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EXPECTED UTILITY WITH PURELY SUBJECTIVE
NON-ADDITIVE PROBABILITIES

by

Itzhak Gilboa

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INTRODUCTION

The problem of "subjective" (or "personalistic") probability, which is at the root of Bayesianity, has aroused interest since the early works of Bayes. The research on subjective probability attained new momentum with the works of F.P.Ramsey and B. De Finetti. The most convincing and well-known axiomatization of subjective probability was given by Savage (1954). He started with a preference relation over acts (i.e., functions from the states of the world into the consequences), in order to end up with a utility function and a probability measure, such that the individual's decisions are being made so as to maximize the expected utility.

However compelling Savage's axioms and results are, they are not immune to attacks. The following argument against his theory, which is the conceptual basis of this paper, was raised by Schmeidler (1982 and 1984a): Consider the tossing of two coins, the one known to be fair, and the other one about which there is no information whatsoever. Symmetry considerations lead us to ascribe the probability 1/2 to both sides of each coin, but there surely is a great difference between the number 1/2 in the first case, and the very same number in the second. This somewhat vague distinction becomes palpable when translated into a decision problem, as is done in the Ellsberg Paradox. (See E llsberg (1961)).

The modification suggested by Schmeidler in order to cope with this difficulty is the introduction of non-additive probabilities. Allowing non-additivity, one may ascribe a probability strictly smaller than 1/2 to each side of the 'unknown' coin, although the coin, when tossed, will fall on one of its sides with probability 1.

Although this was not the primary motive for developing the non-additive expected utility theory, it turned out that Schmeidler's model may explain some of the 'paradoxes' or counterexamples to the traditional expected utility theory, which have already stimulated many studies of various generalizations of expected utility theory. Some of the latest of these studies (most of which are in a purely objectivistic context) such as Quiggin's (1982), Yaari's (1984) and others, lead to results that are special cases of the non-additive theory.

Schmeidler's works provide an axiomatization for expected utility maximization, where the probability measure is not necessarily additive, in the framework of Anscombe and Aumann (1963). Their model, as opposed to Savage's, involves both 'objective' ('physical') and 'subjective' probabilities, while only the latter are derived from a preference relation over acts ('Horse Lotteries'), and the former are primitives of the model. This model is mathematically simpler than that of Savage, but it has the drawback of using the controversial concept of 'objective' probabilities.

This paper is the non-objectivistic counterpart of that of Schmeidler. That is to say, it axiomatizes expected utility maximization with a non-additive subjective probability in a Savageian spirit. It does not purport to be conceptually innovative; it merely justifies the same results in the more satisfactory framework not involving objective probabilities. The connections among the models will be clarified by the following table:

	Objective and Subjective Probabilities	Only Subjective Probabilities
Additive Probabilities	Auscombe-Aumann (1963)	Savage (1954)
Non (necessarily) Additive Probabilities	Schmeidler (1982)	The paper you're now reading

Mathematically speaking, there is a great difference between the right and the left columns of the table, since the mathematical objects involved in them are quite different, whereas there is a considerable similarity in the nature of the mathematical work within each column. However, it should be pointed out that this paper, although constantly comparing itself to that of Savage, differs significantly from the latter. In fact, almost none of Savage's results were proved applicable, and even the fundamental von Neumann-Morgenstern expected utility theorem (1947), which is at the basis of all three existing theories, could not be used here.

The paper is organized as follows. Section 1 deals with some preliminaries, namely: the framework of the model and some useful definitions; The Choquet integral; Savage's theorem (for comparison purposes) and Statement and brief discussion of the axioms for the non-additive theory. Sections 2 and 3 contain the proof of the main representation theorem. In section 2 the probability measure is almost constructed, or rather, something that is almost a measure is constructed. Section 3 develops the utility theory, by defining a utility function and proving some representation theorems. One of the stages is, of course, the completion of the construction of the measure. However, the distinction between these sections, which is undeniably somewhat arbitrary, is based on their subject-matter: section 2 goes as far as the theory proceeds without mentioning the word 'utility', and there begins section 3. Section 4 contains some technical details or, more specifically, examples which prove the necessity of the technical axioms. Section 5 presents some further results regarding continuity and quasi-continuity of the measure. (These terms are defined in it). These results are also valid in Savage's model since once the representation theorem has been proved, the additive theory becomes a special case of the non-additive one.

1. PRELIMINARIES

1.1. Framework and Definitions

Let S be the set of states of the world, X the set of consequences, and $F = \{f: S \to x\}$ the set of acts. Subsets of S will be called events. For $f, g \in F$ and $A \subset S$ we will define $f / {}^{g}_{A}$ to be the element of F satisfying:

$$f / {}^{g}_{A}(s) = f(s) \; \forall s \in A^c; \quad f / {}^{g}_{A}(s) = g(s) \; \forall s \in A.$$

For $x \in X$ we will define $x \in F$ to be the constant act:

$$x(s) = x \quad \forall \, s \in S.$$

Since no confusion may result, we will not distinguish between the notations of the two entities (the consequence and the act). $\geq$ will denote a binary relation over F: $\geq \, \subset F \times F$, to be interpreted as the preference relation. $(>, \leq, <, \sim)$ are defined in the usual way. An act assuming only finitely many values is said to be a simple act or a step function.

We will use the following

Notation For $x_1 > x_2 > \ldots > x_n$ $(x_i \in X)$ and $\phi = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = S$, $(x_1, A_1; \; x_2, A_2; \ldots; x_n, A_n)$ denotes the simple act f satisfying:

$$f(s) = x_i \qquad \forall s \in A_i - A_{i-1}, \quad 1 \leq i \leq n.$$

Using this notation will henceforth presuppose that $x_1 > x_2 > \ldots > x_n$ and $A_1 \subset A_2 \subset \ldots A_n$. That is to say, any statement involving the act

$(x_1, A_1;\ldots;x_n,A_n)$ should be read as follows: "$x_1 > \ldots > x_n$, $A_1 \subset \ldots \subset A_n$, and... ".

A set function $v: 2^S \to R$ will be called a measure iff it satisfies:

(i) $E \subset F \Rightarrow V(E) \leq (F)$;

(ii) $v(\phi) = 0$; $v(S) = 1$.

If not explicitly stated, a "measure" is not assumed to be additive. A measure v is said to have a convex range if for any $B \subset A \subset S$ and any $\alpha \in [0,1]$ there is an event C, $B \subset C \subset A$, such that

$$v(C) = \alpha v(B) + (1-\alpha)v(A).$$

A real function over X will be called a utility.

Two acts $f, g \in F$ are said to be comonotonic iff there are no $s, t \in S$ such that

$$f(s) > f(t) \text{ and } g(s) < g(t).$$

An event A will be said to be comonotonic with an act f iff there are no $s \in A$, $t \in A^c$ such that

$$f(t) > f(s).$$

An event A will be said to be f-convex for an act f, iff the following condition holds:

For any $s, t \in A$, $r \in S$ such that $f(s) < f(r) < f(t)$, it is true that $r \in A$.

1.2. The Choquet Integral

The introduction of non-additive probabilities poses some difficulties. First of all, the integration w.r.t. (with respect to) such (probability) measures is not well-defined: Consider a constant function over $[a,b] \subset R$, and note that the partial Riemann sums (which are all supposed to equal the integral) depend upon the specific partition of the domain. Straightforward definitions of the integral (such as summation over maximal sets, on which the integrand is constant) are bound to face problems of non-monotonicity and/or discontinuity of the functional. It turns out that the natural integral for the non-necessarily additive measures is the Choquet integral, defined as follows:

Let S be the domain of the integrands, and v – a measure on S. The integral of $w: S \to R$ w.r.t. v (over S) is defined to be

(*) $$\int w dv = \int_0^\infty v(\{s \,/\, w(s) \geq t\})dt + \int_{-\infty}^0 [v(\{s \,/\, w(s) \geq t\}) - 1]dt$$

This integral was defined in Choquet (1955), and is used and discussed in Schmeidler (1984b). In this paper the symbol $\int w dv$ will always stand for this functional.

Note that, since the integrands in the two extended Riemann integrals in (*) are monotone functions, the Choquet integral always exist, which is a lot to ask of an integral.

A useful definition will be the following:

the utility u and the measure v are said to consist an Integral-Representation (IR) of $\geq$ over $\bar{F} \subset F$, iff

$$f \geq g \iff \int u(f)dv \geq \int u(g)dv \quad \forall f,g \in \bar{F}.$$

1.3. Savage's Theorem

To formulate Savage's theorem, one has to cite the axioms and definitions involved in it: (The symbol Pn $(1 \leq n \leq 7)$ denotes an axiom).

P1. $\geq$ is complete and transitive.

P2. (Sure Thing Principle). For all $f, g, h_1, h_2 \in F$ and any $A \subset S$,

$$f \,/\, \overset{h_1}{A^c} \geq g \,/\, \overset{h_1}{A^c} \iff f \,/\, \overset{h_2}{A^c} \geq g \,/\, \overset{h_2}{A^c} \;.$$

Definition: If $f \,/\, \overset{h}{A^c} \geq g \,/\, \overset{h}{A^c}$ for some ($\iff$ all by p_2) $h \in F$, we shall say that $f \geq g$ given A.

Definition: If for all $f, g \in F$, $f \geq g$ given A, A will be said to be null.

P3. If $A \subset S$ is not null, then for all $f \in F$; $x,y \in X$,

$$f \,/\, \overset{x}{A} \geq f \,/\, \overset{y}{A} \iff x \geq y$$

P4. For all $x_1, y_1, x_2, y_2, \in X$ and all $A, B \subset S$,

$$(x_1,A;\, y_1,S) \geq (x_1,B;\, y_1,S) \quad \text{iff}$$
$$(x_2,A;\, y_2,S) \geq (x_2,B;\, y_2,S)$$

(Recall that the above notation presupposes that $x_1 > y_1$, $x_2 > y_2$).

Definition: If for some (<=> all by P4) $x, y \in X$, $(x,A;y,S) \geq (x,B;y,S)$. Then $A \geq \cdot B$.

P5. There are $x^*, x_* \in X$ such that $X^* > x_*$.

P6. For any $f,g,h \in F$ such that $f > g$, there is a finite partition of S $(B_1,\ldots,B_n)$ such that

$$f \,/\, \overset{h}{B_i} > g \quad \text{and} \quad f > g \,/\, \overset{h}{B_i} \quad \forall i.$$

P7. If $f \underset{(\geq)}{\leq} g(s)$ given A, for all $s \in A$,
then $f \underset{(\geq)}{\leq} g$ given A.

Savage's Theorem. Suppose $\geq$ satisfies P1-P7. Then there are a unique (finitely) additive probability measure P on S with a convex range, and a bounded utility u, unique up to positive linear transformation (p.l.t.) such that $\geq$ is integral-represented by (u,P) over all F.

