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A CORRESPONDENCE THEOREM BETWEEN EXPECTED UTILITY AND SMOOTH UTILITY

by

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1. Introduction

There has been much recent dissatisfaction with the expected utility hypothesis due primarily to the systematic practical violation of the independence axiom. (See Machina (1984a) for a survey of the relevant literature.) Since expected utility theory continues to dominate the economics of uncertainty, it is important to investigate the robustness of its results to failures of the independence axiom. Machina (1982) provides several demonstrations of such robustness. In this note, we provide a new class of robustness results. The analysis proceeds in two stages. First, we establish a correspondence theorem which translates any appropriate theorem in expected utility analysis into a theorem in a generalized preference framework. Then, we present several illustrative applications of this theorem. The examples cover a broad range of models that arise in the economics of uncertainty -- optimal stochastic growth, portfolio diversification and optimal insurance design. Many other instances will no doubt occur to the reader where our correspondence theorem applies.

Intuitively, the metatheorem hinges on the fact that since "smooth" preference functionals are locally expected utility, first order conditions in optimization problems involving such general functionals "resemble" those arising when the expected utility specification is adopted. The only difference is that a local utility function replaces the von Neumann-Morgenstern utility index. Machina (1982, p.288) points out this formal analogy in one context. Here, the analogy is viewed in a more general context and several new implications are derived from it.

Section 2 presents a definition of upper-semi-smoothness of a preference functional and some examples of such preference functionals. The correspondence theorem is proved in Section 3. Its applications are developed in the final three sections. Some technical details are collected in an appendix.

2. Upper-Semi-Smooth Utility

Let X be a convex subset of $\mathbb{R}^n$, where $n = \infty$ is allowed, and denote by D(X) the set of all c.d.f.'s having support in X. A path is a function $H_{(\cdot)}$ that maps [0,1] into D(X). The set of all paths is denoted by Π. Subsets of Π will be of interest. In particular, consider

$$C_d = \{H_{(\cdot)} \in \Pi \mid H_\alpha = (1-\alpha)F+\alpha G,\ F \text{ and } G \in D(X),\ \alpha \in [0,1]\},$$

$$C_r = \{H_{(\cdot)} \in \Pi \mid H_\alpha = F_{(1-\alpha)\tilde{x}+\alpha\tilde{y}},\ F_{\tilde{x}} \text{ and } F_{\tilde{y}} \in D(X),\ \alpha \in [0,1]\},$$

and

$$C_r' = \{H_{(\cdot)} \in C_r \mid F_{\tilde{x}} \text{ has a density function}\}.$$

Paths in C_d are linear in the space of distributions. Paths in C_r are linear in the space of r.v.'s.

If V is a preference functional defined on D(X), we say that V is upper-semi-smooth in the path sense with respect to $C \subset \Pi$, if $\exists$ u : $X \times D(X) \rightarrow \mathbb{R}$ such that $\forall\ H_{(\cdot)} \in C$,

$$\int_X u(\cdot,H_0)d[H_\alpha-H_0] > 0 \quad \text{for } \alpha \text{ small}$$

$$\Rightarrow \quad V(H_\alpha) - V(H_0) > 0 \quad \text{for } \alpha \text{ small.} \tag{1}$$

Refer to the path derivative u as a local utility function for V

A special case of (1) occurs when V is linear on D(X) and hence

u is independent of its last argument. Then V is an expected utility functional and u is its von Neumann-Morgenstern utility index.

When X is a compact subset of the real line (or indeed of any $\mathbb{R}^n$ with $n < \infty$), Machina (1982) considered the implications of a Fréchet differentiable preference functional in the L^1 norm and showed the existence of u, referred to as a local utility function, having the property that

$$V(G) - V(F) = \int_X u(\cdot,F)d[G-F] + o(\|G-F\|). \tag{2}$$

Clearly, a Fréchet differentiable functional satisfies (1) for any path $H_{(\cdot)}$ that is smooth in the L^1 norm (i.e., $\|H_\alpha - H_0\|$ is differentiable in α). In particular, such a functional is upper-semi-smooth with respect to C_d and C_r. For example, consider the following quadratic preference functional (Machina, 1982):

$$V(F) = \int_X RdF + \frac{1}{2}[\int_X SdF]^2, \tag{3}$$

with local utility function given by

$$u(x,F) = R(x) + [\int_X SdF]S(x).$$

The relation between upper-semi-smoothness and Gâteaux differentiability is considered in the appendix.

