DYNAMIC EFFICIENCY IN OVERLAPPING GENERATIONS MODELS WITH STOCHASTIC PRODUCTION

by

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Working Paper No.23-87

September 1987

* I would like to thank B.Peleg for some helpful discussions

This research was supported by funds granted to the Foerder Institute for Economic Research by the **JOHN RAUCH FUND**

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INTRODUCTION

There has been considerable literature in economic theory dealing with efficiency of intertemporal allocation of resources over time. Characterization of dynamically efficient programs, using efficiency prices, has been a central issue in an infinite horizon economies with production and consumption over time. In the growth models case let us mention few, out of many, such contributions by Malinvaud [1953], Cass and Yaari [1971], Cass [1972], Peleg [1972], Majumdar]1972], Benveniste and Gale [1975] and Mitra [1979]. In models with overlapping generations (OLG) efficient competitive equilibria have been characterized by Okuno and Zilcha [1980], Balasko and Shell [1981] and others. However, perhaps surprisingly, very little has been done in generalizing these results characterizing efficient allocations to stochastic dynamic models. Peleg [1974] studied Malinvand prices in multisector growth model with finite number of states of nature. Peled [1982, 1984] has discussed (conditional) optimality of equilibria in a stationary OLG model. Zilcha [1984] has applied some results about efficient random variables to a stochastic growth model. More recently Abel Mankiw Summers and Zeckhauser [1986] discuss the issue of dynamic efficiency in a stochastic version of Diamond's [1965] model. Given the utility functions of all generations, they have obtained a condition that guarantees dynamic inefficiency and one sufficient for dynamic efficiency. However, as we see in this work, their condition about "overinvestment", for example, is far too strong from that characterizing dynamic inefficiency.

Our model is an overlapping generations where consumers live for two periods, as in Diamond [1965], with random production. Thus, investment and consumption over time are random variables. Our concern is with the issue of overaccumulation of capital and not the intergenerational risk sharing. We define two types of efficient production-consumption allocations. The first criterion uses first degree stochastic dominance, while the second efficiency criterion uses the second degree stochastic dominance. We obtain a complete characterization of dynamic inefficiency: The interest rates $r_t(\omega)$ should be below the population growth rate n "most of the time" with positive probability. In particular, the (stochastic) future value of a unit of capital in period 0 goes "fast" to zero with positive probability.

Diamond [1965] showed that a competitive economy could reach a steady state in which the population growth rate exceeds the steady state marginal product of capital, i.e. dynamically inefficient equilibrium. However, we demonstrate here that any given efficient (of type II) production consumption allocation can be obtained as a competitive equilibrium for some risk averse individuals in each generation. We also prove existence of competitive equilibria from a given initial capital stocks and show that each competitive equilibrium is short-run efficient.

Our research has been motivated by the feeling that the characterization of efficiency in stochastic models of OLG (with production) is a significant tool in analyzing many economic problems. OLG models have been used in the literature in studying the effects of fiscal policies, pricing of capital assets, etc. For example, it was shown by Tirole [1985] that dynamic efficiency rules out the possiblity that speculative bubbles arise as rational expectations equilibria. Also, the Ricardian equivalence theorem (Barro [1974]) fails to hold when an equilibrium is dynamically inefficient (see Weil [1985]).

The paper is organized as follows. Section 2 contains notations and definitions. In section 3 we describe the economy. A complete characterization of inefficiency is brought in section 4. Existence of competitive equilibria and the relationship between competitive equilibria and efficiency of type II appear in section 5. All the proofs are in section 6.

2. NOTATIONS AND PRELIMINARIES

Let R^n be the n-dimensional Euclidean space. For $x, y \in R^n$ $x \geq y$ iff $x_i \geq y_i$ for all i. $x > y$ if $x \geq y$ and $x \neq y$, $x >> y$ if $x_i > y_i$ for $1 \leq i \leq n$. Let $I = [\alpha, \beta]$ where $0 < \alpha < \beta < \infty$, and let μ be the Lebesgue measure on I. Define $\Omega = X_{k=0}^{\infty} I_k$ where $I_k = I$ for all k. Denote by $\mathcal{F}$ the Borel sigma-field on Ω and let σ be a probability measure on sequences in Ω, i.e. $\mathcal{F}$ is the sigma-field generated by cylinder sets in Ω. Let $\mathcal{F}_t$ be the sigma-field generated by all the cylinder sets $X_{k=0}^{\Omega} B_k$ where $B_k = I$ for all $k > t$. $L^1(\Omega, \mathcal{F}, \sigma)$ is the set of all integrable function $g(\omega)$ from Ω into R^1. Let $L_t^1(\Omega, \mathcal{F}_t, \sigma)$, denoted as L_t, be the set of all integrable functions which are $\mathcal{F}_t$-measurable. Thus L_t is the set of all integrable functions $g(\omega)$ which depend upon the first t coordinates of $\omega = (\omega_0, \omega_1, \ldots)$. L_t^+ stands for the non-negative functions in L_t. A sequence $\{g_k(\omega)\}_{k=0}^{\infty}$ is an **adapted** stochastic process if $g_k \in L_k$ for $k = 0,1,\ldots$.

Let E be the expected value operator. For $g \in L_t$ and $k < t$ $E_k g(\omega) = E[g(\omega)|\mathcal{F}_{k-1}]$ i.e. the expectation with respect to $(\omega_k, \ldots, \omega_t)$; Thus $E_k g \in L_{k-1}$.

Let $g \in L_k$. We write $g > 0$ if $g(\omega) \geq 0$ almost surely (a.s.), and $g \neq 0$. $g >> 0$ if $g(\omega) > 0$ a.s.

Denote by U^1 the set of all continuous nondecreasing functions from R^2 to R^1. U^2 is the subset of U^1 which contains all the **concave** functions.

Let $(X_1, X_2) \in L_t^+ \times L_{t+1}^+$, $(Y_1, Y_2) \in L_t^+ \times L_{t+1}^+$. We say that (X_1, X_2) **dominates** (Y_1, Y_2) in the **first degree stochastic dominance** (FDSD), and denote it by $(X_1, X_2) >_1 (Y_1, Y_2)$ if

(1) $\quad E_t v(X_1(\omega), X_2(\omega)) \geq E_t v(Y_1(\omega), Y_2(\omega))$ a.s. $\forall v \in U^1$ and

(2) $\quad E_t \hat{v}(X_1(\omega), X_2(\omega)) > E_t \hat{v}(Y_1(\omega), Y_2(\omega))$ on some set of positive measure and some $\hat{v} \in U^1$.

If only (1) holds we write $(X_1, X_2) \geq_1 (Y_1, Y_2)$. We say that (X_1, X_2) **dominates** (Y_1, Y_2) in the **second degree stochastic dominance** (SDSD), and denote it by $(X_1, X_2) >_2 (Y_1, Y_2)$ if

(3) $\quad E_t u(X_1(\omega), X_2(\omega)) \geq E_t u(Y_1(\omega), Y_2(\omega))$ a.s. $\forall u \in U^2$ and

(4) $\quad E_t \hat{u}(X_1(\omega), X_2(\omega)) > E_t \hat{u}(Y_1(\omega), Y_2(\omega))$ on some set of positive measure and some $\hat{u}$ in U^2.

