_w + bx - c)dG(x,u)$

Lemma B3 (the effects of the expected prices of w and z)

(B7a) $$\frac{\partial}{\partial \mu_w}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) > 0$$

(B7b) $$\frac{\partial}{\partial \mu_z}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) > 0$$

Proof

(B8) $$\frac{\partial}{\partial \mu_w}\iint (x)^{\gamma-1}\lambda(q(x,u))(\mu_w + bx-c)dG(x,u) =$$
$$\frac{\partial}{\partial \mu_w}\int y(x)^{\gamma-1}\delta(x)(\mu_w + bx-c)dG^1(x) =$$
$$= \int\{(\gamma-1)y(x)^{\gamma-2}(\mu_w + bx-c)F\delta(x) + y(x)^{\gamma-1}\delta(x)$$
$$+ \mu_1 y(x)^{\gamma-1}\delta(x)(\mu_w+bx-c)\frac{\delta'(x)}{\delta(x)}\}dG^1(x) = \int y(x)^{\gamma-1}\delta(x)(\mu_w bx-c)\frac{(\gamma-1)}{y(x)}FdG'(x)$$
$$+ \int y(x)^{\gamma-1}\delta(x)dG^1(x) + \mu_1\int y(x)^{\gamma-1}\delta(x)(\mu_w + bx-c)\frac{\delta'(x)}{\delta(x)}dG^1(x).$$

The first equality is obtained by substitution of (B2). All three terms are positive. The first and third by Lemma A2. The second term's integrand is always positive.

(B9) $$\frac{\partial}{\partial \mu_z}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) =$$
$$\mu_2\iint y(x)^{\gamma-1}\lambda'(q(x,u))(\mu_w+bx-c)dG(x,u)=$$
$$= \mu_2\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)\frac{\lambda'(q(x,u))}{\lambda(q(x,u))}dG(x,u) > 0.$$

The inequality follows from Lemma A1, since $\frac{\lambda'}{\lambda}$ is monotone increasing in x and u.

Lemma B4

(B10a) $$\iint y(x)^{\gamma-1}(\mu_w+bx-c)\frac{\partial}{\partial b}\lambda(q(x,u))dG(x,u) \begin{array}{l} < 0 \quad \text{for any } c \text{ if } d = 0 \\ < 0 \quad \text{for any } d \text{ if } c \le \mu_w \end{array}$$

(B10b) $\iint y(x)^{\gamma-1}(\mu_w+bx-c)\frac{\partial}{\partial b}\lambda(q(x,u))dG(x,u)$ may be positive if d and c are sufficiently large (i.e. if F is negative, but sufficiently large in absolute values and the variance of P_z is sufficiently large).

Proof

(B11)
$$\iint y(x)^{\gamma-1}(\mu_w+bx-c)\frac{\partial}{\partial b}\lambda(q(x,u))dG(x,u) =$$
$$= \mu_1\iint y(x)^{\gamma-1}(\mu_w+bx-c)\lambda'(q(x,u))xdG(x,u) =$$
$$= \mu_1\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)\frac{\lambda'(q(x,u))}{\lambda(q(x,u))}xdG(x,u) =$$
$$= \mu_1\Bigg[\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)\Big(\frac{\lambda'(q(x,u))}{\lambda(q(x,u))}x - \frac{\lambda'(q(\tilde{x},u))}{\lambda(q(\tilde{x},u))}\tilde{x}\Big)dG(x,u) +$$
$$+ \tilde{x}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)\Big(\frac{\lambda'(q(\tilde{x},u))}{\lambda(q(\tilde{x},u))} - \frac{\lambda'(q(\tilde{x},\tilde{u}))}{\lambda(q(\tilde{x},\tilde{u}))}\Big)dG(x,u)$$

where

(B12) $\tilde{x} = -\dfrac{(\mu_w-c)}{b}$ is the value of x for which $\mu_w + bx-c = 0$.