Machina's analysis however does not readily extend to infinite dimensional outcome sets X. Moreover, smoothness in the sense of (1) is weaker than Fréchet differentiability in that the one-sided local approximation of the preference functional V by the path derivative u is required only on some set of paths in $D(X)$ and not on open neighborhoods of H_0 defined by some norm. Moreover, unlike Fréchet differentiability, smoothness in the sense of (1) is invariant to arbitrary increasing transformations of V, i.e., it is <u>ordinal</u>.

A class of preferences that are always upper-semi-smooth is given by preference functionals that are induced by expected utility maximization problems; that is, suppose that

$$V(F) = \max_{a \in B} \int_X \phi(x,a)dF(x), \tag{4}$$

where "a" is the choice variable and B is a constraint set. The probability distribution F represents a temporal risk in the sense of Kreps and Porteus (1979) and Machina (1984b). Kreps and Porteus showed that induced preferences over temporal risks are generally nonlinear in F. Machina provided conditions under which such preferences will be Fréchet differentiable. But upper-semi-smoothness for V always holds as long as the optimization problem defining V has a solution for each F in D(X). The proof is as follows: Let $H_{(\cdot)}$ be any path, and denote by a*(F) a solution given F. Let $u(\cdot,F) = \phi(\cdot,a^*(F))$. If $\int_X u(\cdot,H_0)d(H_\alpha - H_0) > 0$ for α small, then

$$V(H_0) = \int_X \phi(\cdot,a^*(H_0))dH_0,$$

and

$$V(H_\alpha) \geq \int_X \phi(\cdot,a^*(H_0))dH_\alpha$$

$$> \int_X \phi(\cdot,a^*(H_0))dH_0 = V(H_0) \quad \text{for } \alpha \text{ small.}$$

Another class of examples that always satisfies (1) is given by implicit weighted utility (Chew, 1983; 1985; Dekel, 1985). In this case, the utility V(F) is defined to be the solution s of:

$$\int_X w(x,s)[v(x)-s]dF(x) = 0, \tag{5}$$

where v is a utility function on X, and $w(\cdot,\cdot) : X \times Rng(v) \to \mathbb{R}^+$ is a weight function which depends on the outcome x as well as the reference utility level s. $w(x,s)[v(x)-s]$ is continuous and strictly decreasing in s. Then V is upper-semi-smooth with respect to Π with local utility function given by

$$u(x,F) = w(x,V(F))[v(x)-V(F)]. \tag{6}$$

This follows from the observation that $V(G) > V(F)$ if (and only if) $\int_X u(\cdot,F)d[G-F] > 0$.

A different example, called rank-dependent utility is the following (see, e.g., Quiggin (1982) and Chew, Karni and Safra (1985)). Assume that X is an interval of $\mathbb{R}$,

$$V(F) = \int_X v(z)d(g \circ F)(z), \tag{7}$$

where $v : X \to \mathbb{R}$ and $g : [0,1] \to [0,1]$. Both v and g are continuous and g is strictly increasing and onto. It is shown in the appendix that when g is differentiable and v is bounded, V is upper-semi-smooth with respect to C_d and C_r'.

3. A Correspondence Theorem

A subset $C \subset \Pi$ is called locally-monotone with respect to a set A of real-valued functions on X if $\forall$ $f \in A$, and $\forall$ $H_{(\cdot)} \in C$,

$$\int_X fd[H_1-H_0] > 0 \quad \Rightarrow \quad \int_X fd[H_\alpha-H_0] > 0 \quad \text{for small } \alpha.$$

For example, C_d is locally monotone with respect to any real-valued function on X, while C_r is locally monotone with respect to the class of concave functions on X.

