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IMPLEMENTATION THEORY IN ECONOMIES WITH INCOMPLETE INFORMATION

by

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by: David Wettstein*

1. Introduction

The theory of mechanisms' design in general and implementation in particular deals with ways by which a society can achieve desired outcomes.

We shall deal with economic environments, the society, or the environment will consist of a set of individuals endowed with initial endowments and preferences. An outcome will be a reallocation of the initial aggregate endowment vector to the various individuals.

A Social Choice Correspondence (SCC) will map environments into sets of allocations, the set corresponding to a given environment can be thought of as the set of "desired allocations". This set will usually depend on the characteristics of the individuals comprising the environment, their preferences, initial endowments, etc.

The basic problem is how can an outside designer make sure the economy reaches a desired outcome. If the designer had complete information on the characteristics of the environment, he could just tell the individuals what they should do and thus reach a "good" outcome.

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The problems start once the designer does not possess all the relevant information, and has to rely on the individuals to supply it. The straightforward solution would seem to be: Let each individual supply the required information, and then, based on the messages received, go ahead and tell the individuals what they should do. However, realizing their messages determine their fates, the individuals might find it advantageous to give out false information, and then there is nothing to guarantee that the outcome arrived at is indeed a good one.

So we are actually dealing with a game. In the complete information framework for the straightforward way to work we actually need truthtelling to be a Nash Equilibrium (NE) for any possible configuration of characteristics which amounts to requiring that truthtelling is a dominant strategy. This is a very stringent requirement severely limiting the class of SCC's that are implementable.

The basic result on implementation with complete information was proved by Maskin [1977]. He employed the Nash Equilibrium (NE) concept and showed that any SCC satisfying monotonicity and no veto power can be implemented provided the number of individuals exceeds three. Maskin's proof had some flaws in it and a complete proof of the above theorem can be found in Saijo [1985]. Since in economic environments with private goods the no veto power assumption is usually satisfied, we get that a monotonic SCC can be implemented. The proof was carried out by explicitly constructing the desired game.

The next step was trying to define and investigate implementation in the presence of incomplete information. Postlewaite and Schmeidler [1984] defined economies with differential information using the following model: There is a space Ω of possible states of nature, an ω in Ω completely determines all the characteristics of the economy. Each individual has a partition on Ω and knows only what member of his partition occurred. An allocation rule associates with each ω in Ω an allocation of commodities to the various individuals. An SCC is a collection of allocation rules. They have shown that if the number of individuals exceeds three and if Non Exclusivity of Information (NEI) is satisfied then any SCC satisfying a strong condition of monotonicity can be implemented. The solution concept is no longer the NE but the Bayesian Nash Equilibrium (BNE) introduced by Harsanyi [1967-68] for games with incomplete information. NEI is the assumption that any N-1 individuals, where N is the number of individuals, can tell what the true state of nature is if they pool together their information. Their proof is constructive as well.

Similar results were proved by Palfrey and Srivastava [1985]. They succeeded in proving their results with a weaker monotonicity condition. The differences in the phrasing of the results stems in part from the fact that they did not make the assumption that the SCC is closed under common knowledge concatenation, which appeared in Postlewaite and Schmeidler [1984].

A major fault with the mechanisms proposed above is their inherent discontinuity, a slight change, which might be interpreted as a slight mistake, in one's strategy might cause a huge change in the outcome reached. Hence the continuity of the mechanism is very important, especially when dealing with implementation since in reality "mistakes" do occur and a discontinuous mechanism is just not suited to deal with them.

In the incomplete information framework the discontinuity problem assumes an additional aspect. The informational structure in the mechanisms proposed by Postlewaite and Schmeidler [1984] and Palfrey and Srivastava [1985] was discrete so that their mechanism was trivially continuous with respect to the information structure. However, once the information structure is taken to be continuous the mechanism is no longer trivially continuous.

We build a "continuous" version of the above models. The informational structure is made "continuous" by assuming the following form. The state of the economy is determined by the realization of an N-dimensional random variable. The distribution of the random variable is common knowledge as in Aumann [1976]. However, individual i observes only the i-th coordinate of the vector, his preferences depend on the whole realization and his consumption. Various interpretations of the random variable are possible. It could be thought of as a way of parametrizing the utility functions of the various individuals.

This formulation is general enough to include economies with complete information as in Maskin [1977] in which case the support of the random variable is the "diagonal" any single coordinate uniquely determines all the others. In the Postlewaite and Schmeidler framework as in Palfrey and Srivastava, the random variable assumes only a finite number of values.

In our framework NEI is the requirement that the support of the random variable is such that any N-1 coordinates uniquely determine the remaining one. Without loss of generality we assume the coordinates must add up to zero.

In the complete information setup, the individuals are much better informed than is the designer. Using this fact the game forms designed usually try to deduce the preferences of the remaining individual from the strategies of N-1 individuals. In the most general case of incomplete information such a deduction process is infeasible. The N-1 individuals have no way of knowing for sure what the remaining individual observed. The NEI assumption makes such a deduction process still possible. It seems to us implementation theory runs into difficulties without that sort of an assumption.

At this stage it is advisable to sum up what are the informational assumptions in our model ("Who knows what?"). The designer and the individuals know the distribution of the random variable, how the individual characteristics depend on the realization of it, and know the SCC, the designer does not observe any part of the realization while the i-th individual gets to observe the i-th component of the realization. The problem facing the designer is finding a game form implementing the SCC.

In the first part of the paper we show that a monotonicity condition, basically the same one used by Palfrey and Srivastava [1985], is necessary if the implementing game form satisfies certain properties. Then we show that if NEI is satisfied any SCC satisfying monotonicity can be implemented if the number of individuals exceeds three and some technical conditions (continuity and strong monotonicity of preferences) are satisfied. The proof is carried out by constructing an "almost" continuous game form implementing the SCC. Some convexity demands regarding the support of the random variable and the SCC will guarantee continuity.

We do not make the assumption that the SCC is closed under common knowledge concatenation as in Postlewaite and Schmeidler [1984]. Instead we assume the game designed can consist of several stages. The first stage is played before any private information is observed, then the private information is observed and the game continues. The assumption that there is an initial stage where everyone possesses symmetric information somewhat restricts the applicability of the results. However, this assumption can be replaced by the common knowledge concatenation assumption at the cost of further complicating the mechanism.

The second part of the paper introduces the notion of signalling structures. An aspect peculiar to implementation theory in economies with incomplete information is the fact that the designer might be able to change the initial distribution of information among the individuals. This might, of course, alter individual preferences. In the complete information framework one could also let the designer have some means of changing tastes, maybe at some cost, but these issues seem to be more natural in economies with incomplete information.

The game forms designed will basically consist of two main stages. At the first stage individuals act, and based on those acts some signals (possibly different ones for different individuals) are sent out. At the second stage individuals act again, but now, however, their evaluation of outcomes might be altered as a result of the signals received. Then the game form based on all the acts taken determines the outcome.

In this framework we shall define monotonicity in the wide sense (MWS) which is implied by monotonicity, and show that if NEI is satisfied any SCC satisfying MWS can be implemented if the number of individuals exceeds three and the previously mentioned technical assumptions are satisfied.

The game with signalling can be reduced to a game with no signalling by using much larger strategy spaces than the ones used in the games proposed so far. Namely the players will be able to condition part of their acts on some aggregates based on other players' acts. This shows that the theorems concerning the necessity of monotonicity are valid only if one restricts the strategy space so that no conditioning of the type mentioned above can occur.