The second equality is obtained upon multiplication and division by $\lambda(x,u)$. The third is obtained by adding and subtracting

$\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)\dfrac{\lambda'(q(\tilde{x},u))}{\lambda(q(\tilde{x},u))}\tilde{x}\ dG^2(x,u)$ and subtracting

$\dfrac{\lambda'(q(\tilde{x},\tilde{u}))}{\lambda(q(\tilde{x},\tilde{u}))}\tilde{x}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u)$ which is zero by the first-order condition. The first integral in (B11*) is negative because the integrand is always negataive by the choice of $\tilde{x}$ and since, by equation (10) $\frac{\partial}{\partial x}\big(\frac{\lambda'(q(x,u))}{\lambda(q(x,u))}x < 0$. We now turn to the second integral in (B11*). Define

(B13) $$\phi(u) = \int y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w + bx-c)dG^1(x).$$

Note that by the first-order condition and the continuity of the integrand in u, there exists a value $\tilde{u}$ such that $\phi(\tilde{u}) = 0$. We show next that $\tilde{u}$ is unique and that $\phi(u) \lessgtr 0$ if $u \lessgtr \tilde{u}$. To that end differentiate equation (B13) at $\tilde{u}$ to obtain

(B14) $$\phi_u(\tilde{u}) = \mu_2 d \int y(x)^{\gamma-1}\lambda(q(x,\tilde{u}))(\mu_w+bx-c)\frac{\lambda'(q(x,\tilde{u}))}{\lambda(q(x,\tilde{u}))}dG^1(x) > 0.$$

(B14) is positive by Lemma 1 since $\frac{\lambda'}{\lambda}$ is monotone increasing.

Thus $\phi(u)$ has a single root at $\tilde{u}$ - it is negative for $u < \tilde{u}$ and positive for $u > \tilde{u}$.

Substitute equation (B12) in the second integral in equation (B11*) to obtain:

(B15) $$\tilde{x}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)\left(\frac{\lambda'(q(\tilde{x},u))}{\lambda(q(\tilde{x},u))} - \frac{\lambda'(q(\tilde{x},\tilde{u}))}{\lambda(q(\tilde{x},\tilde{u}))}\right)dG(x,u) =$$

$$= \tilde{x}\int \phi(u)\left[\left(\frac{\lambda'(q(\tilde{x},u))}{\lambda(q(\tilde{x},u))} - \frac{\lambda'(q(\tilde{x},\tilde{u}))}{\lambda(q(\tilde{x},\tilde{u}))}\right]dG^2(u) \gtrless 0 \quad \text{if} \quad \tilde{x} \gtrless 0.$$

The inequality in equation (B15) follows from Lemma A2 since $\frac{\lambda'(q(\tilde{x},u))}{\lambda(q(\tilde{x},u))}$ is monotone increasing. Note now that:

(a) $\tilde{x} \gtrless 0$ if $\mu_w - c \lessgtr 0$ (this follows from equation (B12) since $b > 0$).

(b) If $d = 0$ the second integral in equation (B11*) is zero.

(c) For $d > 0$ the second integral in equation (B11*) increases in absolute value with $\tilde{x}$. Thus when $\tilde{x}$ is positive and increases, equation (B11) may become positive. (a) and (b) and the negativity of the first integral in equation (B11*) imply equation (B10a). (c) and the negativity of equation (B11*) imply equation (B10b).

Lemma B5

(B16) $$\int \delta(x)\frac{\partial}{\partial b}[y(x)^{\gamma-1}(\mu_w+bx-c)]dG^1(x) \lessgtr 0 \quad \text{if} \quad F \gtrless F^*(\mu_1)$$

where F^* is strictly negative if $\mu_1 > 0$, $\frac{\partial}{\partial \mu_1} F^* < 0$ and

$F^* = 0$ if $\mu_1 = 0$.

Proof

(B17) $$\frac{\partial}{\partial b}[y(x)^{\gamma-1}(\mu_w+bx-c)] = (\gamma-1)y(x)^{\gamma-2}Fx(\mu_w+bx-c) + xy(x)^{\gamma-1} =$$
$$y(x)^{\gamma-1}(\mu_w+bx-c)(\gamma-1)F\frac{x}{y(x)} + xy(x)^{\gamma-1}.$$

Substituting (B17) in (B16) yields:

(B18) $$\int \delta(x)\frac{\partial}{\partial b}[y(x)^{\gamma-1}(\mu_w+bx-c)]dG^1(x) = \int [y(x)^{\gamma-1}\delta(x)(\mu_w+bx-c)](\gamma-1)F\frac{x}{y(x)}dG^1(x)$$
$$+ \int y(x)^{\gamma-1}\delta(x)xdG^1(x)$$

Note that (a) if $F > 0$ the first integral on the r.h.s. is negative by Lemma A2 (because $(\gamma-1)F\frac{x}{y(x)}$ is monotone decreasing).