If only (3) holds we write $(X_1, X_2) \geq_2 (Y_1, Y_2)$. Thus all risk averse decision makers who are given any realization of $(\tilde{\omega}_0, \ldots, \tilde{\omega}_{t-1})$ (except a set of measure zero) either prefer (X_1, X_2) to (Y_1, Y_2) or are indifferent; and at least one risk-averse decision maker prefers (X_1, X_2) for a set of positive measure of histories $(\omega_0, \ldots, \omega_{t-1})$. We shall also use stochastic dominance for one-dimensional random variables. Let X and Y belong to L_t. We write $X \geq_1 Y$ if $E_t\, v(X(\omega)) \geq E_t v(Y(\omega))$ a.s. for all continuous nondecreasing functions v, $v : R^1 \to R^1$. Namely if $X >_1 Y$ then for each given history $(\omega_0, \ldots, \omega_{t-1})$, $X(\omega)$ dominates $Y(\omega)$ with respect to the random ω_t.

3. DESCRIPTION OF THE ECONOMY

Our model is basically a stochastic version of Diamond's [1965] overlapping generations economy with production. The discrete-time economy starts at period 0 and has indefinite horizon. There is consumption and production in each date where the aggregate production function is assumed to exhibit constant returns-to-scale and it is subject to random shocks. Let $F: R^3_+ \to R^1_+$, where $F(K_t, L_t, \omega_t)$ represents the output at the end of period t, K_t is the capital stock invested at the beginning of this period, L_t is the labor input and ω_t is a random variable representing the state of the environment at date t. There is a perishable homogeneous good which can be either consumed or used as productive capital input. In each period t there are N(t) identical individuals born at date t (hence called generation t, G_t) and who live for two periods t and t+1. Each member of G_t is endowed with one unit of labor (supplied inelastically) in the first period of his lifetime and has no labor endowment in his second period, the retirement period, where he consumes his savings. At period 0 there are members of generation G_{-1} which are engaged in consumption only in date 0, however their consumption may depend upon the state of nature ω_0. The population growth rate is assumed to be a constant n; hence the total labor supply at date t is $L_t = L_0(1+n)^t$.

Since our analysis is unaffected by this constant n we shall assume that $n=0$, i.e. no population growth. Also, to simplify the model, we assume that the capital depreciation rate is 1, hence $F(K_t, L_t; \omega_t)$ is the total stock of capital at the end of period t. The production function satisfies the following assumptions,

(A1) $F(K,L; \omega_t)$ is homogeneous of degree 1, concave in (K,L), and twice continuously differentiable in K, L for all ω_t; $F(0,L; \omega_t) = F(K,0; \omega_t) = 0$, $F_1 > 0$, $F_2 > 0$, $F_{KK} < 0$, $F_{LL} < 0$ $F_1(0,L; \omega_t) = \infty$ and $F_1(\infty,L; \omega_t) = 0$ for all ω_t. F and F_1 are continuous in ω_t.

Let $f(k_t, \omega_t)$ be the **per-capita** production function i.e. $f(k_t, \omega_t) = F(\frac{K_t}{L_t}, 1; \omega_t)$. From (A1) we see that $f' = \frac{\partial f}{\partial k} > 0$, $f'' < 0$, $f'(0,\omega_t) = \infty$ and $f'(\infty, \omega_t) = 0$ for all ω_t. The random shocks to production are given by a sequence of random variables $\{\tilde{\omega}_t\}_{t=0}^{\infty}$ where each $\tilde{\omega}_t$ assumes values in the interval $[\alpha,\beta] = I$, $0 < \alpha < \beta < \infty$. The probability measure σ over the sequences $\omega = (\omega_0, \omega_1, \omega_2, \ldots) \in \Omega$ is known and it satisfies,

(A2) For any $A \subset \Omega$, $A = \Pi_{i=0}^{\infty} A_i$, where $\mu(A_i) > 0$ for $i = 0,\ldots,k$ and $A_i = I$ for $i > k$, then $\sigma(A) > 0$. σ is a nonatomic measure.

(A2) is a mild assumption and it holds, for example, if $\tilde{\omega}_t$ $t = 0,1,\ldots$ are i.i.d.

Given the per-capita capital stock at the outset of period 0, k_0^*, a **feasible** (per-capita) **production consumption allocation** (FPCA) from k_0^* is a consumption c^0_{-1} for the G_{-1} members and a sequence $\{k_t, (c^y_t, c^0_t)\}^\infty_{t=0}$, where k_t is the aggregate capital stock at date t, (c^y_t, c^0_t) the (per-capita) consumption allocation for G_t, which satisfies:

(a) $c^0_{-1} \in L^+_0$, $k_{t+1} \in L^+_t$, $c^y_t \in L^+_t$ and $c^0_t \in L^+_{t+1}$ for $t = 0,1,2,\ldots$.

(b) $k_{t+1} + c^y_t + c^0_{t-1} = f(k_t, \omega_t)$ a.s. for $t = 0,1,\ldots$.

Condition (a) indicates that $k_{t+1}(\omega)$ and $c^y_t(\omega)$ depend upon the history of the states of the environment from period 0 to period t only, while $c^0_t(\omega)$ depends upon the realizations of ω_τ from $\tau = 0$ up till $\tau = t+1$ (since this consumption takes place at date $t+1$). Condition (b) is a material balance condition. Denote by $P(k^*_0)$ the set of all FPCA from initial (per-capita) capital k^*_0. For a given $<c^0_{-1}, \{k_t, (c^y_t, c^0_t)\}^\infty_{t=0}>$ in $P(k^*_0)$ define the aggregate consumption

$$c_t = c^y_t + c^o_{t-1} \qquad t = 0,1,2,\ldots ,$$

and denote by $\underset{\sim}{k} = (k^*_0, k_1, k_2, \ldots)$, $\underset{\sim}{c} = (c_0, c_1, c_2, \ldots)$. Thus, $\underset{\sim}{c}$ is the aggregate consumption program corresponding to this given FPCA. A FPCA is called **interior** if its aggregate consumption $\underset{\sim}{c}$ satisfies $c_t >> 0$ for all t. Note that this implies also that $k_t >> 0$ for all t.