In the complete information framework much work has been done on implementing the Walrasian, Lindahl and some other market related correspondences - Hurwicz [1979], Schmeidler [1980] and Postlewaite and Wettstein [1983]. We shall show that the SCC induced by Constrained Rational Expectations Equilibria can be implemented by a continuous and feasible mechanism provided that the usual assumptions are satisfied, NEI, continuity, strong monotonicity of preferences and the number of individuals exceeds three. The game form constructed will naturally differ from the general game forms constructed in the previous proofs. It will require much less information on the part of the designer and the individuals. One part of the game will use the game described in Postlewaite and Wettsein [1983]. The Constrained Rational Expectations Equilibria (CREE) differ from the Rational Expectations Equilibria (REE) only on the boundary of the feasible set. We do not discuss the existence of REE, a discussion of this and many other issues with all the relevant references can be found in Jordan and Radner [1982].

The paper is organized as follows. In the first section several notations and some terminology are introduced. In the next section we prove the two theorems concerning monotonicity and implementation. In the third section we introduce some additional notations and prove the theorem concerning MWS and implementability. In the fourth section a game form implementing the CREE is constructed. The last section discusses possible extensions and some further lines of research.

2. NOTATIONS AND DEFINITIONS

ϵ - An n-dimensional random variable. It will sometimes be referred to as the profile.

$\epsilon = (\epsilon_1, \ldots, \varepsilon_n)$ where $\epsilon_i \in R$ for all $i = 1, \ldots, n$

J - The support of ϵ.

$J_{i_1, \ldots, i_m}$ - The projection of J onto the $(i_1, \ldots, i_m)$ axis. For instance, J_{23} is the set of all pairs of numbers that individuals 2 and 3 could have observed.

The economy E will consist of the following:

1. An n-tuple of utility functions

$$U_i : R^k_+ \times R^n \to R \qquad i = 1,\ldots,n;$$

$U_i(X,\epsilon)$ - X denotes the commodity bundle in R^k_+ consumed by the individual, and ϵ is the realization of the random variable.

Hence the economy contains n individuals and k commodities.

2. An n-tuple of initial endowments $(w_1,\ldots,w_n)$ where $w_i \in R^k_+$ is the initial bundle of individual i. Hence we assume the endowments are certain and do not vary with ϵ. $w = \sum_{i=1}^{n} w_i$ and assume $w \in R^k_{++}$.

We assume that the distribution of the random variable and the U_i's are common knowledge as in Aumann [1976], but the particular realization of ϵ_i observed by the various individuals is private information.

X - An allocation for E

$X \in R^{nk}_+$, $X = (X_1,\ldots,X_n)$ where X_i denotes the commodity bundle allocated to individual i.

A - The set of feasible allocations for E

$$A = \{(X_1,\ldots,X_n) \in R^{nk}_+ \mid \sum_{i=1}^{n} X_i^j \le \sum_{i=1}^{n} W_i^j \text{ for all } j = 1,\ldots,k\}$$

$\hat{A}$ - The set of feasible net trades

$$\hat{A} = \{(z_1,\ldots,z_n) \in R^{nk}_+ \mid \sum_{i=1}^{n} z_i^j \le 0 \text{ for all } j = 1,\ldots,k\}$$

Remark: With this definition of a feasible net trade we implicitly assume free disposal.

f - An allocation rule for E

$f: J \to A$, associates a feasible allocation with each possible profile

$\tilde{F}$ - The set of all allocation rules

a - A trade rule

$a: J \to \hat{A}$, associates a feasible net trade with each possible profile

$\tilde{A}$ - The set of all trade rules

$\tilde{0}$ - The zero trade rule which maps every ϵ in J into the null vector in R_+^{nk}.

F - A Social Choice Correspondence (SCC).

$F \subset \tilde{F}$

$f^* \in F$ The default rule.

An SCC F will be called monotonic if the following holds:

For any allocation rule $f \in F$, if there exists a mapping $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_i: J_i \to J_i$ that satisfies the following:

(i) For any ϵ in J, $\sigma(\epsilon)$ is in J where $\sigma(\epsilon) = (\sigma_1(\epsilon_1), \ldots, \sigma_n(\epsilon_n))$

(ii) For all $i = 1, \ldots, n$, for any allocation rule a that satisfies for all ϵ_i in J_i

$$E(U_i(f_i(\epsilon_{-i},\sigma_i(\epsilon_i)), \epsilon_{-i}, \sigma_i(\epsilon_i)) \mid \sigma_i(\epsilon_i)) \geq$$
$$E(U_i(a_i(\epsilon_{-i}, \sigma_i(\epsilon_i)), \epsilon_{-i}, \sigma_i(\epsilon_i)) \mid \sigma_i(\epsilon_i))$$

We have for all ϵ_i in J_i

$$E(U_i(f_i(\sigma(\epsilon)), \epsilon) \mid \epsilon_i) \geq E(U_i(a_i(\sigma(\epsilon)),\epsilon) \mid \epsilon_i)$$

where $(\epsilon_{-i}, \sigma_i(\epsilon_i)) = (\epsilon_1,\ldots,\epsilon_{i-1},\sigma_i(\epsilon_i), \epsilon_{i+1},\ldots,\epsilon_n)$

Then we have $\tilde{f} \in F$ where

$$\tilde{f}(\epsilon) = f(\sigma(\epsilon))$$

A game form G will consist of the following:

1. An n-tuple of strategy sets $(S_1,\ldots,S_n)$ where S_i denotes the strategy set of individual i.

   $S_i = B_i \times D_i$

   B_i denotes acts that have to be decided upon before any private information is observed. The acts in D_i are taken after the private information is observed.

   $S = \prod_{i=1}^{n} S_i$; $B = \prod_{i=1}^{n} B_i$; $D = \prod_{i=1}^{n} D_i$

2. An outcome function $g : B \times D \to A$

   $g = (g_1,\ldots,g_n)$ where g_i denotes the bundle received by the i-th individual.

The timing of actions is fixed. First the individuals choose an act in B_i. Then they observe their ϵ_i and decide on their action in D_i. A strategy for individual i will be a choice of b_i in B_i and a function $d_i : J_i \to D_i$. We must remark that the case where the individuals can condition their acts on a function of acts taken by others is excluded.

3. In defining the Bayesian Nash Equilibrium (BNE) of such a game we use the following notations.

   $s = (s_1,\ldots,s_n)$; $b = (b_1,\ldots,b_n)$; $\delta = (\delta_1,\ldots,\delta_n)$ will respectively denote generic elements of S, B and D.

   For $\hat{s}$ in S_i we define

   $(s_{-i},\hat{s}) = (s_1,\ldots,s_{i-1}, \hat{s}, s_{i+1},\ldots,s_n)$

   $(b_{-i},\hat{b})$ is similarily defined.

   $d(\epsilon) = (d_1(\epsilon_1),\ldots,d_n(\epsilon_n))$

   For all ϵ in J and all $\hat{\delta}$ in D_i define:

   $(d_{-i},\hat{\delta}) = (d_1(\epsilon_1),\ldots,d_{i-1}(\epsilon_{i-1}),\hat{\delta},d_{i+1}(\epsilon_{i+1}),\ldots,d_n(\epsilon_n))$

   A BNE of the game is an n-tuple of strategies $\tilde{s} = (\tilde{s}_1,\ldots,\tilde{s}_n)$ where $\tilde{s}_i = (\tilde{b}_i,\tilde{d}_i)$ satisfies for all $i = 1,\ldots,n$ the following:

(i) $E\,(U_i(g_i(\tilde{s}),\epsilon)) \geq E(U_i(g_i(\tilde{s}_{-i},\hat{s}),\epsilon))$ for all $\hat{s}$ in S_i.

   Where the expectation is taken with respect to ϵ, note that s in part, depends on ϵ.