(b) if $F \geq 0$, the second integral on the r.h.s. is negative by Lemma A2.

(c) if $F < 0$ the first integral on the r.h.s. is positive by Lemma A2 (because $(\gamma-1)F\frac{x}{y(x)}$ is now monotone increasing).

(d) there exists a boundary $F^*(\mu_1) < 0$ such that for any $F < F^*$ the second integral on the r.h.s. is positive. For $F^* < F < 0$ this integral is negative.

(e) $\frac{\partial}{\partial \mu_1}F^*(\mu_1) < 0$ because the larger μ_1 the stronger is the effect of $\delta(x)$.

(f) at $F = 0$, the first integral on the r.h.s. is zero.

(g) if $\mu_1 = 0$ both integrals are negative (positive) if $F >(<) 0$.

Lemma B6 (the effect of the variance of the price of wheat)

(B19) $$\frac{\partial}{\partial b}\int y(x)^{\gamma-1}\delta(x)(\mu_w+bx-c)dG^1(x) \lessgtr 0 \quad \text{if} \quad F \gtrless \tilde{F}(\mu_1)$$

where $\tilde{F}(\mu_1)$ is negative and $\frac{\partial \tilde{F}}{\partial \mu_1} < 0$, $\tilde{F} = 0$ if $\mu_1 = 0$.

Proof

(B20) $$\frac{\partial}{\partial b}\int y(x)^{\gamma-1}\delta(x)(\mu_w+bx-c)dG^1(x) = \mu_1\int y(x)^{\gamma-1}\delta'(x)(\mu_w+bx-c)dG^1(x) + \int \delta(x)\frac{\partial}{\partial b}[y(x)^{\gamma-1}(\mu_w+bx-c)]dG^1(x).$$

The first term on the r.h.s. is negative on the positive quadrant and at least on a portion of the negative quadrant by Lemma B4. The second term on the r.h.s. is negative for $F > F^*(\mu_1)$ (where $F^* < 0$) and $\frac{\partial F^*}{\partial \mu_1} < 0$) and positive for $F < F^*(\mu_1)$ by Lemma B5. Thus equation (B20) is clearly negative for $F > F^*$. For $F < F^*$ the second term on the r.h.s. is positive and increasing in $|F|$. Therefore there exists an $\tilde{F}(\mu_1)$ such that either both terms are positive or that the second term dominates the first for $F < \tilde{F}(\mu_1)$. The larger is μ_1 the smaller is F* (larger in absolute value) at which the second term overtakes the first (in a leftward direction) since $\frac{\partial F^*}{\partial \mu_1} < 0$ (see Lemma B5), and since μ_1 multiplies the first term on the r.h.s. of equation (B20).

Lemma B7 (the effect of the variance of the price of the all purpose factor)

(B21) $$\frac{\partial}{\partial d}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) \le 0 \quad \text{if} \quad \mu_1 \ge 0$$

Proof

(B22) $$\frac{\partial}{\partial d}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) =$$
$$(*)\ \mu_2\iint y(x)^{\gamma-1}\lambda'(q(x,u))(\mu_w+bx-c)udG(x,u) =$$
$$\mu_2\int y(x)^{\gamma-1}\lambda(q(x,\hat{u}))(\mu_w+bx-c)\left[\int\frac{\lambda'(q(x,u))}{\lambda(q(x,\hat{u}))}udG^2(u)\right]dG^1(x)$$

where the last equality is obtained by multiplying and dividing by $\lambda(x,\hat{u})$, where $\hat{u}$ is an arbitrary constant.

Denote by $\psi(x)$ the term in the square brackets. Note that by Lemma A2:

(B23) $$\psi(x) \equiv \int\frac{\lambda'(q(x,u))}{\lambda(q(x,\hat{u}))}udG^2(u) = \frac{1}{\lambda(q(x,\hat{u}))}\int\lambda'(q(x,u))udG^2(u) > 0$$

since $\lambda'(q(x,u))$ is monotone increasing.

Differentiating equation (B23) by x yields:

$$(B24)\ \psi_x(x) = \mu_1 b\left[-\lambda(q(x,\hat{u}))^{-2}\int\lambda'(q(x,u))udG^2 + \lambda(q(x,\hat{u}))^{-1}\int\lambda''(q(x,u))^2\right] =$$

$$\mu_1 b\ \lambda(q(x,\hat{u}))^{-2}\left[\lambda(q(x,\hat{u}))\int\lambda''(q(x,u))udG^2 - \int\lambda'(q(x,u))udG^2\right] < 0.$$

The inequality follows from Lemma A2 because $\lambda'' > 0$ and $\lambda''' < 0$.