Given two FPCAs from initial capital k_0, we say that $<c^0_{-1}, \{k_t, (c^y_t, c^0_t)\}>$ **dominates** $<\bar{c}^0_{-1}, \{\bar{k}_t, (\bar{c}^y_t, \bar{c}^0_t)\}>$ in the **first degree stochastic dominance** if $c^0_{-1} \geq_1 \bar{c}^0_{-1}$, $(c^y_t, c^0_t) \geq_1 (\bar{c}^y_t, \bar{c}^0_t)$ for $t = 0,1,2,\ldots$ and for some τ we have strict $>_1$. A FPCA in $P(k_0)$ is **efficient of type I** if it is not dominated in the FDSD by any other FPCA in $P(k_0)$. Similarly $<c^0_{-1}, \{k_t, (c^y_t, c^0_t)\}>$ **dominates** $<\bar{c}^0_{-1}, \{\bar{k}_t, (\bar{c}^y_t, \bar{c}^0_t)\}>$ in the **second degree stochastic**

dominance if $c^0_{-1} \geq_2 \bar{c}^0_{-1}$, for all t $(c^y_t, c^0_t) \geq_2 (\bar{c}^y_t, \bar{c}^0_t)$ and for some τ $(c^y_\tau, c^0_\tau) >_2 (\bar{c}^y_\tau, \bar{c}^0_\tau)$. A FPCA $<c^0_{-1}, \{k_t, (c^y_t, c^0_t)\}>$ in $P(k_0)$ is called **efficient of type II** if it is not dominated in the SDSD by any other FPCA in $P(k_0)$. Since $U^2 \subset U^1$ it is easy to see that any type II efficient FPCA is also efficient of type I.

We shall use in the sequel **efficient** instead of "efficient of type I".

Given a particular preferences for individuals in all generations we can define competitive equilibrium in our economy which generalizes Diamond's [1965] equilibrium concept to this stochastic model. Let the function u_t, $u_t: R^2_+ \to R^1$, represent the preferences of individuals in G_t, $t = 0,1,2,\ldots$. For a given k^*_0 $<c^{0*}_{-1}, \{k^*_t, (c^{y*}_t, c^{0*}_t), r^*_t, W^*_t\}^\infty_{t=0}>$ is a **competitive equilibrium** from k^*_0 if

(5) $c^{0*}_{-1} = k^*_0 f'(k^*_0, \omega_0)$ a.s. .

(6) $<c^{0*}_{-1}, \{k^*_t, (c^{y*}_t, c^{0*}_t)\}>$ is a FPCA from k^*_0.

(7) $1 + r^*_{t+1} = f'(k^*_{t+1}, \omega_{t+1})$ a.s., $t = 0,1,2,\ldots$.

(8) $W^*_t = f(k^*_t, \omega_t) - k^*_t f'(k^*_t, \omega_t)$ a.s., $t = 0,1,\ldots$

and for all generations G_t $t = 0,1,\ldots,$ the solution to the maximiation:

$$(9) \quad \begin{aligned} &\max_{s.t.} \; E_t\, u_t\,(c^y_t(\omega), c^0_t(\omega)) \\ &c^y_t + s_t = W^*_t \quad \text{a.s.} \\ &c^0_t = (1 + r^*_{t+1}) s_t \quad \text{a.s.} \\ &c^y_t \geq 0 \qquad c^0_t \geq 0 \end{aligned}$$

is attained at $(c^{y*}_t(\omega), c^{0*}_t(\omega))$, except for a set of histories $(\omega_0,\ldots,\omega_{t-1})$ of measure zero.

The (stochastic) competitive interest factors are the marginal product of capital in all states of nature. The wages W_t^* are the marginal product of labor in all states of the environment, hence W_t^* is a random variable which depends upon the realizations of $(\tilde{\omega}_0, \ldots, \tilde{\omega}_t)$. Thus in equilibrium the capital stock in each state t equals the aggregate savings by individuals in the previous period, i.e. $k_{t+1}^* = s_t^*$ can be shown using our assumption (A1) about the production functions (s_t^* is the optimal saving of G_t obtained from (9)). Condition (5) guarantees that the older individuals at $t=0$, who invested k_0^* receive the competitive rate of return $f'(k_0^*, \omega_0)$. The material balance condition holds in each date in probability 1 due to (6).

Given an efficient $<c_{-1}^0, \{k_t, (c_t^y, c_t^0)\}_{t=0}^{\infty}>$ in $P(k_0^*)$. Let us define a system of intertemporal profit maximizing (IPM) prices (ψ_t) as follows:

(10) $$\psi_{t-1}(\omega) = f'(k_t^*(\omega), \omega_t)\psi_t(\omega) \quad \text{a.s.}, \quad t = 0,1,2,\ldots .$$

For each t $\psi_t \in L_t^+$ and let us set $\psi_{-1} = 1$. Moreover it follows from the definition that under the prices $\{\psi_t\}$ the intertemporal expected profits are maximized along the path $\underset{\sim}{k}^*$, i.e.

(11) $$E_t[\psi_t f(k_t^*, \omega_t) - \psi_{t-1} k_t^*] \geq E_t[\psi_t f(k, \omega_t) - \psi_{t-1} k]$$
$$\text{a.s.} \quad \text{for all } k \in L_{t-1}^+ \quad t = 0,1,2,\ldots .$$

4. A Complete Characterization of Inefficiency

Let us write some necessary and sufficient conditions for inefficiency of type I.

Lemma 1: Let $<c_{-1}^0, \{k_t, (c_t^y, c_t^0)\}_{t=0}^{\infty}> \in P(k_0)$, then it is inefficient (of type I) if and only if there exists an adapted stochastic process $(\epsilon_t)_{t=\tau}^{\infty}$ such that $\epsilon_t \in L_{t-1}^{+}$ for all $t \geq \tau$ and some positive measure set A, $A \in \mathcal{F}$ $\sigma(A) > 0$, such that:

(12a) $\epsilon_\tau(\omega) > 0$ for all $\omega \in A$ $\epsilon_\tau(\omega) = 0$ for $\omega \notin A$

(12b) $\epsilon_{t+1}(\omega) \geq f(k_t(\omega), \omega_t) - f(k_t(\omega) - \epsilon_t(\omega), \omega_t)$ for $\omega \in A$,

$\epsilon_t(\omega) < k_t(\omega)$ for all $\omega \in A$, while $\epsilon_t(\omega) = 0$ for $\omega \notin A$ $\forall t \geq \tau$.

This lemma is a stochastic version of Cass' [1972] result for deterministic aggregative growth model. We relegate all proofs to the last section.

To derive a complete characteriztaion of inefficiency (of type I) we make the following assumption about the elasticities of the production function and the marginal product,

(A3) There are positive constants m_1, m_2, m_3, m_4 such that for all $k > 0$ and all θ in $[\alpha, \beta]$ the following conditions hold:

$$m_1 \leq \frac{kf'(k,\theta)}{f(k,\theta)} \leq m_2 \quad \text{and} \quad m_3 \leq \frac{-k^2 f''(k,\theta)}{f(k,\theta)} \leq m_4. \tag{13}$$

This is a stochastic version of the assumption made by Benveniste and Gale [1975]. This assumption can be weakened (see Mitra [1979]) without affecting the characterization we obtain. (A3) implies that (see for example Mitra [1979]):

There exists constants $0 < m < M < \infty$ such that for all θ in $[\alpha,\beta]$ and all $0 < \epsilon < k$,

(14) $$\frac{m\epsilon}{k} \leq \frac{f(k,\theta) - f(k-\epsilon,\theta)}{\epsilon f'(k,\theta)} - 1 \leq \frac{M\epsilon}{k}.$$

Let us state our characterization theorem for **interior** feasible production consumption allocations.