(This is a slight rephrasing of the original (Savage's) theorem. The axioms are basically the original, rewritten with some new notations, whereas the conclusion is based on that appearing in Fishburn (1970).)

1.4. Axioms for a non-additive theory

The main difference between the additive and non-additive theories is the sure-thing-principle, accepted by Savage, but rejected by the non(necessarily) additive theory. This means that we cannot accept Savage's P2, and consequently have to replace it by a weaker version.

However, it turns out that there are some technical differences between the two theories, which call for modifying or replacing other axioms as well:

P3 as phrased, turns out to be too strong an axiom, excluding some of the measures we have no reason to object to;

P4 is implied by P2's substitute:

P5 is too weak, since the minimal number of $\geq$-distinguishable consequences needed for the uniqueness of the measure is 3 in the non-additive theory (rather than 2);

P6 is simply insufficient for any kind of continuity in a non-additive context. Here it will be replaced by two axioms, of non-atomicity and archimedianity;

P7 is used in the sequel in a slightly different version than Savage's, but the difference stems mainly from terminological reasons.

In order to facilitate comparison, we will name the axioms after those of Savage. An asterisk will indicate that the axiom differs from its Savageian counterpart. When more than one axiom is used to replace a single axiom in the original model, the number of asterisks will increase monotonically.

The axioms we will need are the following:

(1) P1. – as Savage's

(2) P2.*– For all $f_1, f_2, g_1, g_2 \in F$, all $A, B \subset S$, and all $x_1, x_2, y_1, y_2 \in X$ such that $y_1 > x_1$ and $y_2 > x_2$, <u>if</u>

(i) $f_1 / {}^{x_1}_{A}$, $f_1 / {}^{y_1}_{A}$, $g_1 / {}^{x_2}_{A}$, $g_1 / {}^{y_2}_{A}$ are pairwise comonotonic (p.c.), and so are

$f_2 / {}^{x_1}_{B}$, $f_2 / {}^{y_1}_{B}$, $g_2 / {}^{x_2}_{B}$, $g_2 / {}^{y_2}_{B}$;

and (ii) $f_1 / {}^{x_1}_{A} \sim f_2 / {}^{x_1}_{B}$, $g_1 / {}^{x_2}_{A} \sim g_2 / {}^{x_2}_{B}$ and $f_1 / {}^{y_1}_{A} \geq f_2 / {}^{y_1}_{B}$

<u>then</u> $$g_1 / {}^{y_2}_{A} \geq g_2 / {}^{y_2}_{B} .$$

1.4.1. Observation. P2* implies Savage's P4.

Proof: Take $f_1 = f_2 = x_1$ and $g_1 = g_2 = x_2$.//

Since P4 justifies the definition of $\geq \cdot$ (over 2^S), we may use Savage's definition.

(3) P3*. For all $A \subset S$, $x, y \in X$, $f \in F$, if $x < y$ then $f \,/\, \overset{y}{A} \geq f \,/\, \overset{x}{A}$.

1.4.2. Observation* If, furthermore, $f(s) \leq x < y$ for all $s \in S$, and $A >\cdot \phi$, then $f \,/\, \overset{y}{A} > f \,/\, \overset{x}{A}$.

Proof: In P2*, take $f_1 = f_2 = x$, $g_1 = g_2 = f \,/\, \overset{x}{A}$, $B = \phi$, and $x_1 = x_2 = x$, $y_1 = y_2 = y$.//

(4) P5*. There are at least three consequences x^*, x, x_* such that $x^* > x > x_*$.

(5) P6*. (Non-atomicity). Let $x, y \in X$, $f, g \in F$ and $A \subset S$ satisfy $f \,/\, \overset{x}{A} > g > f \,/\, \overset{y}{A}$, where $f \,/\, \overset{x}{A}$ and $f \,/\, \overset{y}{A}$ are comonotonic. Then there exists an event $B \subset A$ such that

$$(f \,/\, \overset{x}{A-B}) \,/\, \overset{y}{B} \sim g.$$

(6) P6**. (Archimedianity). Let there be a sequence $\{f_n\}_{n \geq 1} \subset F$, which for some $x, y \in X$, $x > y$, and $A \subset S$ satisfies the following two conditions:

(i) $\forall s \in S$, $\forall n \geq 1$, $f_n(s) \leq y$;

(ii) $f_n \,/\, \overset{x}{A} \sim f_{n+1}$

– then $A \sim\cdot\, \phi$.

* This observation follows from P2*, but it is closely connected to P3*, since both mean monotonicity. In the sequel we shall refer to P3* and 1.4.2. together as "P3*".

(7) P7*. Let A be an f-convex event for $f \in F$, and suppose that for some $g \in F$,

$$f \, / \, \underset{A}{f(s)} \; \underset{(\leq)}{\geq} \; g \;\; \forall s \in A.$$

– Then $f \underset{(\leq)}{\geq} g$.

The main theorem is: P1–P7* hold iff there are a measure v with a convex range and a bounded utility u, such that (u,v) consist an IR of $\geq$ over F. Furthermore, v is unique and so is u, up to p.l.t. (The proof is given in sections 2 and 3.) The difference between this theorem and that of Savage is that v is not necessarily additive, and, consequently, the integration operation refers to the Choquet integral.

A discussion of the axioms

Considering the axioms, one should distinguish between conceptually-essential axioms, such as P1, and technical ones, such as P6*. The "essential" axioms are those that are easily defendable on philosophical grounds, and it usually turns out to be the case that they are also easily defended on mathematical grounds, since one can construct simple examples of preference orders, satisfying all the axioms but the one under discussion, but having no IR.

The axioms we will consider to be "essential" are P1, P2* and P3*, and they will be discussed first.

P1 is identical to that of Savage, and we will not expatiate on it.

P3* is a weaker version of P3, and is easily justifiable since it means monotonicity. A technical point should, however, be clarified: Under Savage's P3, if $A \succ \phi$ and $A \cap B = \phi$, then $A \cup B \succ B$. (This fact is also implied by P2). This is not necessarily true in the non-additive case, so that P3 must be modified in order to include probability measures not satisfying the above condition.

P2* is a new axiom, and deserves some deliberation. First, suppose that none of the eight acts involved in it are required to be comonotonic. The axiom simply states that there is a preference order between events: suppose $f_1 / {}^{x_1}_{A} \sim f_2 / {}^{x_1}_{B}$ and $f_1 / {}^{y_1}_{A} \geq f_2 / {}^{y_1}_{B}$ where $y_1 > x_1$. This means that the improvement on A is more weighty than the same improvement on B, so that in some sense A is preferred to (or considered more likely than) B. This statement would be reversed if there were $g_1 / {}^{x_2}_{A} \sim g_2 / {}^{x_2}_{B}$ such that $g_1 / {}^{y_2}_{A} < g_2 / {}^{y_2}_{B}$ with $y_2 > x_2$. The axiom basically state that this reversal is impossible. (For simple acts it is equivalent to Savage's P2 and P4). However, this condition is restricted to the case where $f_1 / {}^{x_1}_{A}, f_1 / {}^{y_1}_{A}, g_1 / {}^{x_2}_{A}, g_1 / {}^{y_2}_{A}$ are p.c., and so are

$$f_2 / {}^{x_1}_{B}, f_2 / {}^{y_1}_{B}, g_2 / {}^{x_2}_{B}, g_2 / {}^{y_2}_{B} .$$

The meaning of comonotonicity is that each event (A or B above) is indeed conceived in the same way in each of the above acts in which it appears. What the Ellsberg paradox shows, in these terms, is that event-assessment is not context-free, i.e., the same event may have different weights when the 'better' or 'worse' events are different.

In Schmeidler (1984a) comonotonicity plays a similar role: restricting one of the essential axioms (Independence) to comonotonic acts allows the probability measure to be non-additive.

Next we turn to the technical axioms, namely, P5*, P6*, P6**, P7*.

P5*, to start with, is the most innocuous of all. It merely states that $|X/_{\sim}| \geq 3$, and should it not hold, one cannot expect to have a unique measure.

P6* is a non-atomicity axiom. It is supposed to sound reasonable. The same can be said about P6**, which is an archimedian axiom: it asserts that an act cannot be indefinitely improved if all 'improvements' are equally weighty.

Both P6* and P6** have very similar counterparts in Luce and Krantz (1971), which is one of the few existing models in the Savageian spirit. However, the justification of these axioms is mainly pragmatic: without each of them, IR of $\geq$ is not guaranteed. This is proved by counterexamples in section 4.

Finally consider P7*. Basically it is similar to P7, only that the latter is phrased in terms of '$\geq$ given an event', which, in the absence of P2, is not well-defined.

P7*, as phrased seems to be the natural way of stating the axiom in our model. However, Savage's example of a preference order, which satisfies P1–P6 but is not integral-represented, may also serve as a justification of P7*, since that preference relation also satisfies P1–P6**.

2. Defining Something like a Measure

We begin with a preliminary lemma which will be used extensively hereafter.

2.1. Lemma. Let a and b be two simple acts such that

$a = (z_1,c_1;\ z_2,c_2;\dots;z_n,c_n)$; $b = (z_1,D_1,\dots,z_n,D_n)$,

with $c_i \sim D_i \quad \forall i \leq n$.

Then $a \sim b$.

Proof: The proof is, naturally, inductive: for $n = 2$ the conclusion of the lemma follows immediately from 1.4.1. and the definition of $\geq$. Assume, therefore, it is true for $k < n$. Let $a' = (z_2, C_2; \ldots; z_n, C_n)$; $b' = (z_2, D_2; \ldots; z_n, D_n)$,

so that $a' \sim b'$.

Denote $f_1 = f_2 = z_2$; $g_1 = a'$; $g_2 = b'$;

$x_1 = x_2 = z_2$; $y_1 = y_2 = z_1$;

$A = C_1$; $B = D_1$.

P2* implies that $a \geq b$ iff $C_1 \geq^{\bullet} D_1$, so that $a \sim b$. //

Throughout the rest of this section and subsection 3.1, we shall assume X to be the triple $T = \{x_*, x, x^*\}$ satisfying $x^* > x > x_*$. Since the least-preferred consequence is always (= until subsection 3.2) x_*, we can write any act f as $(x^*, A; x, B)$, meaning

$$A = \{s \,/\, f(s) = x^*\} \quad \text{and} \quad B = \{s \,/\, f(s) = x\} \cup A.$$

All definitions made in this context should be understood as dependent upon the triple T. However, for convenience of notation, the subscript T will be omitted.