Choose $\hat{u}$ so that $\int y(x)^{\gamma-1}\lambda(q(x,\hat{u}))(\mu_w+bx-c)dG^1(x) = 0$. Such a $\hat{u}$ exists by the first-order condition and the continuity (in u) of the integral of the first-order condition.

Note that given this choice of $\hat{u}$ we can establish:

$$(B25)\quad \frac{\partial}{\partial d}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) =$$
$$\mu_2\int y(x)^{\gamma-1}\lambda(q(x,\hat{u}))(\mu_w+bx-c)\psi(x)dG^1(x) \le 0 \quad \text{if} \quad \mu_1 \ge 0.$$

The inequality follows from Lemma A2 since $\hat{u}$ is chosen so that $\int y(x)^{\gamma-1}\lambda(q(x,\hat{u}))(\mu_w+bx-c)dG^1(x) = 0$ and since $\psi(x)$ is monotone decreasing if $\mu_1 > 0$.

If $\mu_1 = 0$, then $\psi(x)$ is a constant in which case equation (B25) equals zero by virtue of the choice of $\hat{u}$.

Lemma B8 (the effect of the wheat factor in the price mechanism)

$$(B26a)\quad \frac{\partial}{\partial\mu_1}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(xu) > 0 \quad \text{if} \quad F < \tilde{F}(\mu_1).$$

$$(B26b)\quad \frac{\partial}{\partial\mu_1}\iint y(x)^{G-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) > 0 \text{ for any } F \text{ if } \frac{\mu_w}{b} > \bar{\xi}.$$

$$(B26c)\quad \frac{\partial}{\partial\mu_1}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) \le 0 \text{ for } F > \tilde{F}(\mu_1) \text{ if } \frac{\mu_w}{b} <$$

ξ^*.

$\bar{\xi}$ and ξ^* are constants which depend on all other parameters (other than μ_w and b) and such that $\xi^* < \bar{\xi}$.

Proof

(B27) $$\frac{\partial}{\partial\mu_1}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) =$$
$$\iint y(x)^{\gamma-1}\lambda'(q(x,u))(\mu_w+bx-c)(\mu_w+bx)dG(x,u) =$$
$$\mu_w\iint y(x)^{\gamma-1}\lambda'(q(x,u))(\mu_w+bx-c)dG(x,u) +$$
$$b\iint y(x)^{\gamma-1}\lambda'(q(x,u))(\mu_w+bx-c)xdG(x,u).$$

The first terms on the r.h.s. is positive by Lemma (B3). The integral of the second term is negative for $F > \tilde{F}(\mu_1)$ by Lemma (B4).

Note now that if μ_w is sufficiently larger than b, the first term on the r.h.s. of (B27) dominates the second and vice versa.

Lemma B9 (the effect of the all purpose factor in the price mechanism)

(B28a) $$\frac{\partial}{\partial\mu_2}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) > 0 \text{ if } \frac{\mu_z}{d} > \bar{\phi}$$

(B28b) $$\frac{\partial}{\partial\mu_2}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) < 0 \text{ if } \frac{\mu_z}{d} < \phi^*.$$

$\bar{\phi}$ and ϕ^* are constants which depend on all other parameters (other than μ_z and d) and such that $\phi^* < \bar{\phi}$.

Proof

(B29) $$\frac{\partial}{\partial\mu_2}\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)dG(x,u) =$$
$$\iint y(x)^{\gamma-1}\lambda'(q(x,u))(\mu_w+bx-c)[\mu_z+du]dG(x,u) =$$
$$\mu_z\iint y(x)^{\gamma-1}\lambda'(q(x,u))(\mu_w+bx-c)dG(x,u) +$$
$$d\iint y(x)^{\gamma-1}\lambda'(q(x,u))(\mu_w+bx-c)udG(x,u).$$

The first term on the r.h.s. is positive by lemma B3; the second term on the r.h.s. has the sign of (B22*), and thus is negative by Lemma B7. An argument similar to the one made in Lemma B8 completes the proof.