Theorem 1: Assume that (A1)-(A3) **hold and let** $<c^0_{-1}, \{k_t, (c^y_t, c^0_t)\}>$ **be an interior FPCA in** $P(k_0)$. **It is inefficient** (of type I) **if and only if for some** A in $\mathcal{F}_\tau$, $\sigma(A) > 0$ **its intertemporal profit maximizing price system** (ψ_t), defined in (10), **satisfy**

(15) $$\Sigma^{\infty}_{t=1}\frac{1}{\psi_t(\omega)k_{t+1}(\omega)} < \infty \text{ for all } \omega \in A.$$

The theorem demonstrates that in this stochastic OLG model when condition (15) holds with positive probability it implies inefficiency of the production-consumption allocation. Moreover (15) implies that the IPM prices should satisfy:

(16) $$\Sigma^{\infty}_{t=1}\frac{1}{\psi_t(\omega)} < \infty \quad \text{on some } A \in \mathcal{F},\ \sigma(A) > 0.$$

Thus (16) generalizes the well-known condition about the prices from the deterministic models (see for example Cass [1972]). Rewriting condition (16) with the interest rates let us define π_t to be the (stochastic) future value of a unit capital in period 0, i.e.

(17) $$\pi_t(\omega) = \Pi^{t-1}_{\tau=0}f'(k_\tau,\omega_\tau) = \Pi^{t}_{\tau=0}(1 + r_\tau(\omega)).$$

Then, Theorem 1 asserts, basically, that an interior feasible program is inefficient (of type I) if and only if the rate at which the future value of a unit of capital in period 0 goes to zero "fast", on a set of ω of positive measure i.e. for some A, $\sigma(A) > 0$.

(18) $\Sigma_{t=0}^{\infty}[\Pi_{\tau=0}^{t}(1+r_{\tau}(\omega))] < \infty$ for all $\omega \in A$.

Thus if, for some interior program the interest rates $r_t(\omega)$ converge, as $t \to \infty$, "fast" to the rate of population growth (which is assumed to be 0) in probability 1 then it is efficient.

5. **COMPETITIVE EQUILIBRIA AND EFFICIENCY OF TYPE II.**

Let us show first the existence of competitive equilibrium from $k_0^* > 0$ when the given utility functions $\{u_t\}_{t=0}^{\infty}$ are concave. We add the following technical assumption in order to simplify the existence proof.

Assumption 4: For all t u_t satisfies:

$$\frac{\partial u_t}{\partial c_t^y}(0,b) = \infty \quad \text{and} \quad \frac{\partial u_t}{\partial c_t^0}(b,0) = \infty \quad \text{for all } b > 0.$$

Theorem 2: Assume that (A.1)-(A.4) hold and that the given utility function for each generation t u_t is continuous increasing and concave. Then there exists a competitive equilibrium from each initial capital $k_0^* > 0$.

Now it is shown that when the utility functions are concave any competitive equilibrium which is efficient of type I is also efficient of type II. Thus the characterization of type I inefficiency (in Theorem 1) is a characterization of type II inefficiency when competitive equilibrium allocations are considered, and $u_t \in U^2$ for all t.

Theorem 3: Under (A.1)-(A.3) any competitive equilibrium which is efficient of type I is also efficient of type II if each utility function u_t is in U^2, for $t = 0,1,2,\ldots$.

It has been indicated earlier, in a deterministic models (see for example

Diamond [1965]) that in a **steady state** equilibrium where the population growth factor exceeds the marginal product of capital the equilibrium is inefficient. Since equilibrium allocations may be inefficient, let us show now that a competitive allocation is **short-run** efficient, i.e.:

Definition: A FPCA $< c^{0*}_{-1}, \{k^*_t, (c^{y*}_t, c^{0*}_t)\}>$ in $P(k^*_0)$ is **short-run efficient** (of type II) if for all T, T = 0,1,2,..., **there exists no** $<c^0_{-1}, \{k_t, (c^y_t, c^0_t)\}>$ in $P(k^*_0)$ such that: $c^0_{-1} \geq_2 c^{0*}_{-1}$, $(c^y_t, c^0_t) \geq_2 (c^{y*}_t, c^{0*}_t)$ for $t = 0,\ldots,T-1$ and $c^y_T \geq c^{y*}_T$, $k_{T+1} \geq k^*_{T+1}$, with at least one **strict** inequality.

Theorem 4: Assume that (A.1)-(A.3) hold and that u_t is increasing for all t. Any competitive equilibrium is **short-run efficient** (of type II).

In an overlapping generations model it has been shown that the second theorem of welfare economics holds (under mild assumptions, see for example Okuno-Zilcha [1980] and Balasko-Shell [1981]), i.e. efficient allocations may be attained as competitive equilibria under some tax-subsidies transfers. We show here that a FPCA which is **efficient of type** II (and where G-1 receives the competitive consumption) can be attained as a competitive equilibrium for **some** concave continuous nondecreasing utility functions $\{u_t\}$.

Definition: We say that in a FPCA $<c^0_{-1}, \{k_t, (c^y_t, c^0_t)\} > \in P(k^*_0)$ G_{-1} **is fairly treated** if $c^0_{-1} = k^*_0 f'(k^*_0, \omega_0)$; i.e. if this older generation receives the competitive returns on its saving k^*_0.

Now we state:

Theorem 5: **Assume that (A1)-(A3) hold. For any interior** $\langle c_{-1}^{0*}, \{k_t^*, (c_t^{y*}, c_t^{0*})\}\rangle$ **in** $P(k_0^*)$ **which is efficient of type II and where** G_{-1} **is fairly-treated, there exist continuous nondecreasing and concave utility functions** $\{u_t\}_{t=0}^{\infty}$ **such that** $\langle c_{-1}^{0*}, \{k_t^*, (c_t^{y*}, c_t^{0*})\}, \{r_t^*\}, \{W_t^*\}\rangle$ **is a competitive equilibrium, where the interest rates and wages are determined from** $\{k_t^*\}_{t=0}^{\infty}$ **by equations (7) and (8).**

6. Proofs.

Proof of Lemma 1: Define for $k_0^* > 0$,

(19) $$\bar{P}(k_0^*) = \{(\underset{\sim}{k},\underset{\sim}{c}) \mid \underset{\sim}{k} = (k_0^*, k_1, \ldots),\ \underset{\sim}{c} = (c_0, c_1, c_2, \ldots)$$

where $k_t \in L_{t-1}^+$, $c_t \in L_t^+$ and $c_t + k_{t+1} = f(k_t, \omega_t)$ a.s.

for $t = 0,1,2,\ldots, k_0 = k_0^*\}$.