The subset $S \subset D(X)$ is said to be connected with respect to $C \subset \Pi$ if $\forall$ F, $G \in S$, $\exists$ $H_{(\cdot)} \in C$ such that $H_0 = F$, $H_1 = G$, and $H_\alpha \in S$ for

$\alpha \in (0,1)$. For example, any convex subset of $D(X)$ is connected with respect to C_d while many standard economic problems (Sections 5 and 6) generate subsets of $D(X)$ that are connected with respect to C_r. The appendix provides an instance of connectedness with respect to a set C different from both C_d and C_r.

Consider optimization problems of the form

$$\max \{ V(F) \mid F \in S \}, \tag{8}$$

where $S \subseteq D(X)$ is a (fixed) constraint set. If a unique optimum exists, we denote it by $F^*(V)$. Theorem 1 below is the central result of this paper.

Theorem 1: *Let A be a set of real-valued functions on* X. *Suppose that* $C \subset \Pi$ *is locally-monotone with respect to* A *and that* S *is connected with respect to* C. *Consider optimization problem* (8). *Suppose the following is true:*

(a) *If* V *is an expected utility functional and if* $u \in A$, *then* $F^*(V)$ *exists and it lies in* R.

Then:

(b) *If* V *is upper-semi-smooth with respect to* C, *if* $F^*(V)$ *exists and if* $u(\cdot,F) \in A \ \forall\ F \in S$, *then* $F^*(V) \in R$.

Statement (a) defines a class of theorems in expected utility analysis. It asserts that under specified assumptions, the solution of the simple optimization problem (8) will have certain properties. Statement (b) describes the corresponding theorem for upper-semi-smooth preference functionals. Existence of $F^*(V)$ is often guaranteed by continuity, compactness and convexity assumptions on V and/or S. The critical hypothesis in (b) is that <u>each</u> local utility function satisfies the assumptions corresponding to A, which renders the independence axiom unnecessary. That $F^*(V) \in R$ can again be deduced.

<u>Proof</u>: Let W be the expected utility functional with utility index $u(\cdot, F^*(V))$. Note that the latter is in A. Thus, by (a), $F^*(W)$ exists. If

$$F^*(W) = F^*(V), \tag{9}$$

then $F^*(V) \in R$ is implied by (a). Suppose (9) is false. Then, $W(F^*(W)) > W(F^*(V))$. (Equality here would contradict the uniqueness of $F^*(W)$ as the W-maximizing distribution in S.) Since S is connected with respect to C, $\exists\ H_{(\cdot)} \in C$ such that $H_0 = F^*(V)$, $H_1 = F^*(W)$ and $H_\alpha \in S$ for $\alpha \in (0,1)$. By local monotonicity of C, $\int_X u(\cdot, F^*(V))d[H_\alpha - F^*(V)] > 0$ for small α. By upper-semi-smoothness, this implies that $V(H_\alpha) > V(F^*(V))$ for small α, contradicting the optimality of $F^*(V)$.

Q.E.D.

4. <u>Optimal Insurance Design</u>

An individual with wealth w faces a random damage $\tilde{x} \geq 0$ with a distribution function $F_{\tilde{x}}$. Let a risk neutral insurance company offer a contract (π, I), where $\pi \geq I > 0$, π is the insurance premium and I is the expected indemnity. Thus the insured pays π with certainty and chooses a payoff function $s(x) \geq 0$ subject to the constraint $E[s(\tilde{x})] = I$. The set of feasible distributions for net wealth is given by

$$S = \{ F_{\tilde{y}} \in D(\mathbb{R}_+) \mid E[\tilde{y}] \leq w-\pi+I-E[\tilde{x}] \}.$$