(ii) For all ϵ_i in J_i

$$E(U_i(g_i(\tilde{b},\tilde{d}),\epsilon) \mid \epsilon_i) \geq E(U_i(g_i(\tilde{b},\tilde{d}_{-i},\hat{\delta}),\epsilon) \mid \epsilon_i) \quad \text{for all } \hat{\delta} \text{ in } D_i.$$

Note that actually (ii) is implied by (i) almost everywhere. The strategies chosen by the individuals yield an allocation rule a.

$$a(\epsilon) = g(b,d(\epsilon))$$

$N_G(E)$ - the set of allocation rules corresponding to BNE of the game form G for E.

We shall say that G implements F if:

$$N_G(E) = F$$

3. **Necessary and Sufficient Conditions for Implementability.**

Theorem 1: If an SCC is implementable by a game form G which satisfies the following condition:

For all f in F there exists a BNE $\tilde{s} = (\tilde{b},\tilde{d})$ satisfying

(C1) For all δ in D, for all $i = 1,\ldots,n$

$$g(\tilde{b}_{-i},\hat{b},\delta) = g(\tilde{b}_{-i},\bar{b},\delta) \quad \text{for all } \hat{b} \text{ and } \bar{b} \text{ in } B_i.$$

(C2) $g(\tilde{b},\tilde{d}(\epsilon)) = f(\epsilon)$ for all ϵ in J.

Then F must satisfy monotonicity.

Proof: Suppose that F is implementable by a game form G with the above property and take an f in F and a mapping σ which satisfies the requirements of the monotonicity definition.

In the game G we must have a BNE satisfying (C1) and (C2) with respect to f.

Denote that BNE by (b,d), thus we have:

(i) $g(b,d(\epsilon)) = f(\epsilon)$

(ii) A change in the B_i strategy on the part of a single individual will have no effect whatsoever on the outcome.

We shall show that the following n-tuple of strategies forms a BNE as well.

$$(\tilde{b},\tilde{d}(\epsilon)) = (b,d(\sigma(\epsilon)))$$

where $d(\sigma(\epsilon)) = (d_1(\sigma_1(\epsilon_1),\ldots,d_n(\sigma_n(\epsilon_n)))$

By (ii) above it is enough to show that no individual can gain by a deviation from his d_i strategy, at the second stage of the game. Suppose by way of contradiction that there exists an individual i and an $\bar{\epsilon}_i$ in J_i for which there exists a $\hat{\delta}$ in D_i that satisfies:

$$\text{(B)}\quad E(U_i(g_i(\tilde{b},\tilde{d}(\epsilon)),\epsilon) \mid \bar{\epsilon}_i) < E(U_i(g_i(\tilde{b},\tilde{d}_{-i}(\epsilon),\hat{\delta}),\epsilon) \mid \bar{\epsilon}_i)$$

Now define an allocation rule a in the following way:

$$a(\epsilon) = f(\epsilon) \quad \text{for} \quad \epsilon_i \neq \sigma_i(\bar{\epsilon}_i)$$
$$a(\epsilon) = g(b,\tilde{d}_{-i}(\epsilon),\hat{\delta}) \quad \text{if} \quad \epsilon_i = \sigma_i(\bar{\epsilon}_i)$$

By (B) we have that this allocation rule satisfies

$$E(U_i(f_i(\sigma(\epsilon)),\epsilon) \mid \bar{\epsilon}_i) < E(U_i(a_i(\sigma(\epsilon)),\epsilon) \mid \bar{\epsilon}_i)$$

where $f_i(\sigma(\epsilon)) = f_i(\sigma_1(\epsilon_1),\ldots,\sigma_n(\epsilon_n))$ and $a_i(\sigma(\epsilon))$ is similarly defined.

Hence for this i, the allocation rule a cannot satisfy

$$E(U_i(f_i(\epsilon_{-i},\sigma_i(\epsilon_i)),\ \epsilon_{-i},\sigma_i(\epsilon_i)) \mid \sigma_i(\epsilon_i)) \geq$$
$$E(U_i(a_i(\epsilon_{-i},\sigma_i(\epsilon_i)),\ \epsilon_{-i},\sigma_i(\epsilon_i)) \mid \sigma_i(\epsilon_i))$$

For all possible ϵ_i's.

(recall that $(\epsilon_{-i},\sigma_i(\epsilon_i)) = (\epsilon_1,\ldots,\epsilon_{i-1},\sigma_i(\epsilon_i),\epsilon_{i+1},\ldots,\epsilon_n))$

Since we have by the definition of a an equality for all $\sigma_i(\epsilon_i)$ different from $\sigma_i(\bar{\epsilon}_i)$ we must have:

$$E(U_i(f_i(\epsilon_{-i},\sigma_i(\bar{\epsilon}_i)),\ \epsilon_{-i},\ \sigma_i(\bar{\epsilon}_i)) \mid \sigma_i(\bar{\epsilon}_i)) <$$
$$E(U_i(a_i(\epsilon_{-i},\sigma_i(\bar{\epsilon}_i)),\ \epsilon_{-i},\sigma_i(\bar{\epsilon}_i)) \mid \sigma_i(\bar{\epsilon}_i))$$

By the definition of a and since the d_i's together with b yielded f we have:

$$E(U_i(g_i(b,d(\epsilon_{-i},\sigma_i(\bar{\epsilon}_i))),\ \epsilon_{-i},\sigma_i(\bar{\epsilon}_i)) \mid \sigma_i(\bar{\epsilon}_i)) <$$
$$E(U_i(g_i(b,d_{-i}(\epsilon),\hat{\delta}),\epsilon_{-i},\sigma_i(\bar{\epsilon}_i)) \mid \sigma_i(\bar{\epsilon}_i)).$$

But this contradicts the fact that the d_i's are part of a BNE. Individual i at the second stage of the game, after observing $\sigma_i(\bar{\epsilon}_i)$, is better off doing $\hat{\delta}$ and not doing his equilibrium strategy when the others follow their equilibrium strategies.

So we get that the allocation rule $\tilde{f}(\epsilon_1,\ldots,\epsilon_n) = f(\sigma_1(\epsilon_1),\ldots,\sigma_n(\epsilon_n))$ is induced by a BNE. Hence $\tilde{f}$ is in F as well, since the game form G implemented F, and F indeed satisfies monotonicity.

□

As we shall see later on, the conclusion of theorem 1 is false if you enlarge the strategy spaces in such a way that agents are able to condition some of their acts on a function of acts taken by others. We shall return to this issue when dealing with signalling structures.

In order to prove that a monotonic F can be implemented we shall have to assume that J satisfies NEI (Non Exclusivity of Information). This assumption will be formulated as follows:

$$\text{(NEI)} \quad \text{For all } \epsilon \text{ in } J \quad \sum_{i=1}^{n} \epsilon_i = 0$$

This is not the most general way of formulating the NEI consumption. We use this convenient form to refrain from additional notations. In the proof we use only the fact that any ϵ_i can be uniquely expressed as a function of all the other coordinates. We also need two technical assumptions that the U_i's are continuous and strictly increasing in their first k arguments, and that no-one ever gets the zero bundle allocated to him. These assumptions are needed, amongst other things, to guarantee that various fines imposed during the game are indeed effective.

Theorem 2: If the following assumptions are satisfied:

(A1) $n \geq 3$

(A2) For all $i = 1,\ldots,n$ U_i is continuous and strictly increasing in its first k arguments.

($x \geq y$, $x \neq y$ implies for all ϵ in J

$U_i(x,\epsilon) > U_i(y,\epsilon)$)

(A3) F satisfies monotonicity

(A4) J satisfies NEI

(A5) For all f in F and all ϵ in J

$f_i(\epsilon) \geq 0$, $f_i(\epsilon) \neq 0$

For all $i = 1,\ldots,n$

where f_i denotes the bundle allocated to individual i.

Then F can be implemented.