Let us show first that the $(\underset{\sim}{k},\underset{\sim}{c})$ corresponding to the given FPCA is inefficient in $\bar{P}(k_0)$ if and only if $\langle c_{-1}^0, \{k_t, (c_t^y, c_t^0)\}\rangle$ is inefficient in $P(k_0)$.

Assume that $(\underset{\sim}{k},\underset{\sim}{c})$ is inefficient in $\bar{P}(k_0)$ while $\langle c_{-1}^0, \{k_t, (c_t^y, c_t^0)\}\rangle$ is efficient. Then for some $(\underset{\sim}{\hat{k}},\underset{\sim}{\hat{c}})$ in $\bar{P}(k_0)$, $\hat{c}_t \geq_1 c_t$ for all t and $\hat{c}_\tau >_1 c_\tau$ for some τ. Note that if τ is the first date where $\hat{c}_\tau$ strictly dominates c_τ then $\hat{c}_t = c_t$ for all $t < \tau$ (since whenever $\hat{c}_t \geq_1 c_t$ but $\hat{c}_t \not>_1 c_t$ we must have $\int u(\hat{c}_t) = \int u(c_t)$ for all nondecreasing and concave u, $u:R^1 \to R^1$). Hence without loss of generality we can assume that $\hat{c}_{-1}^0 = c_{-1}^0$ and $\hat{c}_0 >_1 c_0$. Since $\hat{c}_0 = \hat{c}_{-1}^0 + \hat{c}_0^y >_1 c_0 = c_{-1}^0 + c_0^y$ we obtain that $\hat{c}_0^y$

$>_1 c_0^{y*}$. Now we can choose $\hat{c}_0^0 \in L_1^+$ which satisfies: $\hat{c}_0^0 \le c_0^0$ a.s., $\hat{c}_0^0 \ne c_0^0$, and $(\hat{c}_0^y, \hat{c}_0^0) >_1 (c_0^y, c_0^0)$. Thus $\hat{c}_1^y - \hat{c}_1 - c_0^0 >_1 c_1 - c_0^0 - c_1^y$. Again we choose $\hat{c}_1^0$ which satisfies $\hat{c}_1^0 \le c_1^0$ a.s. and also $(\hat{c}_1^y, \hat{c}_1^0) \ge_1 (c_1^y, c_1^0)$. Continuing this process indefinitely (perhaps taking equalities from some period on) results in contradiction to the efficiency of $<c_{-1}^0, \{k_t, (c_t^y, c_t^0)\}>$ in $P(k_0)$.

Now suppose that $(\underset{\sim}{k}, \underset{\sim}{c})$ is efficient in $\bar{P}(k_0)$ while $<c_{-1}^0, \{k_t, (c_t^y, c_t^0)\}>$ is inefficient in $P(k_0)$. Therefore, it is easy to see that the following set is **not** empty,

$$Q = \{<c_{-1}^{0*}, \{k_t^*, (c_t^{y*}, c_t^{0*})\}> \in P(k_0) \mid c_{-1}^{0*} = c_{-1}^0 \text{ and } (c_t^{y*}, c_t^{0*}) \ge_1 (c_t^y, c_t^0) \text{ for all } t \ge 0 \text{ with some strict inequality}\}.$$

Moreover, Q contains some $<c_{-1}^{0*}, \{k_t^*, (c_t^{y*}, c_t^{0*})\}>$ which satisfies: $c_t^{y*} \ge_1 c_t^y$ and $c_t^{0*} \ge_1 c_t^0$ for $t = 0,1,2,\ldots$ with at least one strict FDSD for some τ. Take $(c_0^{y*}, c_0^{0*}) >_1 (c_0^y, c_0^0)$ and let us assume that $c_0^{y*} \ge_1 c_0^y$ and $c_0^{0*} >_1 c_0^0$, while $(c_t^{y*}, c_t^{0*}) = (c_t^y, c_t^0)$ for $t = 1,2,3,\ldots$. Thus, for some set of positive measure B and some $\bar{\theta} > 0$ small enough we can guarantee that $c_0^{0*}(\omega) - \bar{\theta}\chi_B \ge_1 c_0^0$ (where χ_B is the characteristic function of B). This implies that $c_0^* \ge_1 c_0$, $c_1^* >_1 c_1$ while $c_t^* = c_t$ for $t > 1$, which is in contradiction to the efficiency of $(\underset{\sim}{k}, \underset{\sim}{c})$ in $\bar{P}(k_0)$.

Now let us prove the necessary part of the Lemma. Assume that $(\underset{\sim}{k}, \underset{\sim}{c})$ is inefficient in $\bar{P}(k_0)$. Then there exists some $(\underset{\sim}{k}^*, \underset{\sim}{c}^*)$ in $\bar{P}(k_0)$ which dominates $(\underset{\sim}{k}, \underset{\sim}{c})$, i.e. $c_t^* \ge_1 c_t$ for all t with strict dominance for some dates. Let τ be the smallest t for which $c_t^* >_1 c_t$. Thus, for any $t < \tau$

we have $c_t^* = c_t$. This implies easily that for any $t \le \tau$ we must have $k_t^* = k_t$ since both programs start from the same k_0. Now, for $t = \tau$ we have

$$f(k_\tau^*, \omega_\tau) - k_{\tau+1}^* >_1 f(k_\tau, \omega_\tau) - k_{\tau+1}.$$

Hence $k_{\tau+1} >_1 k_{\tau+1}^*$. Thus, on some set A of positive measure (in $\mathcal{F}$) we have $k_{\tau+1}(\omega) > k_{\tau+1}^*(\omega)$ for all $\omega \in A$. Now define

$\epsilon_{\tau+1}(\omega) = k_{\tau+1}(\omega) - k_{\tau+1}^*(\omega)$ for all $\omega \in A$ and

$\epsilon_{\tau+1}(\omega) = 0$ for $\omega \notin A$.

For $t > \tau+1$ define,

$$\epsilon_{t+1}(\omega) = f(k_t(\omega), \omega_t) - f(k_t(\omega) - \epsilon_t(\omega), \omega_t) \quad \text{for all } \omega.$$

(Sufficiency). Given the adapted stochastic process $(\epsilon_t)_{t=\tau}^{\infty}$ as in the Lemma. Define the following $(\hat{\underset{\sim}{k}}, \hat{\underset{\sim}{c}})$ in $P(k_0)$ as follows: $\hat{k}_t = k_t$ for all $t < \tau$, $\hat{k}_t = k_t - \epsilon_t$ for all $t \ge \tau$. $\hat{c}_t = f(\hat{k}_{t-1}, \omega_t) - \hat{k}_t$ for all t, (where $\hat{k}_{-1} = k_0$). Now clearly $\hat{c}_t = c_t$ for all $t < \tau$, and for $t \ge \tau$

$$\hat{c}_{t+1} = f(k_t - \epsilon_t, \omega_t) - (k_{t+1} - \epsilon_{t+1}) \ge f(k_t,\omega_t) - k_{t+1} = c_{t+1}.$$

For $t = \tau$ we have $\hat{c}_{t+1}(\omega) > c_{t+1}(\omega)$ for all $\omega \in A$. Thus $(\hat{\underset{\sim}{k}},\hat{\underset{\sim}{c}}) \in \bar{P}(k_0)$ dominates $(\underset{\sim}{k},\underset{\sim}{c})$, i.e. $(\underset{\sim}{k},\underset{\sim}{c})$ is inefficient in $\bar{P}(k_0)$.