Note that S is connected with respect to C_d which is always locally monotone. Let

$$A = \{ u : \mathbb{R}_+ \to \mathbb{R} \mid u \text{ is increasing and strictly concave}\}.$$

For any expected utility functional with utility index in A, a unique optimum in S exists (this is a special case of the result in Raviv (1979) which extends Arrow (1974)), and it lies in

$$R = \{F_{\tilde{y}} \in S \mid \exists\ d \geq 0 \text{ and } \tilde{y}(x) = w-\pi-\min(\tilde{x},d)\ \forall \text{ realizations } x \text{ of } \tilde{x}\}.$$

That is, the optimal insurance contract has full coverage beyond some deductible $d \geq 0$.

By Theorem 1, such contracts are also optimal for upper-semi-smooth (with respect to C_d) preference functionals whose local utility functions $u(\cdot,F) \in A\ \forall\ F \in S$. In particular, the optimality of such contracts is established for the realistic case of temporal risks if each "primitive" von Neumann-Morgenstern utility index $\phi(\cdot,a)$, from (4) is increasing and strictly concave.

In the particular example of the rank-dependent utility (7), the restriction $u \in A$ is equivalent to (i) v is strictly increasing on $\mathbb{R}_+$ (hence $u(x,F)$ increasing in x $\forall\ F \in D(\mathbb{R}_+)$); (ii) v and g are both concave and at least one is strictly concave (hence $u(x,F)$ strictly concave in x $\forall\ F \in D(\mathbb{R}_+)$). This follows immediately from expression (A.4) for $u(\cdot,F)$ given in the appendix. (Note that (i) and (ii) correspond to first and second degree stochastic dominance respectively.) The general statement of the above equivalence for Fréchet smooth utility functionals was shown by Machina (1982; 1983).

The corresponding conditions for implicit weighted utility are those which correspond to the increasingness and concavity of $u(\cdot,F)$ as defined in (6). Note that, when $w(x,s)$ is not differentiable in s, implicit weighted utility may not be Fréchet (or even Gâteaux) differentiable, even though it is always upper-semi-smooth.

5. Stochastic Optimal Growth

Consider the aggregative optimal growth model with uncertain technology as in Brock and Mirman (1972) and Mirman and Zilcha (1975). Assume that the initial stocks of capital vary in $[a,b]$, $0 < a < b < \infty$, and let $X \subset \mathbb{R}_+^\infty$ be the set of all realizations of feasible consumption plans $\tilde{c} = (\tilde{c}_0, \tilde{c}_1, \cdots)$ from some initial stock $x_0 \in [a,b]$. Then, $\exists$ $0 < B < \infty$ such that $0 \leq c_t \leq B$ for $t = 0, 1, 2, \cdots$ if $(c_0, c_1, \cdots) \in X$. Fix x_0 in $[a,b]$, let $S = \{F \in D(X) \mid F$ is the distribution function for some feasible consumption $\tilde{c} = (\tilde{c}_0, \tilde{c}_1, \cdots)\}$, and define the class of von Neumann-Morgenstern utility indices

$A = \{u : X \to \mathbb{R} \mid u(c) = \Sigma_{t=0}^{\infty} \alpha^t \hat{u}(c_t)$ on X, where $0 < \alpha < 1$, $\hat{u} : [0,B] \to \mathbb{R}$, $\hat{u}$ is increasing, strictly concave, continuously differentiable with $\hat{u}'(0) = \infty\}$.

If preferences are expected utility with utility index in A, a unique optimal consumption plan exists with distribution function F^* and t-th marginal F_t^*. Moreover, $F^* \in R$, where

$$R = \{F \in D(X) \mid \exists \text{ a distribution function } G \text{ on } [0,B] \text{ such that } F_t \underset{t\to\infty}{\to} G \text{ uniformly on } [0,B]\}.$$

Thus, the distribution functions of the optimal consumption levels converge asymptotically to some limiting distribution.