Proof: We shall construct an explicit game form implementing F. The strategy space for individual i will be:

$$S_i = B_i \times D_i$$

where

$$B_i = \tilde{F}$$

$$D_i = J_i \times \tilde{A} \times N \times M$$

N denotes the set of all positive natural numbers and M the set of all strictly positive numbers.

A generic element of the strategy space will be denoted by

$$(f^i, r_i, a_i, n_i, m_i)$$

The first component belongs to B_i and the last four to D_i. The strategy of individual i can be given the following interpretation:

f^i - An allocation rule individual i would like to have. This has to be decided upon before the observation of any private information

r_i - The ϵ_i he "observed".

a_i - A trade rule individual i would like to have

n_i - An "indication" as to how much weight should be assigned to the trade rule demanded

m_i - A number affecting the "fines" imposed on individual i for any detected lies (declared profiles outside J) and deviations from allocations in F.

The outcome function is defined as follows:

Stage 1

Construct a weighted average of the f^i's and denote it by $\bar{f}$. We shall explicitly show how $\bar{f}_1^j(\epsilon_1,\ldots,\epsilon_n)$, which indicates how much of the first commodity does individual j get, given some realization of ϵ, is constructed:

Define:

$$\alpha_i(\epsilon_1,\ldots,\epsilon_n) = \sum_{t,t'\neq i} \left| f_j^{t_1}(\epsilon_1,\ldots,\epsilon_n) - f_j^{t'_1}(\epsilon_1,\ldots,\epsilon_n) \right|$$

where $f_j^{t_i}$ indicates how much of the i^{th} commodity does individual j get in the allocation rule proposed by t.

Now define: $\alpha = \sum_{i=1}^{n} \alpha_i$.

$$\beta_i = \begin{cases} \dfrac{\alpha_i}{\alpha} & \text{if } \alpha > 0 \\ \dfrac{1}{n} & \text{if } \alpha = 0 \end{cases} \qquad i = 1,\ldots,n$$

and finally

$$\bar{f}_1^j(\epsilon_1,\ldots,\epsilon_n) = \sum_{i=1}^{n} \beta_i f_j^{i_1}$$

In this way we get an allocation rule $\bar{f}$, if $\bar{f}$ is in F we call it f and move to the next stage. Otherwise we choose the f in F which minimizes

$$\int_J \|\bar{f}(\epsilon) - f(\epsilon)\|^2 d\ell(\epsilon)$$

where $\ell(\epsilon)$ denotes the distribution function of ϵ. If this problem has more than one solution we just choose one of them. If it has no solution then we choose the default allocation rule as f. In any case we come up with an $f \in F$.

As said previously this stage takes place before any private information is observed. The next stages take place after the individuals observe their ϵ_i's.

The individuals, after observing their respective ϵ_i's, announce r_i, a_i, n_i, m_i.

Stage 2.

n profiles in J one for each individual are constructed. The i-th profile denoted by $(\tilde{r}_1^i,\dots,\tilde{r}_i^i,\dots,\tilde{r}_n^i)$ is constructed in the following way. The closest point in $J_{1,\dots,i-1,i+1,\dots,n}$ to $(r_1,\dots,r_{i-1},r_{i+1},\dots,r_n)$ is denoted by:

$$(\tilde{r}_1^i,\dots,\tilde{r}_{i-1}^i,\tilde{r}_{i+1}^i,\dots,\tilde{r}_n^i)$$

and $\tilde{r}_i^i$ is defined by

$$\tilde{r}_i^i = -\sum_{j\neq i}\tilde{r}_j^i$$

This way we get a profile in J given by

$$\tilde{r}^i = (\tilde{r}_1^i,\dots,\tilde{r}_i^i,\dots,\tilde{r}_n^i)$$

This construction has two properties which we shall use later on.

(i) If $(r_1,\ldots,r_n)$ is in J then all the n profiles are identical and equal to r.

(ii) A change in the strategy of individual i will not change the i-th profile.

Stage 3.

Construct a trade rule $\tilde{a}$. It will depend on the announced trade rules a_i and the various n_i's.

We shall explicitly show how

$$\tilde{a}_j^1(\epsilon_1,\ldots,\epsilon_n) = \tilde{a}_j^1(\epsilon) \quad \text{for} \quad j = 1,\ldots,n$$

are constructed, where $\tilde{a}_j^1(\epsilon)$ shows how much of the first commodity is traded by individual j given some realization of ϵ. Define $\bar{N} = \sum_{i=1}^{n} n_i$.

Then solve:

$$\begin{aligned}
&\min(\tilde{a}_1^1(\epsilon))^2 + \ldots + (\tilde{a}_n^1(\epsilon))^2 \\
&\text{s.t.} \\
&\quad \tilde{a}_1^1(\epsilon) + \ldots + \tilde{a}_n^1(\epsilon) = 0 \\
&\frac{n_1+0.1}{\bar{N}}(\tilde{a}_1^1(\epsilon) - a_1^1(\epsilon)) + \ldots + \frac{n_n}{\bar{N}}(\tilde{a}_n^1(\epsilon) - a_n^1(\epsilon)) = 0
\end{aligned}$$

K such problems are solved and a trade rule $\tilde{a}$ is thus constructed.

$a_j^i(\epsilon)$ - shows how much did individual j trade in commodity i, in the allocation rule proposed by individual j.

Remarks: If the $a^i_j(\epsilon)$ are all zeros then the $\tilde{a}^i_j(\epsilon)$ will be zeros as well.

Writing $(n_1 + 0.1)$ ensures the set of feasible points for the minimization problem is nonempty. This, in turn, will guarantee the existence of a unique solution to the minimization problem (since the objective function is continuous, strictly convex and bounded from below).

Stage 4.

Define an allocation rule τ by $\tau = \tilde{a} + f$, where $\tilde{a}$ and f are respectively the trade rule and allocation rule constructed in the previous stages.

Now we shall construct parts of a set of new allocation rules, one for each individual.

For all $i = 1,\ldots,n$ define μ_i by:

$$\mu_i(\epsilon_1,\ldots,\tilde{r}^i_i,\ldots,\epsilon_n) = \delta_i \cdot \tau_i(\epsilon_1,\ldots,\tilde{r}^i_i,\ldots,\epsilon_n)$$

where $0 < \delta_i \leq 1$ and is the largest such number for which:

$$E(U_i(f_i(\epsilon_{-i},\tilde{r}^i_i),\ \epsilon_{-i},\tilde{r}^i_i)\ |\ \tilde{r}^i_i) \geq$$

$$E(U_i(\mu_i(\epsilon_{-i},\tilde{r}^i_i),\ \epsilon_{-i},\tilde{r}^i_i)\ |\ \tilde{r}^i_i)$$

Notice that for any individual i, μ_i is defined for all ϵ_{-i} compatible with $\tilde{r}_i^i$.

Now define

$$\eta_i(\tilde{r}^i) = \gamma \cdot \mu_i(\tilde{r}^i)$$

where $0 < \gamma \leq 1$ is the largest such number for which

$$\sum_{i=1}^{n} \eta_i(\tilde{r}^i) \leq W$$

Stage 5.

Define $\tilde{\epsilon}$ as the closest point in J to $(r_1,\ldots,r_n)$ and let:

$$g_i(s_1,\ldots,s_n) = t_i \cdot \eta_i(\tilde{r}^i)$$

where $t_i = (1 + \frac{m_i}{\sum_{j=1}^{n} m_j}(\operatorname{Sup}_J \|\tilde{a}(\epsilon)\| + \|r-\tilde{\epsilon}\|))^{-1}$

First we shall prove that $F \subset N_G(E)$. Given f in F we construct the following BNE:

$f^i = f$ for all $i = 1,\ldots,n$

$r_i = \epsilon_i$; $a_i = \tilde{0}$; $n_i = m_i = 1$ for all $i = 1,\ldots,n$ and for all ϵ in J.