We shall need some lemmas:

2.2. Lemma. Let $E \subset E'$, $F \subset F'$ be events, and $a, a', b, b', c, c', d, d'$ be acts satisfying one of the following three sets of conditions:

(i) $a = (x^*, E; x, A) \sim b = (x^*, F; x, B)$

$a' = (x^*, E'; x, A)$; $b' = (x^*, F'; x, B)$

$c' = (x^*, C; x, E') \sim d = (x^*, D; x, F)$

$c' = (x^*, C; x, E')$; $d' = (x^*, D; x, F')$

(ii) $a = (x^*, E;\ x,A) \sim b = (x^*, F;\ x,B)$

$a' = (x^*, E';\ x,A); \quad b' = (x^*, F';\ x,B)$

$c = (x^*, E;\ x, C) \sim d = (x^*, F;\ x,D)$

$c' = (x^*, E';\ x,c); \quad d' = (x^*, F';\ x,D)$

or

(iii) $a = (x^*, A;\ x,E) \sim b = (x^*, B;\ x,F)$

$a' = (x^*, A;\ x,E'); \quad b' = (x^*, B;\ x,F')$

$c = (x^*, c;\ x,E) \sim d = (x^*, D;\ x,F)$

$c' = (x^*, c;\ x,E'); \quad d' = (x^*, D;\ x,F')$

then $a' \geq b'$ iff $c' \geq d'$.

Proof: First consider the case (i). a, a', c, c' are obviously p.c. (pairwise comonotonic), and so are b, b', d, d' – whence one may apply P2* (with $A: = E' - E$; $B: = F' - F$) to get the sought-after conclusion.

Now assume (ii) holds. If $A \sim^{\cdot} C$, $a \sim c$ and $a' \sim c'$ (by 2.1). If $\neg(A \sim^{\cdot} C)$, assume w.l.o.g. (without loss of generality) that $A >^{\cdot} C$. Since $E' \subset C$, we have $E' \,^{\cdot}\!\leq C$ (by P3*). Applying P6*

will yield the existence of an event $\bar{C} \sim^{\cdot} C$, such that $E' \subset \bar{C} \subset A$. Replacing C by $\bar{C}$ will not change preferences (by 2.1), but will make a, a', c, c' p.c. A symmetric argument for B and D implies the required conclusion. For the case of (iii), the proof is, of course, very similar.//

We shall now define a partial binary operation on $2^S/\sim\cdot$, which is to be thought of as an addition. It will, eventually, be equivalent to summation of the measure. In order to simplify notations and facilitate the discussion, we will not define the operation on equivalence classes of events formally, but rather use the following

<u>Notation</u> If there are events H_0, H_1, H_0', H_1' such that

$H_0 \sim\cdot H_0'$, $H_1 \sim\cdot H_1^1$ and

$$(x^*, B_1; x, H_0) \sim (x^*, \phi; x, H_1)$$

$$(x^*, B; x, H_0') \sim (x^*, B_2; x, H_1')$$

then we shall write $B \sim\cdot B_1 \oplus B_2$ (henceforth read "B is the circle-

sum of B_1 and B_2").

Note that, as defined, circle-addition need not be commutative, nor should it be defined for all pairs of events (B_1, B_2).

Next, let us observe the following facts:

2.3. <u>Lemma</u> (i) If $B \sim\cdot B_1 \oplus B_2$, then $B \geq\cdot B_1$ and $B \geq\cdot B_2$;

(ii) If $B' \sim\cdot B$, $B_1' \sim\cdot B_1$, $B_2' \sim\cdot B_2$

and

$B \sim\cdot B_1 \oplus B_2$, then $B' \sim\cdot B_1' \oplus B_2'$;

(iii) Let $B \sim\cdot B_1 \oplus B_2$, and suppose that $F_0 \sim\cdot F_0'$, $F_1 \sim\cdot F_1'$, where $B_1 \subset F_0$, $B \subset F_0'$ and $B_2 \subset F_1'$.

If $(x^*, B_1; x, F_0) \sim (x^*, \phi; x, F_1)$

then $(x^*, B; x, F_0') \sim (x^*, B_2; x, F_1')$.

Proof: Using 2.1, P3* and P6* shows (i) and (ii) to be trivial, whereas (iii) becomes a direct application of Lemma 2.2.//

Lemma 2.3 (iii) means, in fact, that the circle-sum of two events B_1, B_2 does not depend upon other events.

Having the circle-addition operation, we wish to construct a measure which is additive w.r.t. (with respect to) it. Constructing the measure is based on the familiar principle of measuring each event with an ever-increasing precision, for which one should have an ever-decreasing measurement unit. We are now about to construct these units.

Since $(x^*, S) > (x,S) > (x_*S)$, there is an event A_1 such that $(x^*, A_1) \sim (x,S)$, whence $\phi \;\dot{<}\; A_1 \;\dot{<}\; S$. Similarly, there is an event $A_2 \subset A_1$ such that $(x^*, A_2) \sim (x,A_1)$, and, arguing inductively, we have a sequence $\{A_k\}_{k\geq 1}$ for which the following conditions hold:

(i) $A_k \supset A_{k+1}$

(ii) $A_k \;\dot{>}\; A_{k+1}$ (which also implies $A_k \;\dot{>}\; \phi$).

(The notation A_k will be reserved for members of this sequence even beyond sub-section 3.1, only that there the subscript T will be added to it). We would like to know that this sequence is indeed fine enough to construct a measure. This is guaranteed by:

2.4. Lemma. Suppose $H \;\dot{>}\; \phi$. Then there exists an integer k such that

$$A_k \;\dot{<}\; H.$$

<u>Proof</u>: Denote the act (x,H) by g_1. Suppose $H \cdot < A_1$, for otherwise the lemma is trivial. Now let H_2 be an event satisfying $(x^*,H) \sim (x,H_2) \equiv g_2$. (P6* assures the existence of such an H_2, because $H \cdot < A_1$). The same argument can now be applied for H_2. Eventually we end up with a sequence $\{g_n = (x,H_n)\}$, which is either finite or not. If the sequence is infinite, define $f_1 = g_1$, $A{=}H$, and f_{n+1} to be such that $f_{n+1}(s) \leq x \ \forall s$, and $f_{n+1} \sim f_n \,/\, \overset{x^*}{A}$. Obviously, $f_n \leq g_n$, whence the $\{f_n\}$ sequence is also infinite. But this, by P6**, should imply $H \sim \cdot\ \phi$, which is known to be false.

Consequently there is a k such that $H_k \geq \cdot\ A_1$. But then we have $H \geq \cdot\ A_k$ ($\Rightarrow H \succ\cdot\ A_{k+1}$), which is what we intended to prove. //

Another notation will be proved useful: for $B, C \subset S$ and $n \in N$, we will say that $B \sim \cdot\ nC$ if there are $C = C_1,\ldots,C_n = B$ such that $C_i \sim \cdot\ C_{i-1} \oplus C$ for $2 \leq i \leq n$. We shall refer to the symbol nC as an event, meaning 'any B such that $B \sim \cdot\ nC$'. If there is no B such that $B \sim \cdot nC$, we will write '$nC > \cdot\ B$' for all B. Now we can formulate:

2.5. <u>Lemma</u>. If $C > \cdot\ \phi$ and $B \subset S$, there is an integer n such that $nC \geq \cdot\ B$.

<u>Proof</u>: This is a straightforward application of P6**.//

The preceding lemma allows us the following

<u>Definition</u> For $B \subset S$ such that $\phi \cdot < B \cdot \leq A_1$ and $k \geq 1$, n_k^B is the unique integer satisfying

$$n_k^B A_k \cdot \leq B \cdot < (n_k^B + 1)A_k.$$

(The existence is implied by the lemma, whereas the uniqueness follows from the fact that $A_k \succ\cdot\ \phi$).

We shall also need

2.6. Lemma. Suppose $A_1 \geq^{\bullet} B \geq^{\bullet} C$. Then there exists an event H such that

$$B \sim^{\bullet} C \oplus H. \text{ If, furthermore } B >^{\bullet} C, \text{ then } H >^{\bullet} \phi.$$

Proof: W.l.o.g. we may assume that $C \subset B \subset A_1$. (For B,C not satisfying this condition, one can always find $\bar{C} \sim^{\bullet} C$ and $\bar{B} \sim^{\bullet} B$ which do satisfy it, and use 2.1. to complete the proof).

Since $(x^*, \phi; x, A_1) \leq (x^*, C; x, A_1) \leq (x^*, A_1) \sim (x^*, \phi; x, S)$, there is an event $D \supset A_1$ such that

$$(x^*, c; \; x, A_1) \sim (x^*, \phi, x, D).$$

Now we know that

$$(x^*, B; \; x, D) \geq (x^*, B; x, A_1) \geq (x^*, C; x, A_1) \sim (x^*, \phi; x, D).$$

P6* implies the existence of an event $H \subset B$ satisfying $(x^*, B; x, A_1) \sim (x^*, H; \; x, D)$, which means that $B \sim^{\bullet} C \oplus H$.

As for the 'furthermore' clause, if $H \sim^{\bullet} \phi$, $(x^*, C; X, A_1) \sim (x^*, B; x, A_1)$. Therefore (by P3*), $B >^{\bullet} C$ implies $H >^{\bullet} \phi$.//

Now we can prove:

2.7. Lemma. If $B \subset S$ is such that $\phi \;^{\bullet}< B \;^{\bullet}< A_1$, then $n_k^B \underset{k\to\infty}{\longrightarrow} \infty$

Proof: It is obvious that $n_k^B \leq n_{k+1}^B$. If $\{n_k^B\}_k$ were a bounded sequence, there would be $K, M \in N$ such that $n_k^B = M$ for all $k \geq K$.

Since $MA_k \;^{\bullet}< MA_\ell$, for all $k > \ell$, there would be $k_1 \geq K$ for which $MA_{k_1} \;^{\bullet}< B$. In that case, let H be an event satisfying

$B \sim^{\bullet} MA_{k_1} \oplus H$ (the existence of which follows from Lemma 2.6), and take $k_2 \geq k_1$ such that $A_{k_2} \,{}^{\bullet}\!< H$ (this can be done because of Lemma 2.4). Obviously, $n_{k_2}^B \geq n_{k_1}^B + 1$, which contradicts the boundedness of the sequence $\{n_k^B\}_k$. //

Now we are in a position to define a set-function for all events B such that $B \,{}^{\bullet}\!\leq A_1$, which will be the measure of these events, up to a scaling factor:

for each $k \geq 1$ define ε_k to be $(n_k^{A_1})^{-1}$. Note that by 2.7, $\varepsilon_k \underset{k\to\infty}{\to} 0$.
Now let there be given an event $B \,{}^{\bullet}\!\leq A_1$.
Define $\tilde{v}(B) = \limsup\limits_{k\to\infty} \varepsilon_k n_k^B$.
To see that this set-function is indeed 'almost' a measure, which is monotonic w.r.t. $\geq^{\bullet}$, we have:

2.8. Lemma. (i) If $C \,{}^{\bullet}\!\leq B \,{}^{\bullet}\!\leq A_1$, then $\tilde{v}(c) \leq \tilde{v}(B)$;
(ii) $\tilde{v}(A_1) = 1$;
(iii) $\tilde{v}(\phi) = 0$.