Lemma B10 (the effect of the concavity)

(B30) $$\frac{\partial}{\partial\gamma}\iint y(x)^{\gamma-1}\theta(q(x,u))^{\gamma}(\mu_w+bx-c)dG(x,u) \gtrless 0 \quad \text{if} \quad F \lessgtr F^*(\mu_1)$$
$$F^*(\mu_1) \geq 0; \quad F^*(\mu_1 = 0) = 0 \quad \text{and} \quad \frac{\partial F^*}{\partial\mu_1} > 0.$$

Proof

(B31) $$\frac{\partial}{\partial\gamma}\iint y(x)^{\gamma-1}\theta(q(x,u))^{\gamma}(\mu_w+bx-c)dG(x,u) =$$
$$\iint y(x)^{\gamma-1}\lambda(q(x,u))(\mu_w+bx-c)\ln(y(x)\theta(q(x,u)))dG(x,u).$$

Note that:

(B32) $$\frac{\partial}{\partial u}\ln(y(x)\phi(x,u)) = \frac{\theta'(q)}{\theta(q)}\mu_2 d < 0.$$

The inequality follows from Lemma A3 since $\frac{\theta'(q)}{\theta(q)} < 0$

and:

(B33) $$\frac{\partial}{\partial x}\ln(y(x)\theta(q(x,u))) = b[\frac{F}{y(x)} + \frac{\theta'(q)}{\theta(q)}\mu_1] \lessgtr 0 \quad \text{if} \quad F \lessgtr \bar{F}(\mu_1)$$

where:

(B33) $$\bar{F}(\mu_1) \geq 0; \ \bar{F}(\mu_1 = 0) = 0 \quad \text{and} \quad \frac{\partial \bar{F}}{\partial \mu_1} > 0.$$

Thus, by Lemma A1, equation (B31) is negative for $F < \bar{F}(\mu_1)$. It becomes positive for some $F^*(\mu_1)$.

APPENDIX C: WELFARE ANALYSIS - FORMAL PROOFS

Lemma C1 (the effect of the expected price of wheat)

(C1a) $$\frac{\partial}{\partial \mu_w}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) \lessgtr 0 \quad \text{if} \quad F \lessgtr \bar{F}(\mu_1)$$

(C1b) $$\bar{F}(\mu_1) \geq 0 \quad \bar{F}(\mu_1 = 0) = 0 \ \frac{\partial \bar{F}}{\partial \mu_1} > 0.$$

Proof

(C2) $$\frac{\partial}{\partial \mu_w}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) = \gamma F\iint y(x)^{\gamma-1}\lambda(q(x,u))dG(x,u) + \mu_1\iint y(x)^{\gamma}\lambda'(q(x,u))dG(x,u).$$

The first term on the r.h.s. is positive (negative) if $F > 0$ $(F < 0)$. The second term is negative anywhere. Thus, for $F < 0$ equation (C2) is negative. For $F > 0$, but sufficiently small relative to μ_1 the second term dominates the first and equation (C2) is negative. If $F > 0$ is sufficient large, relative to μ_1 the first term on the r.h.s. dominates the second and equation (C2) is positive. Thus there exists $\bar{F}(\mu_1)$ which is monotone increasing in

μ_1 and equals zero for $\mu_1 = 0$ such that equation (C2) is negative below it and positive above it.

Lemma C2 (the effect of the expected price of z)

(C3) $$\frac{\partial}{\partial \mu_z}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) < 0$$

Proof

(C4) $$\frac{\partial}{\partial \mu_z}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) = \mu_1\iint y(x)^{\gamma}\lambda'(q(x,u))dG(x,u) < 0$$

since $\lambda' < 0$.

Lemma C3 (the effect of the variance of the price of wheat)

(C5a) $$\frac{\partial}{\partial b}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) > 0 \quad \text{if} \quad F^*(\mu_1) \leq F \leq F^{**}(\mu_1)$$

(C5b) $$\frac{\partial}{\partial b}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) < 0 \quad \text{otherwise}$$

where

(C6a) $$F^*(\mu_1) \leq 0; \quad F^*(\mu_1 = 0) = 0; \quad \frac{\partial F^*}{\partial \mu_1} < 0$$

(C6b) $$F^{**}(\mu_1) \geq 0; \quad F^{**}(\mu_1 = 0) = 0; \quad \frac{\partial F^{**}}{\partial \mu_1} > 0$$

Proof

(C7) $$\frac{\partial}{\partial b}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) = \frac{\partial}{\partial b}\int y(x)^{\gamma}\left[\int \lambda(q(x,u))dG^2(u)\right]dG^1(x) =$$
$$\frac{\partial}{\partial b}\int y(x)^{\gamma}\delta(x)dG^1(x) = \gamma F\int y(x)^{\gamma-1}\delta(x)xdG^1(x) + \mu_1\int y(x)^{\gamma}\delta'(x)xdG^1(x)$$

where the second equality is obtained upon substitution of equation (B2). We first sign the first term on the r.h.s.