Proof of Theorem 1: By Lemma 1, if $(\underset{\sim}{k},\underset{\sim}{c})$ is inefficient, there is a sequence of adapted random variables $(\epsilon_t)_{t=\tau}^{\infty}$ which satisfy conditions (12) for all t. Without loss of generality let $\tau = 1$ and for $t \ge 1$, assume that,

(20) $$\epsilon_{t+1} = f(k_t, \omega_t) - f(k_t - \epsilon_t, \omega_t) \quad \text{a.s.} .$$

Since $\epsilon_t(\omega) > 0$ if and only if $\omega \in A$ we can rewrite (20) as follows for all $\omega \in A$,

(21) $$\epsilon_{t+1}\psi_t = \epsilon_t\psi_t f'(k_t,\omega_t)[1 + (\frac{f(k_t,\omega_t)-f(k_t-\epsilon_t,\omega_t)}{\epsilon_t f'(k_t,\omega_t)} -1)].$$

By (A3), or its equivalent form (14), and (10) we obtain

(22) $$\epsilon_{t+1}(\omega)\psi_t(\omega) \geq \epsilon_t(\omega)\psi_{t-1}(\omega)\ (1 + \frac{m\epsilon_t(\omega)}{k_t(\omega)}),\ \omega \in A,\quad t=1,2,\ldots$$

Thus for all ω in A and $t \geq 1$,

$$\frac{1}{\epsilon_{t+1}\ \psi_t} \leq \frac{1}{\epsilon_t\ \psi_{t-1}}(1 + \frac{m\epsilon_t}{k_t})^{-1} =$$

$$= \frac{1}{\epsilon_t\ \psi_{t-1}}(1 - \frac{m\epsilon_t\ /k_t}{1+m\ \epsilon_t\ /k_t}) \leq \frac{1}{\epsilon_t\ \psi_{t-1}} - \frac{m}{1+m}\frac{1}{\psi_{t-1}k_t}.$$

Hence we derive the following inequality

(23) $$\frac{m}{1+m}\ \Sigma_{t=2}^{\infty}\ \frac{1}{\psi_{t-1}(\omega)k_t(\omega)} \leq \frac{1}{\epsilon_1(\omega)\ \psi_0(\omega)} < \infty \text{ for } \omega \in A.$$

which completes the proof of the necessity part of the theorem.

Sufficiency: Let $<c^0_{-1}, \{k_t, (c^y_t, c^0_t)\}> \ \in\ P(k_0)$ be a program which satisfies conditions (15) for all $\omega \in A$, $A \in \mathcal{F}_\tau$ for some τ.

For $\omega \in A$ let $\gamma(\omega) = \min\ \{k_0, \Sigma_{t=1}^{\infty} \frac{1}{\psi_t(\omega)k_{t+1}(\omega)}\}$ and $\gamma(\omega) = 0$ for $\omega \notin A$. Assume that $\tau = 1$ and let δ_1 be in L^+_0 which satisfies for ω in A, $0 < \delta_1(\omega) < \min(\frac{M\gamma(\omega)}{\psi_0(\omega)}, \frac{M}{\psi_0(\omega)k_0}, \frac{1}{2}\ k_0)$, and $\delta_1(\omega) = 0$ for $\omega \notin A$. For each t, $t \geq 1$, define δ_{t+1} in L^+_t as follows: $\delta_{t+1}(\omega) = 0$ for $\omega \notin A$ and for $\omega \in A$ it is defined by

(25) $$\frac{1}{\psi_t(\omega)\delta_{t+1}(\omega)} = \frac{1}{\psi_0(\omega)\delta_1(\omega)} - \Sigma_{\tau=0}^{t}\ \frac{M}{\psi_{\tau-1}(\omega)k_\tau(\omega)}$$

By our choice of δ_1 (25) implies that $\delta_{t+1}(\omega) > 0$ on A. Now

(26) $$\frac{1}{\psi_t\ \delta_{t+1}} = \frac{1}{\psi_{t-1}\delta_t}(1 - \frac{M\delta_t}{k_t}) \leq \frac{1}{\psi_{t-1}\delta_t}(1 + \frac{M\delta_t}{k_t})^{-1} \leq$$

$$\leq \frac{1}{\psi_{t-1}\ \delta_t}[\frac{f(k_t,\omega_t)\ -\ f(k_t-\delta_t,\omega_t)}{\delta_t\ f'(k_t,\ w_t)}]^{-1}$$

Thus from (26) we reach

(27) $$\psi_t(\omega)\delta_{t+1}(\omega) \geq \frac{\psi_{t-1}(\omega)}{f'(k_t,\omega_t)}[f(k_t,\omega_t) - f(k_t - \delta_t, \omega_t)] \quad \text{for } \omega \in A \text{ i.e.}$$

$$\delta_{t+1}(\omega) \geq f(k_t,\omega_t) - f(k_t - \delta_t, \omega_t) \quad \text{for } \omega \in A, \quad t = 1,2,\ldots,.$$

Also it is clear from (25) that for all t $\delta_t(\omega) < k_t(\omega)$ a.s. . Thus, by Lemma 1, $(\tilde{k},\tilde{c})$ is inefficient in $\bar{P}(k_0)$ which proves the theorem.

Proof of Theorem 2: Let us sketch here the proof indicating the main steps only. The existence of a competitive equilibrium here follows from a sequence of generational optimization problems. Given k_0^* the wages for G_0, which are random, are given by $W_0^* = f(k_0^*,\omega_0) - k_0^* f'(k_0^*,\omega_0)$. Hence for each given interest rate r_1, on savings in date 0, G_0 solves the following problem:

$$\max Eu_0(c_0^y, c_0^0) \text{ s.t. } c_0^y(\omega_0) - W_0^*(\omega_0) - s_0(\omega_0) \geq 0 \text{ a.s. and}$$
$$c_0^0(\omega_0, \omega_1) - (1 + r_1(\omega_1))\, s_0(\omega_0) \geq 0 \text{ a.s. .}$$

Writing the first-order condition and guaranteeing at the same time that the optimal s_0^* satisfies $1 + r_1(\omega) = f'(s_0^*, \omega_1)$ a.s. can be achieved by solving the following conditions for s_0^*(E_1 is the expectation with respect to ω_1),

(28) $$E_1\{-u_{01}(W_0^*-s_0, f'(s_0,\omega_1)s_0) + f'(s_0, \omega_1)u_{02}(W_0^*-s_0, f'(s_0,\omega_1)s_0)\} = 0$$
for almost all ω_0.