Proof: Trivial. //

The main property of the function $\tilde{v}$ is its circle-additivity:

2.9. Theorem. Let $B_1, B_2, B \,{}^{\bullet}\!\leq A_1$ satisfy $B \sim^{\bullet} B_1 \oplus B_2$.
Then $\tilde{v}(B) = \tilde{v}(B_1) + \tilde{v}(B_2)$.

Proof: First we note that for any $k \geq 1$

$$n_k^{B_1} A_k \,{}^{\bullet}\!\leq B_1 \,{}^{\bullet}\!\leq (n_k^{B_1} + 1)A_k$$

$$n_k^{B_2} A_k \,{}^{\bullet}\!\leq B_2 \,{}^{\bullet}\!\leq (n_k^{B_2} + 1)A_k$$

whence

$$(n_k^{B_1} + n_k^{B_2})A_k \cdot \leq B \cdot \leq (n_k^{B_1} + n_k^{B_2} + 2)A_k.$$

The left-hand side inequivalence implies

$$n_k^{B_1} + n_k^{B_2} \leq n_k^B ,$$

while the right-hand-side one implies

$$(n_k^{B_1} + n_k^{B_2} + 2) \geq n_k^B + 1,$$

so that we may write

$$n_k^{B_1} + n_k^{B_2} \leq n_k^B \leq n_k^{B_1} + n_k^{B_2} + 1.$$

Now suppose that $\{k_i\}_{i\geq 1}$ is a sub-sequence of N such that $\lim_{i\to\infty} \varepsilon_{k_i} n_{k_i}^B = \tilde{v}(B)$. (Such a sub-sequence exists because of the definition of $\tilde{v}$ as limsup).

Obviously, $\exists \lim_{i\to\infty} (\varepsilon_{k_i} n_{k_i}^{B_1} + \varepsilon_{k_i} n_{k_i}^{B_2}) = \tilde{v}(B)$. But, considering the definition of ε_k, one may easily see that $\varepsilon_{k_i} n_{k_i}^B \in [0,1]$ for all k_i, i.e. the sequence is bounded. Hence $\{k_i\}_i$ has a sub-sequence $\{k_{i_j}\}_j$ for which $\{\varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_1}\}_j$ converges.

Since $\{\varepsilon_{k_{i_j}} n_{k_{i_j}}^B\}_j$ also converges (to $\tilde{v}(B)$), we deduce that

$$\exists \lim_{j\to\infty} \varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_2} = \tilde{v}(B) - \lim_{j\to\infty} \varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_1} .$$

Since $\{k_{i_j}\}_j$ is a converging sub-sequence for both events B_1 and B_2 we may write

$$\tilde{v}(B_1) \geq \lim_{j\to\infty} \varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_1} \quad ; \quad \tilde{v}(B_2) \geq \lim_{j\to\infty} \varepsilon_{k_{i_j}} n_{k_{i_j}}^{B_2} ,$$

and, as a conclusion, $\tilde{v}(B_1) + \tilde{v}(B_2) \geq \tilde{v}(B)$.

Now we wish to prove that the converse inequality holds as well. For this we shall need the following

2.9.1. Lemma. Let there be $I = \{k_i\}_{i \geq 1}$ and $J = \{k_j\}_{j \geq 1}$, two indices sequences, such that

$$\exists \lim_{i \to \infty} \varepsilon_{k_i} n_{k_i}^{B_1} \equiv v_1^I$$

and

$$\exists \lim_{j \to \infty} \varepsilon_{k_j} n_{k_j}^{B_2} \equiv v_2^J \quad .$$

Furthermore, assume that

$$\exists \lim_{i \to \infty} \varepsilon_{k_i} n_{k_i}^{B_2} \equiv v_2^I \quad \text{and} \quad \exists \lim_{j \to \infty} \varepsilon_{k_j} n_{k_j}^{B_1} \equiv v_1^J .$$

Then it is impossible that $v_1^I > v_1^J$ and $v_2^I < v_2^J$.

Proof: Assume the contrary, i.e., that indeed $v_1^I > v_1^J$ and $v_2^I < v_2^J$. W.l.o.g. assume $B_1 \geq \cdot B_2$, whence, by Lemma 2.6, there is an event B_3 such that $B_1 \sim \cdot B_2 \oplus B_3$. We already know that

$$n_k^{B_2} + n_k^{B_3} \leq n_k^{B_1} \leq n_k^{B_2} + n_k^{B_3} + 1,$$

and therefore

$$\exists \lim_{i \to \infty} \varepsilon_{k_i} n_{k_i}^{B_3} \equiv v_3^I \quad ; \exists \lim_{j \to \infty} \varepsilon_{k_j} n_{k_j}^{B_3} \equiv v_3^J,$$

which satisfy

$$v_1^I = v_2^I + v_3^I \quad ; \quad v_1^J = v_2^J + v_3^J .$$

Subtraction will yield

$v_1^I - v_1^J = (v_2^I - v_2^J) + (v_3^I - v_3^J),$

or

$v_3^I - v_3^J = (v_1^I - v_1^J) + (v_2^J - v_2^I) > v_1^I - v_1^J$, and,

in particular,

$v_3^I > v_3^J$.

Now we have B_2 and B_3, and we may proceed in this way to construct a sequence $\{B_n\}_{n \geq 1}$ such that $(v_{n-1}^I - v_{n-1}^J)(v_n^I - v_n^J) < 0$. It is important to note that $v_n^I, v_n^J > 0$ for all n. (To see this, note that if

$B_n \sim\cdot B_{n+1}$, $v_n^I = v_{n+1}^I$ and $v_n^J = v_{n+1}^J$,

contrary to the induction assumption. If, for instance, $B_n >\cdot B_{n+1}$ (the case $B_n \cdot< B_{n+1}$ is identical), both

$v_n^I > v_{n+1}^I$ and $v_n^J > v_{n+1}^J$, and consequently $v_{n+2}^I, v_{n+2}^J > 0$).

This sequence satisfies

$$\max(v_n^{I(J)}, v_{n+1}^{I(J)}) - \min(v_n^{I(J)}, v_{n+1}^{I(J)}) = v_{n+2}^{I(J)},$$

that is, any number in $\{v_n^I\}_{n \geq 3}$ (or in $\{v_n^J\}_{n \geq 3}$) is equal to the absolute difference between its two consecutive predecessors. This implies $v_n^I, v_n^J \underset{n \to \infty}{\to} 0$. (for instance: for any $n \geq 1$ there is a finite M such that $v_{n+M}^{I(J)} \leq 1/2\, v_n^{I(J)}$). But this means that $|v_n^I - v_n^J| \underset{n \to \infty}{\to} 0$, while we have shown that

$$\left|v_n^I - v_n^J\right| > \min\{(v_1^I - v_1^J),\ (v_2^J - v_2^I)\}.$$

– This contradicts our assumption and thereby proves the Lemma.//

We return now to the proof of the theorem: Let $I = \{k_i\}_{i\geq 1}$, and $J = \{k_j\}_{j\geq 1}$ be subsequences such that

$$\exists \lim_{i\to\infty} \varepsilon_{k_i} n_{k_i}^{B_1} = \tilde{v}(B_1) \quad\text{and}\quad \exists \lim_{j\to\infty} \varepsilon_{k_j} n_{k_j}^{B_2} = \tilde{v}(B_2).$$

Since $\varepsilon_k n_k^B \in [0,1]$ (for all $k \geq 1$, $B \cdot\!\leq A_1$), one can choose subsequences of I and J, to be denoted by $\bar{I} = \{k_{i_s}\}_{s\geq 1}$ and $\bar{J} = \{k_{j_r}\}_{r\geq 1}$ respectively, such that

$$\exists \lim_{s\to\infty} \varepsilon_{k_{i_s}} n_{k_{i_s}}^{B_2} \equiv v_2^{\bar{I}} \quad\text{and}\quad \exists \lim_{r\to\infty} \varepsilon_{k_{j_r}} n_{k_{j_r}}^{B_1} \equiv v_1^{\bar{J}}\ .$$

If $v_2^{\bar{I}} = v_2^J$, then $\bar{I}$ is a subsequence attaining $\tilde{v}(B_1)$ and $\tilde{v}(B_2)$ simultaneously, and this implies

$$\lim_{s\to\infty} \varepsilon_{k_{i_s}} n_{k_{i_s}}^{B} = \tilde{v}(B_2) + \tilde{v}(B_1),$$

whence $\tilde{v}(B) \geq \tilde{v}(B_2) + \tilde{v}(B_1)$. Therefore we may assume $V_2^{\bar{I}} < v_2^{\bar{J}} = v_2^J$. But according to the lemma, this is possible only if $v_1^{\bar{I}} \leq v_1^{\bar{J}}$. Since $v_1^{\bar{I}} = v_1^I = \limsup_{k\to\infty} \varepsilon_k n_k^{B_1}$, we have $v_1^{\bar{J}} = \tilde{v}(B_1)$, while we already know that $v_2^{\bar{J}} = \tilde{v}(B_2)$. In that case again $\tilde{v}(B) \geq \tilde{v}(B_1) + \tilde{v}(B_2)$.

Combining the two inequalities we have

$$\tilde{v}(B) = \tilde{v}(B_1) + \tilde{v}(B_2),$$

which completes the proof.//

Another important property of the function $\tilde{v}$ is that it agrees with $\geq$:

2.10. Theorem. Let $B, C \cdot\leq A_1$.

Then $B \cdot\leq C$ iff $\tilde{v}(B) \leq \tilde{v}(c)$.

Proof: In Lemma 2.8 we have seen that

$$B \cdot\leq c \Rightarrow \tilde{v}(B) \geq \tilde{v}(c).$$

(In Savage's terms, $\tilde{v}$ 'almost agrees' with $\geq\cdot$). Suppose, then, that $B \cdot< C$, and let H be an event satisfying $B \sim\cdot C \oplus H$ and $H >\cdot \phi$. (As assured by Lemma 2.6).

By theorem 2.9 we know that $\tilde{v}(B) = \tilde{v}(C) + \tilde{v}(H)$.

All we need to prove is, therefore, that for $H >\cdot \phi$, $\tilde{v}(H) > 0$.

But taking $f_1 = (x, A_1)$ and $A = H$ in P6**, one proves the existence of an integer M such that $MH \geq\cdot A_1$, and $(M-1)H \cdot< A_1$.