Note that: (a) $F > 0$ implies that $y(x)^{\gamma-1}\delta(x)$ is monotone decreasing in x and thus the first term on the r.h.s. is negative if $F > 0$.

(b) If $F < 0$ then $y(x)^{\gamma-1}$ is monotone increasing in x and thus if $F < 0$ and $\mu_1 = 0$ (In which case $\delta(x) =$ constant) the first term on the r.h.s. is negative.

(c) If $F < 0$ but sufficiently close to zero and $\mu_1 > 0$, then $y(x)^{\gamma-1}\delta(x)$ is monotone decreasing. In this case the first term on the r.h.s. is positive.

(d) If $F < 0$ but $|F|$ is sufficiently large and $\mu_1 \geq 0$, then $y(x)^{\gamma-1}\delta(x)$ is monotone increasing. In this case the first term on the r.h.s.is negative.

We now turn to the second term on the r.h.s.

Note that: (e) If $F = 0$, $y(x)^{\gamma}$ is a positive constant and thus $y(x)^{\gamma}\delta'(x)$ is monotone increasing. In this case the second term on the r.h.s. is positive. This remains true for $F > 0$ provided F is sufficiently small.

(f) There exists an $F > 0$ sufficiently large such that $y(x)^{\gamma}\delta'(x)$ is monotone decreasing. In this case the second term on the r.h.s. is negative.

(g) If $F < 0$ then $y(x)^{\gamma}\delta'(x)$ is monotone increasing. In this case the second term on the r.h.s.is positive.

(a) - (g) imply equations (C5) and (C6).

Lemma C4 (the effect of the variance of the price of z)

(C8) $$\frac{\partial}{\partial d}\iint y(x)^{\gamma}\lambda(q(x,u))\,dG(x,u) > 0.$$

Proof

(C9) $\frac{\partial}{\partial d}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) = \mu_2\int y(x)^{\gamma}[\int\lambda'(q(x,u))udG^2(u)]dG^1(x) > 0.$

The term in the square brackets is positive by Lemma A2 because $\lambda'(x,u)$ is monotone increasing in u. Hence equation (C9) is positive anywhere.

Lemma C5 (the effect of μ_1 - the wheat factor)

(C10a) $\frac{\partial}{\mu_1}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) < 0$ if $c < N$

where c is the cost of F

(C10b) $\frac{\partial}{\mu_1}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) < 0$ anywhere for $\frac{\mu_w}{b} > T$

(C10c) $\frac{\partial}{\mu_1}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) > 0$ for $c > N$ if $\frac{\mu_w}{b} < S$

where

(C11) N, T and $S < T$ are constants.

Proof

(C11)
$$\frac{\partial}{\mu_1}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) = \frac{\partial}{\partial\mu_1}\int y(x)^{\gamma}[\int\lambda(q(x,u))dG^2(u)]dG^1(x) =$$
$$\frac{\partial}{\partial\mu_1}\int y(x)^{\gamma}\delta(x)dG^1(x) = \int y(x)^{\gamma}\delta'(x)(\mu_w+bx)dG^1(x) =$$
$$\mu_w\int y(x)^{\gamma}\delta'(x)dG^1(x) + b\int y(x)^{\gamma}\delta'(x)xdG^1(x).$$

The second equality is obtained upon substitution of (B2). The integral of the first term on the r.h.s. is negative anywhere. The integral of the second term is identical to the integral of the second term on the r.h.s. of equation (C7). Thus it is characterized by (e), (f) and (g) described in Lemma C3 (it is negative only for large values of F). Note next that if μ_w is sufficiently larger than b, the first term on the r.h.s. of equation (C11) dominates the second and vice versa. Finally note that F is large if c is small. This completes the proof.