At stage 1 we end up with f and no single individual can change the f agreed upon by deviating and declaring some other allocation rule. At stage 2 we get the profile $(\epsilon_1,\ldots,\epsilon_n)$ for all the individuals.
At stage 3 we get the zero trade rule.
At stage 4 we get $\tau = f$ and μ where defined equals τ. Finally since the profile is the same for all individuals, η equals μ and hence f.
At stage 5 we get for all $i = 1,\ldots,n$

$$g_i(b,d(\epsilon)) = f_i(\epsilon) \quad \text{for all } \epsilon \text{ in } J$$

since $t_i = 1$.
So this n-tuple of strategies does yield the allocation rule f.
It remains to be shown that this n-tuple of strategies does form a BNE.

Individual i cannot affect the allocation rule f or the $(\tilde{r}^i)$ profile by changing his strategy. Since all the others report truthfully $\tilde{r}^i$ is the "true" profile. He can change the $\tilde{a}$ but this will never yield him an allocation which is strictly preferred to the f allocation given the **true** ϵ_i observed by him.

In order to prove $N_G(E) \subset F$ we shall first show that all BNE must have:

$$\tilde{a} = \tilde{0} \qquad \text{for all } \epsilon \text{ in } J$$

$$(r_1(\epsilon_1),\ldots,r_n(\epsilon_n)) \in J \qquad \text{for all } \epsilon \text{ in } J.$$

If one of these conditions is violated, we get, by stage 5, that no choice of m_i would yield an equilibrium point. m_i must be strictly positive, but the i-th individual would like to make it as small as possible.

We shall now show that in a BNE the set of mappings

$$r_1(\epsilon_1),\ldots,r_n(\epsilon_n)$$

satisfies the requirements of the monotonicity condition. r_i maps J_i into J_i and as remarked before

$$r(\epsilon) \text{ is in } J \text{ for all } \epsilon \text{ in } J.$$

Now, suppose by way of contradiction, that there exists as individual i and an allocation rule a that satisfies for all ϵ_i in J_i.

$$E(U_i(f_i(\epsilon_{-i},r_i(\epsilon_i)),\ \epsilon_{-i},r_i(\epsilon_i)) \mid r_i(\epsilon_i)) \geq$$
$$E(U_i(a_i(\epsilon_{-i},r_i(\epsilon_i)),\ \epsilon_{-i},r_i(\epsilon_i)) \mid r_i(\epsilon_i))$$

But there exists an $\bar{\epsilon}_i$ in J_i for which

$$E(U_i(f_i(r(\epsilon)),\epsilon) \mid \bar{\epsilon}^i) < E\ (U_i(a_i(r(\epsilon)),\epsilon) \mid \bar{\epsilon}_i)$$

If that is the case, when observing $\bar{\epsilon}_i$, individual i can improve his position by changing his strategy.

By declaring an n_i large enough and an m_i small enough and the trade rule

$$\tilde{a}(\epsilon) = a(\epsilon) - f(\epsilon)$$

The i-th individual can get arbitrarily close to the allocation rule

$$a_i(\epsilon_{-i}, r_i(\bar{\epsilon}_i))$$

So the i-th individual can get a sequence of allocations

$$a_i^n(\epsilon_{-i}, r_i(\bar{\epsilon}_i)) \underset{n\to\infty}{\to} a_i(\epsilon_{-i}, r_i(\bar{\epsilon}_i))$$

Since U_i is continuous we have that

$$\lim_{n\to\infty} U_i(a_i^n(\epsilon_{-i}, r_i(\bar{\epsilon}_i)), \epsilon_{-i}, \bar{\epsilon}_i) =$$
$$= U_i(a_i(\epsilon_{-i}, r_i(\bar{\epsilon}_i)), \epsilon_{-i}, \bar{\epsilon}_i)$$

for all ϵ_{-i} compatible with $r_i(\bar{\epsilon}_i)$.

So this sequence of allocations satisfies

$$E(U_i(f_i(r(\epsilon)),\ \epsilon) \mid \bar{\epsilon}_i) < E(\lim_{n\to\infty} U_i(a_i^n(r(\epsilon)),\epsilon) \mid \bar{\epsilon}_i)$$

Since U_i can be bounded from above by an integrable function we can take the limit out of the expectation operator and obtain:

$$E(U_i(f_i(r(\epsilon)),\epsilon) \mid \bar{\epsilon}_i) < \lim_{n\to\infty} E(U_i(a_i^n(r(\epsilon)),\ \epsilon) \mid \bar{\epsilon}_i)$$

So we see the i-th individual can, by getting close enough to the a allocation (by a suitable change of his strategy), improve his position contradicting the fact that we were at a BNE.

Hence $r_1(\epsilon_1),\dots,r_n(\epsilon_n)$ satisfy the requirements of the monotonicity definition. However, the allocation yielded by the BNE is precisely

$$f(r_1(\epsilon_1),\dots,r_n(\epsilon_n))$$

and since F satisfies monotonicity and f was in F we have

$$f(r_1(\epsilon_1),\dots,r_n(\epsilon_n)) \in F \text{ as well.}$$

□

Remark: If we assume J and f are closed and convex sets then all the projections we perform are continuous and we have a "truly" continuous game form.

4. Monotonicity in the Wide Sense and Implementability

In order to define monotonicity in the wide sense we shall have to introduce some more notations.

A signalling structure will be an n-tuple of functions $P = (P_1, \ldots, P_n)$ where

$$P_i : F \times J \to \Gamma_i \quad i = 1, \ldots, n$$
$$P_i^f : J \to \Gamma_i$$

and $P_i^f(\epsilon)$ is the signal individual i should get if allocation rule f in F is used and ϵ in J occurred. The signals sent to individual i are taken from some arbitrary space denoted by Γ^i. For most applications this can be taken as some Euclidean space. More complex signalling structures where the signals themselves are realizations of some random variables can be handled at the cost of more notations and the definitions and the results will basically remain the same.

An SCC F is said to satisfy Monotonicity in the Wide Sense (MWS) if there exists a signalling structure P such that the following holds:

For any allocation rule $f \in F$ if there exists a mapping $\sigma = (\sigma_1, \ldots, \sigma_n)$ where σ_i: $J_i \to J_i$ that satisfies the following:

(i) For any ϵ in J, $\sigma(\epsilon)$ is in J

(ii) For all $i = 1, \ldots, n$ for any allocation rule a that satisfies for all ϵ_i in J_i

$$E(U_i(f_i(\epsilon_{-i}, \sigma_i(\epsilon_i)), \epsilon_{-i}, \sigma_i(\epsilon_i)) \mid \sigma_i(\epsilon_i), P_i^f(\tilde{\epsilon}_{-i}, \sigma_i(\epsilon_i)))$$
$$\geq E(U_i(a_i(\epsilon_{-i}, \sigma_i(\epsilon_i)), \epsilon_{-i}, \sigma_i(\epsilon_i)) \mid \sigma_i(\epsilon_i), P_i^f(\tilde{\epsilon}_{-i},\sigma_i(\epsilon_i)))$$

for all $\tilde{\epsilon}_{-i}$ compatible with $\sigma_i(\epsilon_i)$

We have that for all ϵ_i in J_i

$$E(U_i(f_i(\sigma(\epsilon)),\epsilon) \mid \epsilon_i, P_i^f(\sigma_{-i}(\tilde{\epsilon}_{-i}), \sigma_i(\epsilon_i)))$$
$$\geq \quad E(U_i(a_i(\sigma(\epsilon)),\epsilon) \mid \epsilon_i, P_i^f(\sigma_{-i}(\tilde{\epsilon}_{-i}), \sigma_i(\epsilon_i)))$$

for all $\tilde{\epsilon}_{-i}$ compatible with ϵ_i.