Let $H^1 \subset H$ be such that $(M-1)H \oplus H^1 \sim\cdot A_1$, and, using theorem 2.9 once more, one gets

$$M\tilde{v}(H) \geq (M-1)\tilde{v}(H) + \tilde{v}(H^1) = \tilde{v}(A_1) = 1,$$

or $\tilde{v}(H) \geq \frac{1}{M} > 0$, which is the desired conclusion.//

We will also be interested in the range of $\tilde{v}$. First we prove

2.11. Lemma. Suppose that H satisfies $A_1 \geq\cdot H >\cdot \phi$; then there is an $H^1 >\cdot \phi$ such that

$$1/2\ \tilde{v}(H) \geq \tilde{v}(H^1) > 0.$$

Proof: By Lemma 2.4 there is an integer k such that $A_k \cdot< H$. By 2.6 there is an event G satisfying $H \sim\cdot A_k \oplus G$, whence $\tilde{v}(H) = \tilde{v}(A_k) + \tilde{v}(G)$. Evidently either G or A_k is the required H^1.//

This last lemma proves useful in:

2.12. Theorem. $\tilde{v}$ has a convex range. (This property was originally defined for a measure, and $\tilde{v}$ fails to be one, but the definition is extended in an obvious manner).

Proof: Let $B, C \overset{\bullet}{\leq} A_1$ satisfy $B \subset C$, and let $\alpha \in [0,1]$. We must prove the existence of an event D, $B \subset D \subset C$, such that

$$\tilde{v}(D) = \alpha\tilde{v}(B) + (1-\alpha)\tilde{v}(c).$$

By the preceding lemma we may choose a sequence of events $\{H_k\}_{k\geq 1}$, satisfying

$$1/2\ \tilde{v}(H_k) \geq \tilde{v}(H_{k+1}) > 0.$$

Now let us define two sequences of events $\{\underline{D}_k\}_{k\geq 1}$, $\{\overline{D}_k\}_{k\geq 1}$ in the following way:

(i) $\underline{D}_0 = B \ ; \ \overline{D}_0 = C.$

(ii) Given $\underline{D}_k$, $\overline{D}_k$, such that $B \subset \underline{D}_k \subset \overline{D}_k \subset C$, and $\tilde{v}(\underline{D}_k) \leq \alpha\tilde{v}(B) + (1-\alpha)\tilde{v}(c) \leq \tilde{v}(\overline{D}_k)$, let m_k be the unique integer for which

$$\tilde{v}(\underline{D}_k) + m_k\tilde{v}(H_k) \leq \alpha\tilde{v}(B) + (1-\alpha)\tilde{v}(c) \leq \tilde{v}(\underline{D}_k) + (m_k+1)\tilde{v}(H_k).$$

Now let $\underline{D}_{k+1}$ be an event satisfying:

(a) $\underline{D}_{k+1} \overset{\bullet}{\sim} \underline{D}_k \oplus m_k H_k$

(b) $\underline{D}_k \subset \underline{D}_{k+1} \subset \overline{D}_k.$

Next choose $\overline{D}_{k+1}$ such that

(c) $\underline{D}_{k+1} \subset \overline{D}_{k+1} \subset \overline{D}_k$

(d) $\tilde{v}(\overline{D}_{k+1}) = \min\{\tilde{v}(\overline{D}_k),\ \tilde{v}(\underline{D}_k) + (m_k+1)\tilde{v}(H_k)\}$

It is obvious that:

(i) $\underline{D} \equiv \cup_k \underline{D}_k \subset \overline{D} \equiv \cap_k \overline{D}_k$;

(ii) for all $k \geq 1$,

$$\tilde{v}(\underline{D}_k) \geq \alpha\tilde{v}(B) + (1-\alpha)\tilde{v}(c) - \tilde{v}(H_k),$$

whence $\tilde{v}(\underline{D}) \geq \alpha\tilde{v}(B) + (1-\alpha)\tilde{v}(c)$;

(iii) for all $k \geq 1$,

$$\tilde{v}(\overline{D}_k) \leq \alpha\tilde{v}(B) + (1-\alpha)\tilde{v}(c) + \tilde{v}(H_k),$$

whence $\tilde{v}(\overline{D}) \leq \alpha\tilde{v}(B) + (1-\alpha)\tilde{v}(c)$.

So that $\tilde{v}(\underline{D}) = \tilde{v}(\overline{D}) = \alpha\tilde{v}(B) + (1-\alpha)\tilde{v}(c)$, and both $\underline{D}$ and $\overline{D}$ can be the required D.//

2.13. Conclusion. $\{\tilde{v}(B)\}_{B \,\cdot\!\leq A_1} = [0,1]$.

So far we have defined $\tilde{v}(B)$ for $B \cdot\!\leq A_1$. Defining a measure for all 2^S should be postponed until after we have said something about integral representation of $\geq$, which will be done in the next section.

3. Integral Representation of the Preference Order.

This section is divided into three subsections: Subsection 3.1. constructs an IR of $\geq$, retaining section 2's assumption of $X = \{x^*, x, x_*\}$. This requires, of course, a definition of a measure for all 2^S.

Subsection 3.2 removes the restriction on X, but constructs an IR of $\geq$ only for step functions. This step includes, however, the comparison of the measures and utilities constructed in 3.1 for any triple of consequences.

Subsection 3.3 proves that the utility and the measure that were constructed in 3.2 consist an IR of $\geq$ over all acts, and not only over simple ones (= 'step functions').

3.1. IR for a three-consequence world.

The steps in constructing the IR of $\geq$ for a specific triple of consequences are:

(a) IR for $T^S \cap \underline{F}_{(x,A_1)}$, where T is the triple of consequences, and $\underline{F}_f = \{g \in F \,/\, g \leq f\}$ (for $f \in F$);

(b) Extending $\tilde{v}$ and normalizing it to construct a measure for 2^S, (this is done in view of (a))

(c) IR for all T^S.

It should be noted that we do not have a measure until step (b), so that the term "IR" is not well-defined. However, the way we will define it will not be surprising:

Since only three consequences are involved, one may safely assume that any utility $u: T \to R$ satisfies $u(x^*) = 1$ and $u(x_*) = 0$. Hence for $f = (x^*, B;\ x, C)$ with $B, C \cdot\!\leq A_1$, we may define

$$\int u(f) d\tilde{v} = [1 - u(x)]\tilde{v}(B) + u(x)\tilde{v}(c).$$

Bearing this definition in mind until we have a 'real' measure, step (a) is no more than

3.1.1. Theorem. For $T = \{x^*, x, x_*\}$ with $x^* > x > x_*$ and the function $\tilde{v}$ attached to it, there exists a $u: T \to R$ such that (u,v) is an IR of $\geq$ (in the sense of the above definition) over $T^S \cap \underline{F}_{(x,A_1)}$.

Proof: For any $f = (x^*, B;\ x,\ c) \in \underline{F}_{(x,A_1)}$ there is (by P6*) an event $D \cdot \leq A_1$ such that $f \sim (x,D)$. Therefore it suffices to show that there is an $\alpha \in (0,1)$, $(\alpha = u(x))$ such that

$$(1-\alpha)\tilde{v}(B) + \alpha\tilde{v}(c) = \alpha\tilde{v}(D)$$

for all $(x^*,B;\ x,c) \sim (x,D)$ with $D \cdot\leq A_1$.

First we observe that, since $\tilde{v}$ agrees with $\geq\cdot$, $\tilde{v}(D)$ depends on B and C only through $\tilde{v}(B)$ and $\tilde{v}(C)$, respectively. That is, if B' and C' are such that $\tilde{v}(B) = \tilde{v}(B')$ and $\tilde{v}(C) = \tilde{v}(C')$, and $B' \subset C'$, then, by theorem 2.10 and 2.1, $(x^*, B';\ x, C') \sim (x,D)$. Denoting by $\tilde{V}_1$ the set $\{(\tilde{v}(B), \tilde{v}(C))\}_{(x^*,B;x,C)\ \varepsilon\ \underline{F}(x,A_1)}$ we have proved the existence of a function $\psi_1 : \tilde{V}_1 \to [0,1]$ such that

$$(x^*,B;\ x,C) \sim (x,D) \iff \tilde{v}(D) = \psi_1(\tilde{v}(B), \tilde{v}(C)),$$

for all $(x^*, B;\ x, C)\ \varepsilon\ \underline{F}(x,A_1)$ and $D \cdot\leq A_1$.

Since $D \geq\cdot C$, $\tilde{v}(D) \geq \tilde{v}(c)$ and we may write

$$\psi_1(\tilde{v}(B), \tilde{v}(c)) = \tilde{v}(c) + \psi_2(\tilde{v}(B), \tilde{v}(c))$$

with $\psi_2 : \tilde{V}_1 \to [0,1]$.

We now wish to prove

3.1.1.1. <u>Lemma</u>. ψ_2 is independent of its second argument.

<u>Proof</u>: Suppose that $(x^*, B;\ x, C) \sim (x,D)$

and $(x^*, B;\ x, c') \sim (x,D')$.

All we need to show is

$$\psi_2(\tilde{v}(B), \tilde{v}(C)) = \psi_2(\tilde{v}(B), \tilde{v}(C')).$$

W.l.o.g. assume $D \supset D'$ and let $E \subset S$ satisfy $D \sim\cdot D' \oplus E$.

By the definition of $\oplus$ and Lemma 2.3, there are $H_0 \sim\cdot A_1$ and $H_1 \geq\cdot A_1$ such that

$$(x^*, D';\ x,H_0) \sim (x^*, \phi;\ x,H_1)$$

$$(x^*, D\ ;\ x,H_0) \sim (x^*, E;\ x, H_1).$$

Let (by P6*) $H_0' \supset H_0$ be such that

$$(x^*, C'; x, H_0') \sim (x^*, D'; x, H_0)$$

so that Lemma 2.2 assures

$$(x^*, C; x, H_0') \sim (x^*, D; x, H_0).$$

The four last equivalences yield

$$(x, C'; x, H_0') \sim (x^*, \phi; x, H_1)$$

$$(x^*, C; x, H_0') \sim (x^*, E; x, H_1)$$

which means that $C \sim\cdot C' \oplus E$. But this implies

$v(C) - v(C^1) = v(E) = v(D) - v(D^1)$.

whence

$$\psi_2(\tilde{v}(B), \tilde{v}(C')) = v(D') - v(C') = v(D) - v(C) =$$
$$= \psi_2(\tilde{v}(B), \tilde{v}(C)).//$$

Consequently there is a $\psi_3 : \tilde{V}_3 \to [0,1]$, with $\tilde{V}_3$ being the projection of $\tilde{V}_1$ onto its first coordinate, such that

$$f = (x^*, B; x, c) \sim (x,D) \Leftrightarrow \tilde{v}(D) = \tilde{v}(c) + \psi_3(\tilde{v}(B))$$
$$\text{for } f, (x,D) \in \underline{F}_{(x,A_1)}.$$

It is obvious that ψ_3 is a nonnegative, monotonically increasing function, with $\psi_3(0) = 0$. Another important fact about ψ_3 is:

3.1.1.2. Lemma. ψ_3 is additive.

Proof: Let there be given $\tilde{v}(B_1)$, $\tilde{v}(B_2)$, and let B satisfy $B \sim\cdot B_1 \oplus B_2$, provided that $\tilde{v}(B_1) + \tilde{v}(B_2) \leq 1$. (Otherwise there is nothing to prove.) Now suppose that $(x^*, B; x, C) \sim (x,D)$. (There are such C and D by the definition of the domain of ψ_3.)