Lemma C6 (the effect of μ_2 - the all-purpose factor)

(C12a) $$\frac{\partial}{\partial\mu_2}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) < 0 \quad \text{if} \quad \frac{u_z}{d} < S$$

(C12b) $$\frac{\partial}{\partial\mu_2}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) > 0 \quad \text{if} \quad \frac{u_z}{d} > T$$

where T and S are boundaries such that T > S.

Proof

(C13) $$\frac{\partial}{\partial\mu_2}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) =$$
$$\mu_z\iint y(x)^{\gamma}\lambda'(q(x,u))dG(x,u) + d\int y(x)^{\gamma}\left[\int\lambda'(q(x,u))udG^2(u)\right]dG^1(x).$$

The first term on the r.h.s. is negative because $\lambda' < 0$. The term in the square brackets of the second term on the r.h.s. is positive because $\lambda'(x,u)$ is monotone increasing. Thus, the second term is positive. Note that if μ_z is sufficiently larger than d the first term dominates the second and vice versa.

Lemma C7

(C14) $$\frac{\partial}{\partial c}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) \lessgtr 0 \quad \text{if} \quad c \gtrless \tilde{c}.$$

Proof

$$\text{(C15)}\ \frac{\partial}{\partial c}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) = -\gamma F\int y(x)^{\gamma-1}\lambda(q(x,u))dG(x,u) \gtrless 0 \quad \text{if} \quad F \lessgtr 0.$$

Note that $\frac{\partial F}{\partial c} < 0$ if the first-order conditions are satisfied. Thus (C15) implies

$$\text{(C16)} \qquad \frac{\partial}{\partial c}\iint y(x)^{\gamma}\lambda(q(x,u))dG(x,u) \gtrless 0 \quad \text{if} \quad c \gtrless \tilde{c}$$

where $\tilde{c}$ is the cost such that the quantity demanded of F (eq.(9)) is zero (i.e. $F(\tilde{c},\ldots,) = 0$).

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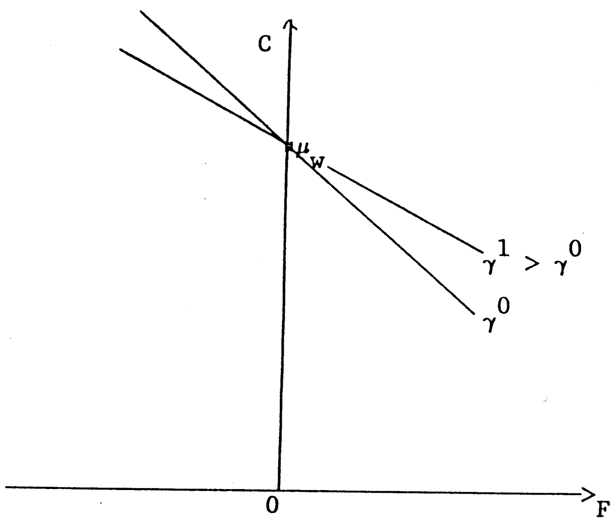
FIGURE 1

The Effect of the Degree of the Concavity (γ)

Case a: $\mu_1 = 0$



Case b: $\mu_1 > 0$

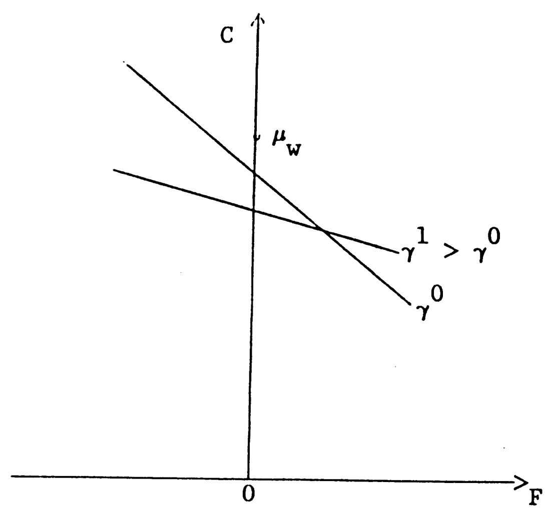
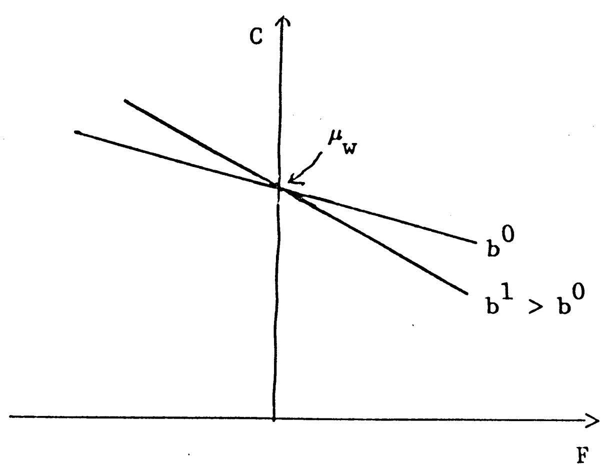