$(\sigma_{-i}(\tilde{\epsilon}_{-i}),\sigma_i(\epsilon_i))$ denotes

$(\sigma_1(\tilde{\epsilon}_1),\ldots,\sigma_{i-1}(\tilde{\epsilon}_{i-1}),\sigma_i(\epsilon_i),\sigma_{i+1}(\tilde{\epsilon}_{i+1}),\ldots,\sigma_n(\tilde{\epsilon}_n))$

Then we have $\tilde{f}$ in F where:

$$\tilde{f}(\epsilon) = f(\sigma(\epsilon)).$$

We shall now define a new game form which uses much larger strategy spaces than the ones used before. We shall define a BNE for that game and then prove that the allocations yielded by BNE will coincide with the allocations in F.

Generally speaking the game will consist of three stages. The first stage is played before any private information is observed and at the end of this stage an allocation rule in F is decided upon. The second stage takes place after the individuals observed their private information and at that stage each one reports the private information he "supposedly" observed. At the end of that stage various signals, based on the allocation rule agreed upon and the private information transmitted, are sent out to the individuals. After observing their respective signals, the individuals act again and this is the last stage of the game after which the outcome is determined. Formally speaking the game form G will consist of the following:

1. An n-tuple of strategy sets $(S_1, \ldots, S_n)$

$$S_i = B_i \times H_i \times D_i$$

B_i denotes acts that have to be taken before the observation of any private information. H_i denotes acts that have to be taken after the observation of the private information. In our framework H_i is just J_i, each one is requested to report the number he supposedly observed. The acts in D_i are taken after the signal is observed

$$S = \prod_{i=1}^{n} S_i, \quad B = \prod_{i=1}^{n} B_i, \quad H = \prod_{i=1}^{n} H_i, \quad D = \prod_{i=1}^{n} D_i$$

2. A signalling structure $P = (P_1, \ldots, P_n)$ where

$$P_i : B \times H \to \Gamma_i$$

P_i is the signal received by individual i and Γ_i is some abstract signal space

$$\Gamma = \Gamma_1 \times \ldots \times \Gamma_n.$$

3. An outcome function

   $g: B \times H \times D \to A$ where A denotes, as before, the set of all feasible allocations

   $g = (g_1,\ldots,g_n)$ where g_i denotes the bundle received by the i-th individual.

A strategy for individual i will be a choice of b_i in B_i, a function $h_i: J_i \to H_i$ and a function $d_i: J_i \times \Gamma_i \to D_i$.

In defining the BNE of such a game we use the following notation:

$s = (s_1,\ldots,s_n)$; $b = (b_1,\ldots,b_n)$; $\alpha = (\alpha_1,\ldots,\alpha_n)$; $\delta = (\delta_1,\ldots,\delta_n)$; $\gamma = (\gamma_1,\ldots,\gamma_n)$ will respectively denote generic elements of S, B, H, D and Γ.

For $\hat{s}$ in S_i we define

$(s_{-i}, \hat{s}) = (s_1,\ldots,s_{i-1}, \hat{s}, s_{i+1},\ldots,s_n)$

$(d_{-i},\hat{d})$ is similarily defined, $\hat{d}$ in any function mapping $J_i \times \Gamma_i$ into D_i

$h(\epsilon) = (h_1(\epsilon_1),\ldots,h_n(\epsilon_n))$

$d(\epsilon,\gamma) = (d_1(\epsilon_1,\gamma_1),\ldots,d_n(\epsilon_n,\gamma_n))$

For all ϵ in J and all $\hat{\alpha}$ in H_i define:

$(h_{-i},\hat{\alpha}) = (h_1(\epsilon_1),\ldots,h_{i-1}(\epsilon_{i-1}),\hat{\alpha},h_{i+1}(\epsilon_{i+1}),\ldots,h_n(\epsilon_n))$

For all (ϵ,γ) in $J \times \Gamma$ and all $\hat{\delta}$ in D_i define:

$$(d_{-i}, \hat{\delta}) = (d_1(\epsilon_1,\gamma_1),\ldots,d_{i-1}(\epsilon_{i-1},\gamma_{i-1}),\hat{\delta},d_{i+1}(\epsilon_{i+1},\gamma_{i+1}),\ldots,d_n(\epsilon_n,\gamma_n))$$

A BNE of the game is an n-tuple of strategies $\tilde{s} = (\tilde{s}_1,\ldots,\tilde{s}_n)$ where $\tilde{s}_i = (\tilde{b}_i, \tilde{h}_i, \tilde{d}_i)$ satisfies for all $i = 1,\ldots,n$ the following:

(i) $E(U_i(g_i(\tilde{s}),\epsilon)) \geq E(U_i(g_i(\tilde{s}_{-i},\hat{s}),\epsilon))$

for all $\hat{s}$ in S_i.

(ii) For all ϵ_i in J_i

$$E(U_i(g_i(\tilde{b},\tilde{h},\tilde{d}), \epsilon) \mid \epsilon_i) \geq$$
$$E(U_i(g_i(\tilde{b},\tilde{h}_{-i},\hat{\alpha},\tilde{d}_{-i},\hat{d}),\epsilon) \mid \epsilon_i)$$

For all $\hat{\alpha}$ in H_i and all $\hat{d}$ mapping $J_i \times \Gamma_i$ into D_i

(iii) For all ϵ_i in J_i and γ_i in Γ_i

$$E(U_i(g_i(\tilde{b},\tilde{h},\tilde{d}),\epsilon) \mid \epsilon_i, \gamma_i) \geq$$
$$E(U_i(g_i(\tilde{b},\tilde{h},\tilde{d}_{-i},\hat{\delta}),\epsilon) \mid \epsilon_i, \gamma_i)$$

for all $\hat{\delta}$ in D_i.

Note that (ii) and (iii) are actually implied by (i) almost everywhere. Now we can state the following theorem.

Theorem 3: If the following assumptions are satisfied:

(A1), (A2), (A4), (A5) of theorem 3

and (A3)' F satisfies MWS

Then one can construct a game form G whose BNE coincide with F.

Proof: We shall use the same notations as in the formal definition of the game form. The strategy set for individual i will be:

$$S_i = B_i \times H_i \times D_i$$

where

$$B_i = \tilde{F}$$

$$H_i = J_i$$

$$D_i = \tilde{A} \times N \times M$$

N and M respectively denote the set of all positive natural numbers and the set of all strictly positive numbers.

A generic element of the strategy space will be denoted by:

$$(f^i, r_i, a_i, n_i, m_i).$$

The first component belongs to B_i, the second to H_i and the last three to D_i.

The interpretation of the strategy is similar to the one in theorem 2. The outcome function is defined as follows. Stages 1 and 2 as in Theorem 2. At the end of stage 2 the individuals get their respective signals. Individual i gets the signal:

$$P_i^f(\tilde{r}^i)$$

Recall that P is the signalling structure that appeared in the definition of MWS, f is the allocation rule arrived at in stage 1, and $\tilde{r}^i$ is the profile constructed for the i-th individual.

At this point of the game the individuals decide on their strategy choice in D_i.

At stage 3 the trade rule $\tilde{a}$ is constructed the same as in theorem 2.

At stage 4 the μ_i is somewhat different.

For all $i = 1,\ldots,n$ define μ_i by:

$$\mu_i(\epsilon_1,\ldots,\tilde{r}_i,\ldots,\epsilon_n) = \delta_i \cdot \tau_i(\epsilon_1,\ldots,\tilde{r}_i,\ldots,\epsilon_n)$$

where $0 < \delta_i \leq 1$ and is the largest such number for which:

$$E(U_i(f_i(\epsilon_1,\ldots,\tilde{r}^i_i,\ldots,\epsilon_n),\ \epsilon_{-i},\tilde{r}^i_i) \mid \tilde{r}^i_i,\ P^f_i(\tilde{r}^i)) \geq$$

$$E(U_i(\mu_i(\epsilon_1,\ldots,\tilde{r}^i_i,\ldots,\epsilon_n),\ \epsilon_{-i},\ \tilde{r}^i_i) \mid \tilde{r}^i_i,\ P^f_i(\tilde{r}^i))$$

Notice that for any individual i, μ_i is defined for all ϵ_{-i}, compatible with $\tilde{r}^i_i$ and the signal $P^f_i(\tilde{r}^i)$.