Let C_1 be an event satisfying

(*) $(x^*, B_1; x, C) \sim (x, C_1)$.

Because $B \sim\cdot B_1 \oplus B_2$, we have

$$(**)\quad (x^*, B_2; x, C_1) \sim (x^*, B; x, C) \sim (x, D).$$

The equivalence (*) yields

$$\tilde{v}(c_1) = \tilde{v}(c) + \psi_3(\tilde{v}(B_1)),$$

and (**) implies $\tilde{v}(D) = \tilde{v}(C_1) + \psi_3(\tilde{v}(B_2))$.

Combining these equalities with $\tilde{v}(D) = \tilde{v}(c) + \psi_3(\tilde{v}(B))$, one gets

$$\psi_3(\tilde{v}(B_1)) + \psi_3(\tilde{v}(B_2)) = \psi_3(\tilde{v}(B)) = \psi_3(\tilde{v}(B_1) + \tilde{v}(B_2)).//$$

In the light of conclusion 2.13, $\tilde{V}_3$ is no more than an interval (either closed or half-closed), so that monotonicity and additivity imply the linearity of ψ_3: There exists a $\lambda > 0$ for which

$$\psi_3(v) = \lambda v \qquad\qquad \forall\ v \in \tilde{V}_3.$$

And therefore λ also satisfies

$$f = (x^*, B; x, c) \sim (x,D) = g \iff \tilde{v}(D) = \tilde{v}(c) + \lambda\tilde{v}(B)$$

$$\text{for all } f, g \in \underline{F}_{(x,A_1)}.$$

Taking $\alpha = (\lambda + 1)^{-1} \in (0,1)$, one concludes that

$$f = (x^*, B; x, c) \sim (x,D) = g \iff \alpha\tilde{v}(D) = \alpha\tilde{v}(c) + (1-\alpha)\tilde{v}(B)$$

$$\text{for all } f, g \in \underline{F}_{(x,A_1)},$$

which completes the proof of theorem 3.1.1.//

Now we may turn to step (b), i.e. finally define the measure v for a given triple T, using the set function $\tilde{v}$ and the number α defined above:

For $B \cdot\leq A_1$, let $v(B) = \alpha\tilde{v}(B)$.

For $B >\cdot A_1$, let $C \cdot\leq A_1$ be such that $(x^*, C; x, A_1) \sim (x, B)$, and define $v(B) = v(A_1) + \frac{1-\alpha}{\alpha}v(c)$.

[v(B) is well defined in this case, since it does not depend upon the choice of C.]

Note that $v(A_1) = \alpha$ and therefore $v(S) = 1$.

Having v defined, we may proceed to the third step, namely, to construct an IR of $\geq$ over all T^S. First we extend theorem 3.1.1. in the following way:

3.1.2. Lemma. If $(x^*, B; x, c) \sim (x,D)$, then $(1-\alpha)v(B) + \alpha v(c) = \alpha v(D)$. (Note that this means integral representation for all $f, g \leq (x, S)$.)

Proof: Suppose that $f = (x^*, B; x, c) \sim (x,D)$. If $f \leq (x,A_1)$, the desired result is the consequence of the previous theorem. Hence we may assume $f > (x, A_1)$, and consequently there is an event E, for which $(x^*, E; x,A_1) \sim (x,D)$. By the definition of v,

$$v(D) = v(A_1) + \frac{1-\alpha}{\alpha} v(E).$$

We should now distinguish between two cases:

Case 1: $A_1 \cdot < C$.

In this case there is an event F satisfying $(x^*, F; x, A_1) \sim (x,C)$. The definition of v for such an event C is

(*) $v(c) = v(A_1) + \frac{1-\alpha}{\alpha} v(F)$.

But since $(x^*, F; x, A_1) \sim (x^*, \phi; x, C)$

and $(x^*, E; x, A_1) \sim (x^*, B; x, C)$,

we have $E \sim\cdot F \oplus B$. Note that $E, F, B \cdot \leq A_1$, whence

$$v(E) = v(F) + v(B),$$

which, in conjunction with (*), yields

$$v(C) = v(A_1) + \frac{1-\alpha}{\alpha} v(E) - \frac{1-\alpha}{\alpha} v(B), \text{ or}$$

$$\alpha v(C) + (1-\alpha)v(B) = \alpha v(A_1) + (1-\alpha)v(E) = \alpha v(D).$$

Case 2: $A \geq^{\bullet} C$.

- Then there is an F such that $(x^*, F; x, C) \sim (x, A_1)$, and, in a similar fashion, $B \sim^{\bullet} E + F$,

$$v(A_1) = v(C) + \frac{1-\alpha}{\alpha}(v(B) - v(E))$$

and finally

$$\alpha v(C) + (1-\alpha)v(B) = \alpha v(A_1) + (1-\alpha)v(E) = \alpha v(D).$$

So that the lemma is true.//

Now we wish to extend the circle-additivity of v over $[0,\alpha]$ to $[0,1]$:

3.1.3. Lemma. v is circle-additive.

Proof: Let there be $D_2 \sim^{\bullet} D_1 \oplus B$ and $E_2 \sim^{\bullet} E_1 \oplus B$. If we prove - as we intend to - that $v(D_2) - v(D_1) = v(E_2) - v(E_1)$, then, by choosing $E_2 = B$, $E_1 = \phi$, we will have $v(D_2) = v(D_1) + v(B)$. Assume w.l.o.g. that $E_1 \, {}^{\bullet}\!\leq D_1$ (and, consequently, $E_2 \, {}^{\bullet}\!\leq D_2$).

By the definition of circle-addition, there are events $C_1 \supset C_0 \supset D_2$ satisfying

$$(x^*, D_1; x, C_0) \sim (x^*, \phi; x, C_1)$$
$$(x^*, D_2; x, C_0) \sim (x^*, B; x, C_1).$$

(W.l.o.g. we assume $E_1 \subset E_2 \subset D_2$ and $E_1 \subset D_1 \subset D_2$.)

Let C_2 be an event satisfying $C_0 \subset C_2 \subset C_1$ and

$$(x^*, E_1; x, C_2) \sim (x^*, \phi; x, C_1),$$

whence (using Lemma 2.3),

$$(x^*, E_2; x, C_2) \sim (x^*, B; x, C_1).$$

Suppose G is an event for which

$$(x^*, G;\ x, E_1) \sim (x^*, \phi;\ x, D_1).$$

(There need not be such an event in the case $(x^*, E_1) < (x, D_1)$. This will be dealt with later on.)

Lemma 2.2. and the above equivalences imply

$$(x^*, G; x, E_2) \sim (x^*, \phi; x, D_2).$$

Using Lemma 3.1.2. and the above two equivalences, one is lead to

$$\frac{1-\alpha}{\alpha} v(G) + v(E_1) = v(D_1) \quad \text{and} \quad \frac{1-\alpha}{\alpha} v(G) + v(E_2) = v(D_2),$$

which means that $v(E_2) - v(E_1) = v(D_2) - v(D_1)$.

If $(x^*, E_1) < (x,D_1)$, so that there is no event G as required, there can be found, by P6**, a finite sequence $\{(F_1^{(i)}, F_2^{(i)})\}_{i=1}^{M}$, such that:

(i) $F_i^{(1)} = E_i \quad i = 1,2, \ ; \quad F_i^{(M)} = D_i \quad i = 1,2 \ ;$

(ii) $F_i^{(k)} \subset F_i^{(k+1)} \quad i = 1,2; \quad 1 \le k \le M-1;$

(iii) $F_2^{(k)} \sim\cdot\ F_1^{(k)} \oplus B;$

(iv) for any $k \le M$, there is a $G^{(k)}$ such that

$$(x^*, G^{(k)};\ x, F_1^{(k)}) \sim (x^*, \phi;\ x, F_1^{(k+1)}).$$

– in which case the Lemma is proved inductively.//

The time is ripe to prove

3.1.4. <u>Theorem</u>. For $T = \{x^*, x, x_*)$ with $x^* > x > x_*$, the measure v and the utility u defined above consist an IR $\geq$ over T^S.

Proof: Let there be given $f = (x^*, B; x, C)$ and $g = (x^*, B'; x, c')$.

If both $f, g \leq (x, S)$, so that there are D, D' satisfying $f \sim (x,D)$ and $g \sim (x,D')$, Lemma 3.1.2. completes the proof.

If both $f, g > (x, S)$, let there be E, E' such that

$$f \sim (x^*, E; x, S) \text{ and } g \sim (x^*, E'; x, S).$$

Let there be an event F for which

$$(x^*, F; x, C) \sim (x^*, \phi; x, S).$$

We already know that $(1-\alpha)v(F) + \alpha v(c) = \alpha$. But $B \sim\cdot F \oplus E$ means because of 3.1.3. that $v(B) = v(F) + v(E)$ and therefore

$$(1-\alpha)\, v(B) + \alpha v(c) = \alpha + (1-\alpha)v(E).$$

Similarly, for g we have

$$(1-\alpha)v(B') + \alpha v(C') = \alpha + (1-\alpha)v(E').$$

It is obvious that $f \geq g$ iff $E \geq\cdot E'$, and that is so iff

$$\int u(f)dv \geq \int u(g)dv.$$

The remaining case, namely, $f \leq (x, S) < g$, is, of course, trivial, so that we may conclude that for $f, g \in T^S$, $f \geq g$ iff $\int u(f)dv \geq \int u(g)dv$. //

3.1.5. Corollary. v has a convex range. (Use theorem 2.12).

3.2. IR for step functions

It is the time to remind ourselves that the utility and the measure we have proved to consist an IR of $\geq$ over T^S for a given triple T, are dependent upon this triple, and should be denoted by u_T and v_T respectively. We now come to the comparison among different triples:

3.2.1. Lemma. Let T_1 and T_2 be non-trivial triples of consequences. (I.E., $|T_i/\sim| = 3$, $i = 1,2$).

Then $v_{T_1} = v_{T_2}$.

Proof: P2* obviously implies

$$B_1 \oplus_{T_1} B_2 \sim^{\bullet} B \Leftrightarrow B_1 \oplus_{T_2} B_2 \sim^{\bullet} B,$$

whence the subscript T of $\oplus$, which has been omitted throughout sections 2 and 3.1, can be omitted again.

Since both v_{T_1} and v_{T_2} agree with $\geq^{\bullet}$, there is a monotonically increasing ψ: $[0,1] \to [0,1]$ such that $v_{T_2}(B) = \psi(v_{T_1}(B))$, for all $B \subset S$, and $\exists \psi^{-1} : [0,1] \to [0,1]$.