FIGURE 2

The Effect of the Variance of p_w

Case a: $\mu_1 = 0$



Case b: $\mu_1 > 0$

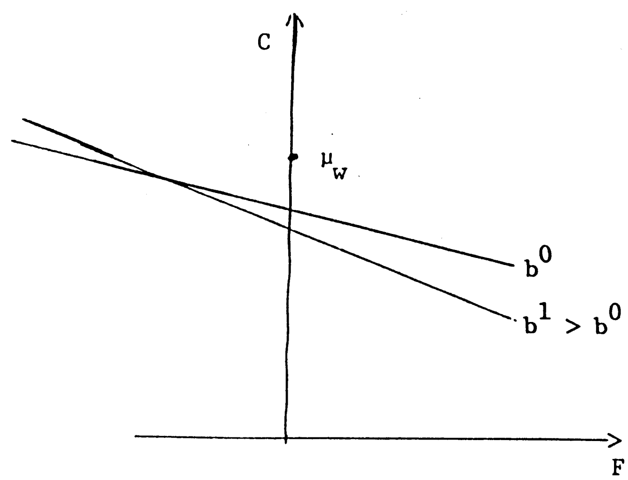


FIGURE 3

The effect of the expected value of P_w

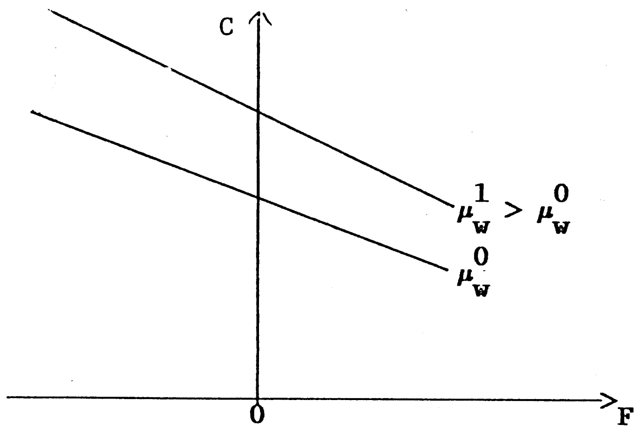


FIGURE 4

The effect of the mean of P_w on the expected utility

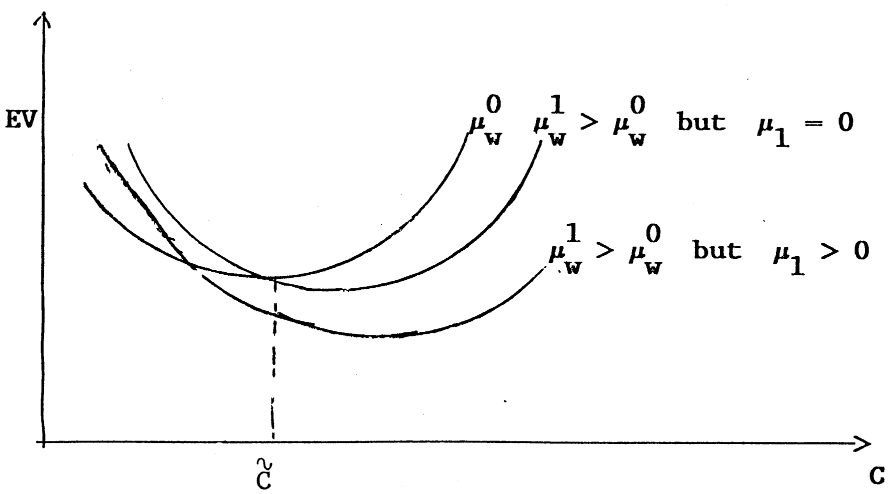


FIGURE 5

The effect of the variance of P_w on the expected utility