η_i and the outcome function g_i are defined as in theorem 2.

First we shall prove that F is contained in the set of BNE.

Given f in F we construct the following BNE:

$f^i = f$ for all $i = 1,\ldots,n$

$r_i = \epsilon_i$; $a_i = \tilde{0}$; $n_i = m_i = 1$ for all $i = 1,\ldots,n$ all ϵ's in J and all possible signals.

Since everyone reports truthfully the $\tilde{r}^i$ will be the "true" profile. Individual i can change the $\tilde{a}$ but this will never yield him an allocation which is strictly preferred to the f allocation given the true ϵ_i and the "true" signal observed by him.

Now we show that $N_G(E) \subseteq F$, the same way as in theorem 2 it can be shown that:

$$(r_1(\epsilon_1),\ldots,r_n(\epsilon_n)) \in J \quad \text{for all } \epsilon \text{ in } J$$

$$\tilde{a} = \tilde{0} \quad \text{for all } \epsilon \text{ in } J \text{ and all possible signals.}$$

Now we show that in a BNE the set of mappings

$$(r_1(\epsilon_1),\ldots,r_n(\epsilon_n))$$

satisfy the requirements of the MWS condition. r_i maps J_i into J_i and as remarked previously $r(\epsilon)$ is in J for all ϵ in J.

Now suppose by way of contradiction, that there exists an individual i and an allocation rule a that satisfies for all ϵ_i in J_i.

$$E(U_i(f_i(\epsilon_{-i},r_i(\epsilon_i)),\ \epsilon_{-i},r_i(\epsilon_i)) \mid r_i(\epsilon_i),\ P_i^f(\tilde{\epsilon}_{-i},\ r_i(\epsilon_i)))$$

$$\geq E(U_i(a_i(\epsilon_{-i},r_i(\epsilon_i)),\ \epsilon_{-i},\ r_i(\epsilon_i)) \mid r_i(\epsilon_i),\ P_i^{\tilde{f}}(\epsilon_{-i},r_i(\epsilon_i)))$$

for all $\tilde{\epsilon}_{-i}$ compatible with $r_i(\epsilon_i)$

But there exists an $(\bar{\epsilon}_{-i}, \bar{\epsilon}_i)$ in J for which

$$E(U_i(f_i(r(\epsilon)),\epsilon) \mid \bar{\epsilon}_i, P_i^f(r_{-i}(\bar{\epsilon}_{-i}), r_i(\bar{\epsilon}_i))) <$$
$$E(U_i(a_i(r(\epsilon)),\epsilon) \mid \bar{\epsilon}_i, P_i^f(r_{-i}(\bar{\epsilon}_{-i}), r_i(\bar{\epsilon}_i)))$$

In that case the i-th individual could at the third stage improve his position by changing his strategy after observing $\bar{\epsilon}_i$ and the signal $P_i^f(r_{-i}(\bar{\epsilon}_{-i}), r_i(\bar{\epsilon}_i))$. Now the same way as in theorem 2 we contradict the fact that we started from a BNE.

Hence $r_1(\epsilon_1),\ldots,r_n(\epsilon_n)$ satisfy the requirements of the MWS definition. The allocation yielded by the BNE is precisely $f(r(\epsilon))$ and since F satisfies MWS and f is in F we have

$$f(r(\epsilon)) \in F \text{ as well.}$$

□

5. Implementation of Constrained Rational Expectations Equilibria

In the previous sections we found sufficient conditions for the implementability of given SCC's. The next issue we wish to address is whether specific SCC's can be implemented more efficiently, i.e. by having smaller strategy spaces or reducing the informational requirements on the part of the designer. The same issue arises in economies with complete information. The typical strategy spaces used in the general implementation theorems asked the individuals to report whole profiles of preferences. In our framework individuals send some real parameters, but since it is assumed the designer knows the basic form of the utility function, i.e. he knows the U_i's, this is equivalent to having individuals send whole profiles of preferences.

In economies with complete information various game forms, using much smaller strategy spaces, were built to implement specific SCC's. Several works were concerned with implementing the SCC induced by competitive equilibrium allocations, namely, the Walrasian correspondence, or some variant of it. The typical game form had individuals announcing prices and net trades.

A natural analogue of the Walrasian correspondence in economies with incomplete information is the SCC induced by Rational Expectations Equilibria (REE).

This concept of equilibrium takes into account the fact that in the presence of incomplete information the individuals will use any information available, be it private or some publicly observed signal, in order to form correct conditional expectations.

In our framework the definition of a REE takes the following form:

A pricing function $P: J \to R^k_+$ and an allocation rule $f: J \to A$ constitute a REE if the following holds:

(i) for all $i = 1,\ldots,n$ $f_i(\epsilon)$ solves the problem

$$\max_X E(U_i(X,\epsilon) \mid \epsilon_i, P(\epsilon))$$

s.t.

$$P(\epsilon)\cdot X \le P(\epsilon)\cdot W_i$$

$$X \ge 0$$

for all ϵ in J.

(ii) For all ϵ in J

$$P(\epsilon)\left(\sum_{i=1}^{n} f_i(\epsilon) - W\right) = 0$$

(J, A, W, respectively denote the support of ϵ, the set of feasible allocations and the aggregate initial endowment.)

This is the usual definition of REE, however it is often required that the above conditions hold just almost everywhere. It is, of course, assumed that the individuals know the distribution of ϵ and the pricing function. One could, of course, let the individuals have different beliefs about the distribution of ϵ, but this is not done in our framework.

If we assume that the utility functions are strictly increasing, P can be taken to map J into R^k_{++} and then $\sum_{i=1}^{n} f_i(\epsilon) = W$ for all ϵ in J.

In economies with complete information the Walrasian correspondence is not monotonic. However, a variant of it, the Constrained Walrasian Correspondence is. The same problem arises here and the definition of a Constrained Rational Expectations Equilibrium (CREE) is as follows:

$P: J \to R^k_+$ and $f: J \to A$ constitute a CREE if the following holds:

(i) For all $i = 1,\ldots,n$ $f_i(\epsilon)$ solves the problem

$$\max_X E(U_i(X,\epsilon) \mid \epsilon_i, P(\epsilon))$$

S.T.

$$P(\epsilon)\cdot X \le P(\epsilon)W_i$$

$$X \le W$$

$$X \ge 0$$

for all ϵ in J

(ii) $P(\epsilon)(\sum_{i=1}^{n} f_i(\epsilon) - W) = 0$

The difference between a CREE and a REE is that in the former, individuals are not allowed to demand bundles that exceed in one, or more, of their coordinates the available aggregate endowment. It is clear that for interior solutions both definitions coincide.

In Postlewaite and Wettstein (1983) a game form implementing the Constrained Walrasian Correspondence was constructed. This game form will be a part (stage 3) of the game we construct to implement CREE.

The game form we construct to implement the CREE will have much less stringent informational requirements than the ones possessed by the general game form in theorem 3. The designer does not know the U_i's, but he will still know the distribution of ϵ and the initial endowments.

Our assumptions will be similar to the ones used in theorem 3, and the game form constructed, as far as timing of actions is concerned, will follow the lines developed in the previous section.