Note that S is connected with respect to C_r and the latter is locally monotone with respect to A. Thus our theorem may be applied to extend this convergence result beyond the expected utility framework. For example, let V be the weighted utility functional defined on D(X) by

$$V(F) = E_F[\Sigma_0^\infty \alpha^t h(\tilde{c}_t)]/E_F[\Sigma_0^\infty \alpha^t w(\tilde{c}_t)],$$

where $0 < \alpha < 1$, $h : [0,B] \to \mathbb{R}$ and $w : [0,B] \to [\underline{m},\bar{m}]$ for some $0 < \underline{m} < \bar{m} < \infty$. Then V is upper-semi-smooth (with respect to C_r) with local utility function:

$$u(c,F) = \Sigma_0^\infty \alpha^t \hat{u}(c_t,F),$$

where $\hat{u}(\cdot,F) = h(\cdot)-V(F)w(\cdot)$. Restrict h and w so that $\hat{u}(\cdot,F)$ is increasing, strictly concave and continuously differentiable with $\hat{u}_1(0,F) = \infty \ \forall \ F \in D(X)$. By Theorem 1, the optimum of V has the limiting property described above (i.e., belongs to R).

Similarly, the result applies to quadratic preference functionals (3) with

$$R(c) = \Sigma\alpha^t r(c_t)$$

and

$$S(c) = \Sigma\alpha^t s(c_t)$$

for r, s increasing, strictly concave, continuously differentiable with $r'(0) = s'(0) = \infty$, $s \geq 0$ everywhere. We have

$$u(c,F) = \Sigma\alpha^t \hat{u}(c_t,F),$$

where

$$\hat{u}(\cdot,F) = r(\cdot) + E_F[\Sigma\alpha^t s(\tilde{c}_t)]s(\cdot).$$

6. Portfolio Diversification

Consider a two-risky-assets portfolio problem where the assets have gross returns $\tilde{z}_1$ and $\tilde{z}_2$ neither of which is degenerate, and the range of each $\tilde{z}_i$ is contained in some compact interval $X \subset \mathbb{R}$. Suppose that $\tilde{z}_1$ and $\tilde{z}_2$ have equal means and are negatively correlated in the sense of Samuelson (1967, p.7) or Hadar and Russell (1974, p.238). The constraint set facing the investor is

$$S = \{F \in D(X) \mid F = F_{(1-\alpha)\tilde{z}_1+\alpha\tilde{z}_2} \text{ for some } \alpha \in [0,1]\}.$$

Let

$$A = \{u : X \to \mathbb{R} \mid u \text{ is continuous, strictly increasing and strictly concave on } X\}.$$

For any expected utility functional with utility index in A, a unique optimum in S exists (Samuelson, 1967; Hadar and Russell, 1974) and it lies in

$$R = \{F \in S \mid F = F_{\alpha\tilde{z}_1+(1-\alpha)\tilde{z}_2} \text{ for some } \alpha \in (0,1)\}.$$

That is, the optimal portfolio is diversified.

By Theorem 1, diversification is also optimal for any upper-semi-smooth preference functional with respect to C_r whose local utility function u satisfies, $\forall\, F \in S$, $u(\cdot,F) \in A$. This includes the examples described in Section 2. In particular, the diversification result is extended to the context of temporal risks.

For rank-dependent utility (6), upper-semi-smoothness is valid with respect to C_r' if g is differentiable and v bounded. Thus diversification is optimal for such rank-dependent utility functions if we consider only r.v.'s $\tilde{z}_1$ and $\tilde{z}_2$ having density functions. Of course, monotonicity and concavity restrictions analogous to those described in Section 4 must be imposed.

Appendix

We consider here the relation between upper-semi-smoothness and Gâteaux differentiability. For a preference functional V defined on D(X), we say that V is <u>smooth</u> in the <u>Gâteaux</u> sense, if there exists $u : X \times D(X) \to \mathbb{R}$ such that $\forall\ F, F_0 \in D(X)$, $\lambda > 0$,

$$V(\lambda F+(1-\lambda)F_0)-V(F_0) = \lambda\int_X u(x;F_0)d[F(x)-F_0(x)]+o(\lambda). \qquad \text{(A.1)}$$

It is clear that Gâteaux differentiability implies upper-semi-smoothness with respect to C_d. It is easy to show that the converse does not hold (see remark at the end of Section 4).