Let s_0 approach 0. If $s_0 f'(s_0,\omega_1) \to 0$ a.s. then by (A4) the LHS of (28) becomes positive. However, if $s_0 f'(s_0, \omega_1)$ does not converge to 0 on some set of ω_1 of positive measure we also obtain that the LHS of (28) is positive since $f'(0,\omega_1) = \infty$ for all ω_1. Now let $s_0(\omega_0) \to W_0^*(\omega_0)$ i.e. allowing $c_0^y(\omega_0)$ to approach 0, then by (A4) the LHS of (28) becomes negative.

Using the continuity property of all the partial derivatives we see that for almost all ω_0 there is some solution $s_0^*(\omega_0)$ to (28).
When s_0^* is determined $k_1^* = s_0^*$. Hence the next period's wage $W_1^* = f(k_1^*, \omega_1) - f'(k_1^*, \omega_1)k_1^*$ is given and the same process yields $s_1^* = k_2^*$. This way we construct the competitive equilibrium from k_0^*.

Proof of Theorem 3: Otherwise some $<\hat{c}_{-1}^0, \{\hat{k}_t, (\hat{c}_t^y, \hat{c}_t^0)\}>$ in $P(k_0^*)$ dominates in the SDSD the equilibrium allocation; particularly,

$$(29) \qquad E_t u_t(\hat{c}_t^y, \hat{c}_t^0) \geq E_t u_t(c_t^{y*}, c_t^{0*}) \qquad \text{a.s.} \qquad \text{for } t = 0,1,2,\ldots$$

and for some τ we have strict inequality. Without loss of generality let $\tau = 0$. Note that whenever u_t is strictly concave (29) has strict inequality if $(\hat{c}_t^y, \hat{c}_t^0) \neq (c_t^{y*}, c_t^{0*})$. Since $Eu_0(\hat{c}_0^y, \hat{c}_0^0) > Eu_0(c_0^{y*}, c_0^{0*})$ we have (using the equilibrium conditions),

$$E(\hat{c}_0^y + \hat{k}_1) > E(c_0^{y*} + k_1^*) = EW_0^*.$$

Since $\hat{c}_{-1}^0 \geq_2 c_{-1}^{0*}$ we also have $E\,\hat{c}_{-1}^0 \geq E\,c_{-1}^{0*}$. But this implies the following contadiction:

$$Ef(k_0^*, \omega_0) = E(\hat{c}_{-1}^0 + \hat{c}_0^y + \hat{k}_1) > E(c_{-1}^{0*} + c_0^{y*} + k_1^*) = Ef(k_0^*, \omega_0)$$

which proves the Theorem.

Proof of Theorem 4: Given a competitive equilibrium $<c_{-1}^{0*}, \{k_t^*, (c_t^{y*}, c_t^{0*})\}> \in P(k_0^*)$ and assume that for some T and some $<c_{-1}^0, \{k_t, (c_t^y, c_t^0)\}> \in P(k_0^*)$, $c_{-1}^0 \geq_2 c_{-1}^{0*}$ and $(c_t^y, c_t^0) \geq_2 (c_t^{y*}, c_t^{0*})$ for $t = 0,\ldots,T-1$ with strict inequality at $t = \tau < T$. Also $c_T^y \geq c_T^{y*}$ and $k_{T+1} \geq k_{T+1}^*$. Since each u_t is increasing and concave whenever $(c_t^y, c_t^0) \neq (c_t^{y*}, c_t^{0*})$ for some history

$(\omega_0,\ldots,\omega_{t-1})$ we have $E_t u_t(c_t^y, c_t^0) > E_t u_t(c_t^{y*}, c_t^{0*})$. Thus by the equilibrium properties we must have

(30) $E_t\{\psi_t c_t^y + \psi_{t+1} c_t^0\} \geq E_t\{\psi_t c_t^{y*} + \psi_{t+1} c_t^{0*}\}$ a.s. for $t = 0,\ldots,T-1$

and at $t = \tau$ the inequality is strict on a set in $\mathcal{F}_{t-1}$ of positive measure.

Also since either $c_{-1}^0 = c_{-1}^{0*}$ or $c_{-1}^0 >_2 c_{-1}^{0*}$ we have $E\psi_0 c_{-1}^0 \geq E\psi_0 c_{-1}^{0*}$.

Now, using (30) we can write,

$$E\psi_0 c_{-1}^0 + \Sigma_{t=0}^{T-1} E[\psi_t c_t^y + \psi_{t+1} c_t^0] + E\psi_T(c_T^y + k_{T+1}) >$$
$$> E\psi_0 c_{-1}^{0*} + \Sigma_{t=0}^{T-1} E[\psi_t c_t^{y*} + \psi_{t+1} c_t^{0*}] + E\psi_T(c_T^{y*} + k_{T+1}^*).$$

Putting $c_t = c_t^y + c_{t-1}^0$ for $t = 0,\ldots,T$ we obtain,

$$\Sigma_{t=0}^T \psi_t c_t + E\psi_T k_{T+1} > \Sigma_{t=0}^T \psi_t c_t^* + E\psi_T k_{T+1}^*.$$

Substituting for each t $c_t = f(k_t, \omega_t) - k_{t+1}$ and rearranging we come to

$$\psi_{-1}k_0^* + \Sigma_{t=0}^T E[\psi_t f(k_t, \omega_t) - \psi_{t-1}k_t] >$$
$$> \psi_{-1}k_0^* + \Sigma_{t=0}^T E[\psi_t f(k_t^*, \omega_t) - \psi_{t-1}k_t^*]$$

which is a contradiction (see (11)).

Proof of Theorem 5: For each τ, $\tau = 0,1,2,\ldots$, let us define a set $A_\tau \subset L_\tau^+ \times L_{\tau+1}^+$ as follows:

$$A_\tau = \{(c_\tau^y, c_\tau^0) \mid \text{for some } s_\tau \in L_\tau^+ \text{ we have}$$
$$c_\tau^y + s_\tau = f(k_\tau^*, \omega_\tau) - c_{\tau-1}^{0*} = W_\tau^*,\quad c_\tau^0 = f(s_\tau, \omega_{\tau+1}) - W_{\tau+1}^*\}.$$

Claim 1: For all τ $(c_\tau^{y*}, c_\tau^{0*})$ is an efficient of type II in A_τ.

Otherwise if this assertion does not hold for some τ^* we can contradict the fact that $<c_{-1}^{0*}, \{k_t^*, (c_t^{y*}, c_t^{0*})\}>$ is a type II efficient in $P(k_0^*)$.

Claim 2: For each τ A_τ is convex and compact in the weak topology of $L_1(\Omega,\mathcal{F}_\tau,\sigma) \times L_1(\Omega,\mathcal{F}_{\tau+1},\sigma)$.