Theorem 4: If the following assumptions are satisfied:

(A1) $n \geq 3$

(A2) For all $i = 1,\ldots,n$ U_i is continuous, strictly increasing and strictly concave in the first k arguments for any ϵ in J.

(A3) For all $i = 1,\ldots,n$ W_i is in R^k_{++}.

(A4) J satisfies NEI.

Then the game form described below implements the CREE correspondence.

The Game Form

The strategy set for individual i will be:

$$S_i = B_i \times H_i \times D_i$$

where

$B_i = \tilde{P}$

$H_i = J_i$

$D_i = R^k \times R_{++} \times R_{++}$

$\tilde{P}$ is the set of all functions from J into R^k_{++}. A generic element of the strategy space will be denoted by:

$$(P^i, r_i, z_i, n_i, m_i)$$

The first component belongs to B_i, and has to be decided upon before the observation of any private information. The second component belongs to H_i and has to be decided upon after the observation of ϵ_i. The last three belong to D_i and have to be decided upon after observing the signal which in our case will be some price vector.

The strategy of individual i can be given the following interpretation:

p^i - The pricing rule individual i would like to prevail.

r_i - The ϵ_i he "observed".

z_i - A net trade individual i would like to have.

n_i - An "indication" as to how much weight should be assigned to the net trade demanded.

m_i - A number affecting the fines imposed on individual i for any detected lies (declared profiles outside J).

The outcome function is defined as follows:

Stage 1

$\bar{P}$ a weighted average of the P^i's is constructed the same way as $\bar{f}$ is constructed in stage 1 of theorem 2.

Stage 2

n profiles in J, one for each individual, are constructed in the same way as in stage 2 of theorem 2. The i-th profile is denoted by $\tilde{r}^i$.

By the end of this stage we have a pricing function $\bar{P}$ and n profiles $(\tilde{r}^i)_{i=1}^n$.

Individual i is now told what $\bar{P}(\tilde{r}^i)$ is, i.e. he is told a certain price vector in R^k_{++}.

Stage 3

After being told some price vector, individual i sends z_i, n_i and m_i. A set K_i is defined for each individual:

$$K^i(\bar{P}(\tilde{r}^i), W_i, W) = \{z \in R^k \mid \bar{P}(\tilde{r}^i)\cdot z = 0, \quad z + W_i \geq 0, \; z + W_i \leq W\}$$

y_i is defined to be the closest point in K^i to z_i. Note that K^i is a closed and convex set.

$$C = \{n \in R_{++} \mid n\cdot n_i \leq 1 \quad \text{for} \quad i = 1,\ldots,n \quad \text{and} \quad n\cdot\sum_{i=1}^{n} n_i(y_i+W_i) \leq W\}$$

n^* will be defined by $n^* = \max_{n\in C} n$. Now define $\tilde{\epsilon}$ as the closest point in J to $(r_1,\ldots,r_n)$ and let

$$g_i(s_1,\ldots,s_n) = n^*\cdot n_i \cdot t_i(y_i + W_i)$$

where $t_i = (1 + \frac{m_i}{\sum_{j=1}^{n} m_j}\|r - \tilde{\epsilon}\|)^{-1}$

Proof: In the first part of the proof we shall show that any CREE allocation can arise as a BNE of this game.

Let $P(\epsilon)$, $f(\epsilon)$ denote a CREE.

We now construct the following BNE which yields f.

$P^i = P$	for all $i = 1,\ldots,n$
$r_i = \epsilon_i$	for all ϵ in J and all $i = 1,\ldots,n$
$z_i = f_i(\epsilon) - W_i$	for all ϵ and all the possible signals for
$n_i = m_i = 1$	all $i = 1,\ldots,n$.

At the first stage we end up with P and no single individual can change it by deviating and declaring some other pricing function.

At the second stage we end up with the true profile $(\epsilon_1,\ldots,\epsilon_n)$ for all the individuals. At the end of this stage each individual is told the "true" price $P(\epsilon_1,\ldots,\epsilon_n)$.

For all ϵ in J z_i is contained in K^i, and hence y_i equals z_i and since f was a CREE we have that for all ϵ in J

$$\sum_{i=1}^{n}(f_i(\epsilon) - W_i + W_i) = W$$

So that $n^* = 1$, since $n_i = 1$ for all $i = 1,\dots,n$.

$t_i = 1$ equals one since they always tell the truth.

Thus

$$g(s_1,\dots,s_n) = f(\epsilon)$$

Now we shall show that this n-tuple of strategies does form a BNE. Individual i cannot change the set K^i and he gets for any ϵ in J the point $f_i(\epsilon) - W_i$ in K^i which, by the definition of the CREE is the most preferred point in K^i. Hence individual i cannot gain by changing his strategy. This holds for all $i = 1,\dots,n$, which shows this n-tuple of strategies forms a BNE.

Now we shall show that any BNE of the game forms a CREE allocation. A BNE gives rise to some pricing function P which is constructed in stage 1. At the second stage we have an n-tuple of strategies

$$r_1(\epsilon_1),\dots,r_n(\epsilon_n)$$

where $r_i\colon J_i \to J_i$.

These functions tell what observation will be reported by individual i as a function of his true observation. Because of the t_i term we get the same as in theorems 2 and 3 that $r(\epsilon) = (r_1(\epsilon_1),\dots,r_n(\epsilon_n))$ must satisfy $r(\epsilon)$ is in J for all ϵ in J. So the "effective" pricing function we have is

$$\bar{P}(\epsilon_1\dots,\epsilon_n) = P(r_1(\epsilon_1),\dots,r_n(\epsilon_n))$$

After stage 2, each individual is told the price vector $\bar{P}(\epsilon)$ and his utility of bundle X if $\tilde{\epsilon}$ occurred is now:

$$V_i(X) = E(U_i(X,\epsilon) \mid \tilde{\epsilon}_i, \bar{P}(\tilde{\epsilon})).$$

It can be shown the same way as in Postlewaite and Wettstein [1983] that for any ϵ in J the BNE must have

$$n^* \cdot n_i = 1 \quad \text{for all} \quad i = 1,\dots,n$$

$$\sum_{i=1}^{n} y_i = 0$$

and as is clear from the properties of $r(\epsilon)$

$$t_i = 1 \quad \text{for all} \quad i = 1,\dots,n.$$

Furthermore, proceeding as in Postlewaite and Wettstein [1983] it can be shown that for any $\tilde{\epsilon}$ in J the bundle received by individual i in the BNE, $f_i(\tilde{\epsilon})$, must solve:

$$\max_{X} \quad V_i(x)$$

s.t.

$$X \in K_i(\bar{P}(\epsilon), W_i, W)$$

and also $\sum_{i=1}^{n} f_i(\epsilon) = W$ for all ϵ in J.

Hence the allocation rule induced by the BNE $f(\epsilon)$, and the pricing function $\bar{P}(\epsilon)$ constitute a CREE.

□

Remark: If J is assumed to be convex the above game form is indeed continuous.

6. **Concluding Remarks**

We have succeeded in continuously implementing various SCC's. However, the strategy space was rigged so as to eliminate bad BNE. This was done by constructing a noncompact strategy space. The question is whether we could obtain similar results using a compact strategy space so that there will always be a best response strategy.

The signalling structures we used were, of course, not the most general ones. One could consider adding some noise to the signals and so on. This will prove interesting when examining the implementability of specific SCC's.

The NEI assumption is indeed quite restrictive. However, it is not clear whether it can be relaxed. Blume and Easley [1985] have shown this is a necessary condition for certain kinds of implementation. It could be that introducing a repeated game structure where some misrepresentation could hurt you in the long run, will enable us to relax that assumption.

One area not mentioned in this work where many of the ideas used here can be applied, has to do with resource allocations within large firms. The designer will be the manager and the players the heads of the various departments. Good allocations will supposedly be profit maximizing ones.

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