We consider upper-semi-smoothness with respect to:

$$C_p = \{H_{(\cdot)} \in \Pi : \tfrac{\partial}{\partial\alpha}H_\alpha(x)\big|_{0^+} \text{ exists } \forall\ x \in X\}. \qquad \text{(A.2)}$$

Note that C_p contains C_d as well as C_r' allowing for a potentially broader class of applications for Theorem 1. Shavell's (1979) paper on optimal insurance under moral hazard provides an example of a feasible set S that is connected with respect to C_p.

The following smoothness definition relative to C_p is due to Hadamard (1923). V is <u>Hadamard differentiable</u> if $\exists\ u : X \times D(X) \to \mathbb{R}$ such that $\forall\ H_{(\cdot)} \in C_p$,

$$V(H_\alpha) - V(H_0) = \int_X u(x,H_0)d[H_\alpha-H_0] + o(\alpha). \qquad \text{(A.3)}$$

Clearly, Hadamard smoothness implies Gâteaux smoothness. It also implies upper-semi-smoothness with respect to C_p. Again, from Lemma 1 below, implicit weighted utility provides a counterexmple for the converse.

Lemma 1: *If V is Gâteaux differentiable whose Gâteaux derivative* $u(\cdot,F)$ *is continuous and* $|u(x,F)| \le M < \infty\ \forall\ x \in X$ *and* $\forall\ F \in D(X)$, *then V is Hadamard differentiable.*

Proof: Consider $H_{(\cdot)} \in C_p$. Then

$$\frac{1}{\delta}\{V(H_\delta) - V(H_0)\} - \int_X u(\cdot,H_0)d[\frac{\partial}{\partial\alpha}H_\alpha|_{0^+}]$$

$$= \frac{1}{\delta}\{V(H_\delta) - V(H_0+\delta[\frac{\partial}{\partial\alpha}H_\alpha|_{0^+}])\} + B,$$

where

$$B = \frac{1}{\delta}\{V(H_0+\delta[\frac{\partial}{\partial\alpha}H_\alpha|_{0^+}]) - V(H_0)\} - \int_X u(\cdot,H_0)d[\frac{\partial}{\partial\alpha}H_\alpha|_{0^+}].$$

Note that Gâteaux differentiability of V implies that B converges to 0 as $\delta \to 0$. Let

$$F_t^\delta = (1-t)[H_0+\delta[\frac{\partial}{\partial\alpha}H_\alpha|_{0^+}]] + tH_\delta.$$

Then

$$\frac{1}{\delta}\{V(H_\delta) - V(H_0+\delta[\frac{\partial}{\partial\alpha}H_\alpha|_{0^+}])\}$$

$$= \frac{1}{\delta}\left[\int_0^1\left[\int_X u(\cdot,F_t^\delta)d(H_\delta-\{H_0+\delta[\frac{\partial}{\partial\alpha}H_\alpha|_{0^+}]\})\right]dt\right]$$

$$\le M\int_X d\{\frac{1}{\delta}[H_\delta-H_0]-\frac{\partial}{\partial\alpha}H_\alpha|_{0^+}\} \underset{\delta\to 0}{\to} 0.$$

This completes the proof.

As shown in Chew, Karni and Safra (1985), when g is differentiable on [0,1], rank-dependent utility (7) is Gâteaux differentiable (but not Fréchet differentiable) with local utility function given by (A.4) below:

$$u(x,F) = \int_{(-\infty,x]\cap X} g'(F(z))dv(z). \tag{A.4}$$

Observe that u in (A.4) is uniformly bounded if v is bounded. In that case, rank-dependent utility is Hadamard differentiable and therefore upper-semi-smooth with respect to C_p, C_d and C_r'.

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