Proof of Claim 2: The convexity of A_τ is clear since $f(\cdot,\omega_{\tau+1})$ is concave for all $\omega_{\tau+1}$. Since the weak topology on L_1 is metrizable (see Dunford and Schwartz [1964] Theorem V.5.1) it is enough to show that the weak limit of each sequence is in A_τ. Assume that $(c_\tau^{yn}, c_\tau^{0n}) \in A_\tau$ for all n and that it converges (in the weak topology) to $(\hat{c}_\tau^y, \hat{c}_\tau^0)$. Some convex combinations of $\{(c_\tau^{yn}, c_\tau^{0n})\}_{n=1}^\infty$, say $\{(\dot{c}_\tau^{ym}, c_\tau^{0m})\}$, converges in the norm topology of $L_\tau \times L_{\tau+1}$ to $(\hat{c}_\tau^y, \hat{c}_\tau^0)$ (see Dunford and Schwartz [1964] Theorem V.3.14). This convergence is in probability (see Neveu [1965] Theorem II.5.4). Since this sequence is uniformly bounded some subsequence of $\{(c_\tau^{ym}, c_\tau^{0m})\}$ converges almost surely to $(\hat{c}_\tau^y, \hat{c}_\tau^0)$. Since each $(c_\tau^{ym}, c_\tau^{0m}) \in A_\tau$ and A_τ is closed under a.s. convergence we obtain that $(\hat{c}_\tau^y, \hat{c}_\tau^0) \in A_\tau$. But A_τ is a bounded set and as was shown closed in the weak* topology hence it is weak*-compact (see Dunford and Schwartz [1964] Theorem V.4.2). Since all functions in A_τ are uniformly bounded this proves the claim.

Claim 3: Let $(\bar{c}_t^y, \bar{c}_t^0)$ and $(\hat{c}_t^y, \hat{c}_t^0)$ be in A_τ and B in $\mathcal{F}_{\tau+1}$. Define

$$(c_\tau^y, c_\tau^0) = \begin{cases} (\hat{c}_\tau^y, \hat{c}_\tau^0) & \text{on } B \\ (\bar{c}_\tau^y, \bar{c}_\tau^0) & \text{on } \sim B \end{cases}$$

Then $(c_\tau^y, c_\tau^0) \in A_\tau$.

Proof of Claim 3: Let $\bar{s}_\tau$ and $\hat{s}_t$ be the functions corresponding to $(\bar{c}_\tau^y, \bar{c}_\tau^0)$ and to $(\hat{c}_\tau^y, \hat{c}_\tau^0)$ (see the definition of A_τ). Define $s_\tau = \begin{cases} \hat{s}_\tau & \text{on } B \\ \bar{s}_\tau & \text{on } \sim B \end{cases}$.

Then it can be verified directly from definitions that

$$c_\tau^y + s_\tau = f(k_\tau^*, \omega_\tau) - c_{\tau-1}^{0*} \text{ a.s. and}$$

$$c_\tau^0 = f(s_t, \omega_{\tau+1}) - W_{\tau+1}^* \text{ a.s.} \quad \text{Hence } (c_\tau^y, c_\tau^0) \in A_\tau.$$

Thus a "mixture" of any two elements in A_τ belongs also to A_τ. Now it is clear that all the assumptions required for Theorem 1 in Zilcha [1984] hold. Therefore, for some continuous non-decreasing concave function u_τ on R_+^2

$\max E_\tau(c_\tau^y, c_\tau^0)$ on A_τ is attained at $(c_\tau^{y*}, c_\tau^{0*})$. It can be shown that in fact we have: For almost any realization $(\omega_0, \ldots, \omega_{\tau-1})$
$\max\ E_\tau\ u_\tau(c_\tau^y(\omega), c_\tau^0(\omega))$ over A_τ is attained at $(c_\tau^{y*}(\omega), c_\tau^{0*}(\omega))$.
Otherwise we can derive a contraction to the construction of u_τ.
Thus we have obtained this way a sequence of utility function u_τ, $\tau = 0,1,2,\ldots$, which satisfy this property.

Now, maximizing $E_t u_t(c_t^y, c_t^0)$ on A_t is equivalent to

$$\text{(31)} \qquad \underset{s_t}{\text{Max}}\ E_t u_t[f(k_t^*, \omega_t) - c_{t-1}^{0*} - s_t,\ f(s_t, \omega_{t+1}) - W_{t+1}^*].$$

Thus the first-order conditions for this problem holds at (c_t^{y*}, c_t^{0*}), i.e.

$$\text{(32)} \qquad E_{t+1}\{-u_{t1}(c_t^{y*}, c_t^{0*}) + f'(s_t^*, \omega_{t+1})u_{t2}(c_t^{y*}, c_t^{0*})\} = 0 \text{ a.s.}$$

Since $c_t^{y*} = f(k_t^*, \omega_t) - c_{t-1}^{0*} - k_{t+1}^*$ a.s. $s_t^* = k_{t+1}^*$ is a maximizer for problem (31).

Now let us consider the maximization problem

$$\text{(33)} \quad \max_{s_t \geq 0}\ E_t\ u_t[f(k_t^*, \omega_t) - c_{t-1}^{0*} - s_t,\ (1+r_{t+1}^*)s_t]$$

where $1+r_{t+1}^* = f'(k_{t+1}^*, \omega_{t+1})$ a.s.. The necessary and sufficient conditions for interior optimum $\hat{s}_t$ for (33), setting $\hat{c}_t^y = f(k_t^*, \omega_t) - c_{t-1}^{0*} - \hat{s}_t$; $\hat{c}_t^0 = (1+r_{t+1}^*)\hat{s}_t$, are

(34) $E_{t+1}[-u_{t1}(\hat{c}^y_t, \hat{c}^0_t) + (1+r^*_{t+1})u_{t2}(\hat{c}^y_t, \hat{c}^0_t)] = 0$ a.s..

Now by induction on t one can show that the lifetime incomes in problems (31) and in (33) are the competitive wages corresponding to $\underset{\sim}{k}^*$. Assuming that $s^*_{t-1} = k^*_t$ implies that $W^*_t = f(k^*_t, \omega_t) - c^{0*}_{t-1} = f(k^*_t, \omega_t) - (1+r^*_t)k^*_t = f(k^*_t, \omega_t) - k^*_t f'(k^*_t, \omega_t)$ which is the competitive income corresponding to the capital stock k^*_t. For $t = 0$ by assumption $c^{0*}_{-1} = k^*_0 f'(k^*_0, \omega_0)$. Thus $\hat{s}_t = k^*_{t+1}$ is a solution to (34). Also note that

$$c^{0*}_t = s^*_t f'(s^*_t, \omega_{t+1}) = k^*_{t+1} f'(k^*_{t+1}, \omega_{t+1}).$$

This completes the proof that for these (u_t) this given efficient of type II allocation is a competitive equilibrium.

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