For $B_1 \oplus B_2 \sim^{\bullet} B$ we have

$$v_{T_2}(B_1) + v_{T_2}(B_2) = \psi(v_{T_1}(B_1) + v_{T_1}(B_2))$$

because both measures are circle-additive. Noting that, by corollary 3.1.5., $\{v_{T_i}(B)\}_{B \subset S} = [0,1]$ $i = 1,2$ (that is, the range of the measures is rich enough), one concludes that $\psi(x) = x$ $\forall x \in [0,1]$. //

Henceforth we shall refer to the measure v (without a subscript), since it does not depend on the defining triple.

We now turn to the comparison among $\{u_T\}_T$. We shall need some new definitions:

$$X_{x_*}^{x^*} = \{x \in X \,/\, x_* \leq x \leq x^*\} \qquad \text{for } x^* \geq x_*;$$

$$F^n = \{f \in F \,/\, \#\{x \in X \,/\, \exists s \in S,\ f(s) = x\} \leq n\} \qquad \text{for } n \geq 1;$$

$$F^* = \bigcup_{n \geq 1} F^n \ ;$$

$$F_{x_*}^{x^*} = \{f \in F \,/\, f(s) \in X_{x_*}^{x^*} \ \forall s \in S\} \qquad \text{for } x^* \geq x_*;$$

and

$I(f) = \int u(f)dv$; (any subscripts, superscripts, apostrophies, and other symbols attached to u will be understood to define their corresponding I's).

– by which it is easier to formulate:

3.2.2. <u>Lemma</u>. For any $x^* > x_*$ there exists a $u_{x^*, x_*} : X_{x_*}^{x^*} \to R$, such that

for any $T = \{x_1^*, x_1, x_*\}$ with $x^* \geq x_1^* > x_1 > x_*$,

u_{x^*,x_*} and v are an IR of $\geq$ over T^S.

<u>Proof</u>: Let there be given an $x \in X_{x_*}^{x^*}$. There is an event B_x satisfying $(x^*, B_x; x_*, S) \sim (x,S)$. Define $u_{x^*,x_*}(x) = v(B_x)$. (Thus $u_{x^*,x_*}(x^*) = 1$ and $u_{x^*,x_*}(x_*) = 0$, whereas for x such that $x^* > x > x_*$, we have $u_{x^*, x_*}(x) = u_{\{x^*, x, x_*\}}(x)$.)

Now consider the non-trivial triple T, for which there exists a (unique) $\alpha = v(A_{1_T}) = u_T(x_1)$, such that $(1,\alpha,0)$ is a utility (that integral represents $\geq$ over T^S with v.)

All we need to show is that

$(u_{x^*,x_*}(x_1^*), u_{x^*,x_*}(x_1), u_{x^*,x_*}(x_*))$ is a p.l.t. of $(1, \alpha, 0)$. This is tantamount to showing that $u_{x^*,x_*}(x_1)/u_{x^*,x_*}(x_1^*) = \alpha$.

Consider the event A_{1_T} satisfying

$$(x_1^*, A_{1_T};\ x_*, S) \sim (x_1, S).$$

By the definition of B_{x_1}, $(x^*, B_{x_1}; x_*, S) \sim (x_1, S)$, which implies

$u_{x^*,x_*}(X_1) = v(B_{x_1})$.

Now consider the triple $T' = \{x^*, x_1^*, x_*\}$. If $x^* \sim x_1^*$,

$u_{x^*,x_*}(x_1^*) = 1$, $v(Bx_1) = v(A_{1_T})$ and the Lemma is proved. Otherwise T' is non-trivial, i.e., $x^* > x_1^* > x_*$, and

$(x_1^*, A_{1_T}; x_*, S) \sim (x^*, B_{x_1}; x_*, S)$,

whence $v(B_{x_1}) = v(A_{1_T})u_{T'}(x_1^*)$. But

$$u_{T'}(x_1^*) = u_{x^*,x_*}(x_1^*),$$

whence $\quad u_{x^*,x_*}(x_1)/u_{x^*,x_*}(x_1^*) = \alpha$

– which is the required result.//

Next we shall need

3.2.3. __Lemma__. Let $a, b, c, d \in F^*$ be p.c., and for some $H \subset S$,

$$a(s) = b(s) \text{ and } c(s) = d(s) \quad \text{for } s \in H,$$

$$a(s) = c(s) \text{ and } b(s) = d(s) \quad \text{for } s \in H^c.$$

Suppose, furthermore, that $a \sim c$. Then $b \sim d$.

__Proof:__ First suppose that a, b, c, d are all constant on H^c:

$a(s) = c(s) = x$, $b(s) = d(s) = y$ $\forall\ s \in H^c$, and assume w.l.o.g. $y > x$.

In P2*, let us define $A = B = H^c$, $g_1 = a$, $g_2 = c$, $f_1 = f_2 = a$, $y_1 = y_2 = y$, $x_1 = x_2 = x$.

Note that $f_1 / {}^{x_1}_{A} = a$; $f_1 / {}^{y_1}_{a} = b$; $g_1 / {}^{x_2}_{A} = a$; $g_1 / {}^{y_2}_{A} = b$;

$$f_2 / {}^{x_1}_{B} = a; \quad f_2 / {}^{y_1}_{B} = b; \quad g_2 / {}^{x_2}_{B} = c; \quad g_2 / {}^{y_2}_{B} = d,$$

and the two quadruples are p.c.

Furthermore, $f_1 / {}^{x_1}_{A} \sim f_2 / {}^{x_1}_{B}$, $g_1 / {}^{x_2}_{A} \sim g_2 / {}^{x_2}_{B}$, and

$f_1 / {}^{y_1}_{A} \sim f_2 / {}^{y_2}_{B}$, so that P2* yields $g_1 / {}^{y_2}_{A} \sim g_2 / {}^{y_2}_{B}$, or $b \sim d$.

Now suppose that $a, b, c, d \in F^*$ but are not necessarily constant on H^c. For an integer n satisfying $a, b, c, d \in F^n$, these functions may assume no more than n different values on H^c, so that there is a partition of H^c into (no more than) $k = n^4$ events, $\{H_j\}_{j=1}^{k}$, such that a, b, c, d are all constant on each H_j.

We shall prove the lemma by induction on k: for $k = 1$ it is already proved. Assume, therefore, that it is true for $k-1$. For each $j \leq k$ define

$$a_j(s) = a(s) \qquad , \; c_j(s) = c(s) \quad \forall \; s \in H_j^c$$

$$a_j(s) = b(s) = d(s), \; c_j(s) = b(s) = d(s) \quad \forall \; s \in H_j,$$

so that

$$a_j(s) = b(s) \text{ and } c_j(s) = d(s) \quad \forall \; s \in H \cup H_j,$$

$$a_j(s) = c_j(s) \text{ and } b(s) = d(s) \quad \forall \; s \in (H \cup H_j)^c,$$

and there is a partition of $(H \cup H_j)^c = \bigcup_{i \neq j} H_i$ into no more than $k-1$ events, on each of which a_j, b, c_j, d are constant.

To complete the proof of the lemma we must show that there is an index j for which a_j, b, c_j, d are p.c., so that P2* may be used in the transition from $k-1$ to k. To verify that this is indeed the case, note that comonotonicity of the four acts implies the existence of a permutation $\{H_{i_\ell}\}_{\ell=1}^{k}$ of $\{H_j\}_{j=1}^{k}$, such that $f(s) \geq f(t)$ for all $s \in H_{i_{\ell_1}}$, $t \in H_{i_{\ell_2}}$ where $\ell_1 < \ell_2$, and all $f \in \{a,b,c,d\}$.

Let us consider the first event in this permutation, i.e. H_{i_1}. Assume, first, that $a[H_{i_1}] \geq b[H_{i_1}]$.
(The other case is dealt with symmetrically.) If a_{i_1}, b, c_{i_1}, d are p.c., we have found the index we are looking for. Otherwise we have $a[H_{i_2}] > b[H_{i_1}] \geq b[H_{i_2}]$. Next take the quadruple a_{i_2}, b, c_{i_2}, d, and test it for pairwise comonotonicity in the very same way. Proceed in this way to show, that if a_j, b, c_j, d are not p.c. for all $j \leq k$, then there is an $\ell^* \leq k$ such that

$$a[H_{i_\ell}] = c[H_{i_\ell}] > b[H_{i_\ell}] = d[H_{i_\ell}] \qquad \forall \quad \ell \leq \ell^*,$$

and

$$a[H_{i_\ell}], b[H_{i_\ell}] > a[H_{i_m}], b[H_{i_m}] \qquad \forall \ \ell \leq \ell^*, m > \ell^*.$$

(That is, the above process terminates in one of the two cases:
(i) All $\{H_j\}_{j=1}^{k}$ have been checked ($\ell^* = k$); or
(ii) There was found an index ℓ^*, which is the maximal to satisfy

$$f[H_{i_{\ell^*}}] \geq f(s) \quad \forall \ s \in H, \quad \forall \ f \in \{a, b, c, d\}.)$$

It is easily seen that ℓ^* is indeed the index we are after, i.e. modifying acts a and c so as to equal b and d on $H_{j_{\ell^*}}$ will yield $a_{j_{\ell^*}}$ and $c_{j_{\ell^*}}$ which are p.c. with a, b, c and d. //

Now we have

3.2.4. <u>Theorem</u>. For any $x^* > x_*$ the utility $u_{x^*,\,x_*}$ and the measure v form an IR of $\geq$ over $F_{x_*}^{x^*} \cap F^*$. (I.E., over all step functions which are bounded by x^* and x_* from above and below, respectively.)

<u>Proof</u>: We shall prove u_{x^*,x_*} and v to be an IR of $\geq$ over $F_{x_*}^{x^*} \cap F^n$, for all $n \geq 1$, which will complete the proof.

For $n = 2$ the conclusion of the theorem follows directly from Lemma 3.2.2. (Note that a two-consequence act $f \in \{x_1^*, x_1\}^S$ is here regarded as a three-consequence act, since it is also true that $f \in \{x_1^*, x_1, x_*\}^S$.) In order to proceed, we would like to prove:

3.2.4.1. <u>Lemma</u>. For any $f \in F^n \cap F_{x_*}^{x^*}$ $(n \geq 3)$ there is an $f' \in F^{n-1} \cap F_{x_*}^{x^*}$ satisfying:

(i) $f' \sim f$

(ii) $I_{x^*,x_*}(f') = I_{x^*,x_*}(f)$.

<u>Proof</u>: Let $x_1 > x_2 > x_*$ be the three least-preferred consequences in $\{f(s)\}_S \cup \{x_*\}$. Denote by C the event $\{s \,/\, f(s) > x_1\}$, and consider $f \,/\, {}^{x_1}_{C} \in T^S$ for $T = \{x_1, x_2, x_*\}$. By Lemma 3.2.2., and the famous P6*, there is an f' satisfying:

(a) $f'(s) = f(s) \quad \forall\, s \in C$

(b) $f' \,/\, {}^{x_1}_{C} \in \{x_1, x_*\}^S$

(c) $f' \,/\, {}^{x_1}_{C} \sim f \,/\, {}^{x_1}_{C}$

(d) $I_{x^*,x_*}(f' \,/\, {}^{x_1}_{C}) = I_{x^*,x_*}(f \,/\, {}^{x_1}_{C})$.

Since f, f', $f \,/\, {}^{x_1}_{C}$, $f' \,/\, {}^{x_1}_{C}$ are p.c., we may apply Lemma 3.2.3., which yields $f \sim f'$. For the same reason we have

$$I_{x^*,x_*}(f) - I_{x^*,x_*}(f') = I_{x^*,x_*}(f \,/\, {}^{x_1}_{C}) - I_{x^*,x_*}(f' \,/\, {}^{x_1}_{C}) = 0$$

so that f' does indeed satisfy (i), (ii), and the Lemma is proved.//

Returning to the proof of Theorem 3.2.4, let us take $f, g \in F^{x^*}_{x_*} \cap F^n$. By the preceding lemma, there are $f', g' \in F^{x^*}_{x_*} \cap F^{n-1}$ such that $f \sim f'$, $g \sim g'$ and

$$I_{x^*,x_*}(f) = I_{x^*,x_*}(f') \quad , \quad I_{x^*,x_*}(g) = I_{x^*,x_*}(g').$$

Hence, if u_{x^*,x_*} and v are an IR of $\geq$ over $F^{x^*}_{x_*} \cap F^{n-1}$, they are also an IR over $F^{x^*}_{x_*} \cap F^n$, and this completes the proof.//

We are now approaching the conclusion of this subsection. At long last we turn to define the utility u:

Choose any $x^* > x_*$, and for $x \in X^{x^*}_{x_*}$, let $u(x) = u_{x^*,x_*}(x)$. (So that $u(x^*) = 1$, $u(x_*) = 0$.) Now let $x \in X$ satisfy $x > x^*$. Consider the triple $T = \{x, x^*, x_*\}$, for which there exists a utility u_T. Define $u(x) = u_T(x^*)^{-1}$, so that $(u(x), u(x^*), u(x_*))$ is a scalar multiplication (and hence a p.l.t) of $(u_T(x), u_T(x^*), u_T(x_*))$. Similarly, for x satisfying $x < x_*$, take the triple $T = \{x^*, x_*, x\}$ and the utility u_T attached to it, and define $u(x) = -u_T(x_*)/(1-u_T(x_*))$, again preserving the equality

$$\frac{u_T(x_*) - u_T(x)}{u_T(x^*) - u_T(x)} = \frac{u(x_*) - u(x)}{u(x^*) - u(x)} .$$

3.2.5. Theorem. The utility u defined above satisfies

$$f \geq g \iff I(f) \geq I(g) \quad \forall f, g \in F^*.$$

Proof: Let $f, g \in F^*$. Since there is an integer n for which $f, g \in F^n$, there are necessarily y^*, y_* in X such that $f, g \in F_{y_*}^{y^*}$, and

$y^* \geq x^* > x_* \geq y_*$. Consider u_{y^*,y_*} which is known to satisfy

$$f \geq g \iff I_{y^*,y_*}(f) \geq I_{y^*,y_*}(g).$$

Re-normalize u_{y^*,y_*} to get $\tilde{u} = \alpha u_{y^*,y_*} + \beta$ $(\alpha > 0)$ such that $\tilde{u}(x^*) = 1$ and $\tilde{u}(x_*) = 0$. It is obvious that $\tilde{u}(y) = u(y)\ \forall y \in X_{y_*}^{y^*}$, whence $f \geq g \iff I(f) \geq I(g)$. //

The results obtained so far may be summarized in:

3.2.6. Theorem. The following two statements are equivalent:

(i) $\geq$ satisfies axioms P1, P2*, P3*, P5*, P6* and P6** for all step functions.

(ii) There is a utility u, which is unique up to p.l.t, and a unique measure v with a convex range, which consist an IR of $\geq$ over F*.

Proof: (i) => (ii) is the conclusion of sections 2, 3.1, 3.2.
(ii)=> (i) is easy to check.//

3.3. IR for all functions.

We now turn to the general case, in which the acts under comparison need not be simple acts. In this section, P7* is assumed to hold, unless otherwise stated.

To begin with, we need:

3.3.1. Theorem. u is bounded.

Proof: The proof is very similar to that of theorem 14.5 in Fishburn* (1970) (pp.206-207): Assume that u is unbounded, say from above. Let $B_0 = S$ and $B_i \subset B_{i-1}$ satisfy $v(B_i) = 1/2\, v(B_{i-1})$.

Take $x_0 < x_1$ and $\{x_n\}_{n \geq 1}$ such that

$$u(x_n) \geq \max \{2^n, u(x_{n-1})\}.$$

Define

$$f(s) = \begin{cases} x_i & s \in B_i - B_{i+1} \quad \forall\, i \geq 0 \\ x_1 & s \in \bigcap_{i \geq 1} B_i \end{cases}$$

Further define for any $n \geq 1$ an act f_n by

$$f(s) = \begin{cases} f(s) & [f(s) \leq x_n] \\ x_n & [f(s) > x_n] \end{cases}$$

so that $f_n \in F^*$.

By P7* $f \geq f_n$, for all $n \geq 1$. Since $I(f_n) \underset{n \to \infty}{\longrightarrow} \infty$, for any $x \in X$ there is an index n such that $f_n \geq x$, whence $f \geq x$. This implies – again by P7* – that $f >\_\ g$ for all $g \in F$. Now consider

---

* This proof is not to be found in Savage (1954). Fishburn notes, that although the theorem is mainly due to Savage, it was not known to him until several years after the publication of "The Foundation of Statistics".

$\tilde{f} = f \,/\, {}^{x_1}_{B_0-B_1}$, which satisfies the same conditions, and therefore $\tilde{f} \geq g$ for all $g \in F$. In particular, $\tilde{f} \sim f$. But by P2* (with P2*'s $A = B_0 - B_1$, $B = \phi$), this should imply $f_1 \sim f_1 \,/\, {}^{x_1}_{B_0-B_1}$, which is obviously untrue.

Unboundedness from below is dealt with in a similar way, so that we may consider the theorem as a fact.//

This theorem allows us to assume henceforth, w.l.o.g., that $\inf_{x \in X} u(x) = 0$ and $\sup_{x \in X} u(x) = 1$.

A crucial property of a preference relation satisfying P7* is

3.3.2. Lemma. Let $\phi = B_0 \subset B_1 \subset \ldots \subset B_n = S$ be events such that $B_i - B_{i-1}$ is f-convex for $i \leq n$. Suppose

$$\underline{u}_i = \inf_{s \in B_i - B_{i-1}} \{u(f(s))\} \quad ; \quad \bar{u}_i = \sup_{s \in B_i - B_{i-1}} \{u(f(s))\}.$$

Let $\bar{f} \in F^*$ satisfy $\bar{f} \sim f$.

Then

$$\textstyle\sum_{i=1}^{n}(\underline{u}_i - \underline{u}_{i+1})v(B_i) \leq I(\bar{f}) \leq \sum_{i=1}^{n}(\bar{u}_i - \bar{u}_{i+1})v(B_i)$$

where $\bar{u}_{n+1} \equiv \underline{u}_{n+1} \equiv 0$.

Proof: We shall prove only one of the two inequalities, say the left-hand-side one, for the other one is proved symmetrically. Assume the contrary, i.e.:

$$\underline{u} \equiv \textstyle\sum_i(\underline{u}_i - \underline{u}_{i+1})v(B_i) > I(\bar{f}).$$

Take $\bar{g} \in F^*$ to be such that $\underline{u} \geq I(\bar{g}) > I(\bar{f})$, whence $\bar{g} > \bar{f} \sim f$. (such a $\bar{g}$ exists because v has a convex range.)

Now for any $(s_1, s_2,\ldots,s_n)$ such that $s_i \varepsilon B_i - B_{i-1}$ and any $k \leq n$, define

$$f^{(s_1,\ldots,s_k)}(s) = \begin{cases} s_i & s \varepsilon B_i - B_{i-1},\quad i \leq k \\ f(s) & \text{otherwise}. \end{cases}$$

Note that $f^{(s_1,\ldots,s_n)} \varepsilon F^*$ for any sequence $(s_1,\ldots,s_n)$, and $I(f^{(s_1,\ldots,s_n)}) \geq \underline{u} \geq I(\bar{g})$ so that $f^{(s_1,\ldots,s_n)} \geq \bar{g}$. This can be written as

$$f^{(s_1,\ldots,s_{n-1})} \Big/ {}^{f(s_n)}_{B_n - B_{n-1}} \geq \bar{g} \quad \forall \quad s_n \varepsilon B_n - B_{n-1},$$

whence, by P7*, $f^{(s_1,\ldots,s_{n-1})} \geq \bar{g}$ for all $(s_1,\ldots,s_{n-1})$. Arguing inductively, $f^{(s_1,\ldots,s_k)} \geq \bar{g}$ for all $k \leq n$ and all $(s_1,\ldots,s_k)$, and, in particular, $f \geq \bar{g}$, which is known to be impossible.//

A straightforward consequence is:

3.3.3. Lemma. Let $f \varepsilon F$, $\bar{f} \varepsilon F^*$ satisfy $f \sim \bar{f}$.

Then $I(f) = I(\bar{f})$.

Proof: For any $n \geq 1$ define $B_i = \{s \varepsilon S \;/\; u(f(s)) \geq 1 - \frac{i}{n}\}$, with $1 \leq i \leq n$. For $\{B_i\}_{i=1}^{n}$ we get

$$\underline{u}_i = \inf_{s\varepsilon B_i - B_{i-1}} \{u(f(s))\} \geq 1 - \frac{i}{n}, \text{ and}$$

$$\bar{u}_i = \sup_{s\varepsilon B_i - B_{i-1}} \{u(f(s))\} \leq 1 - \frac{i-1}{n}.$$

Denote $\bar{\sigma}_n = \sum_{i=1}^{n} \frac{1}{n} v(B_i)$, $\underline{\sigma}_n = \sum_{i=1}^{n-1} \frac{1}{n} v(B_i)$,

and note that

$$\underline{\sigma}_n \leq \sum_{i=1}^{n} (\underline{u}_i - \underline{u}_{i+1}) v(B_i) \leq \sum_{i=1}^{n} (\bar{u}_i - \bar{u}_{i+1}) v(B_i) \leq \bar{\sigma}_n .$$

By Lemma 3.3.2, $\underline{\sigma}_n \leq I(f),\ I(\bar{f}) \leq \bar{\sigma}_n$, but $\bar{\sigma}_n - \underline{\sigma}_n = \frac{1}{n} \underset{n\to\infty}{\to} 0$.

This implies the desired result.//

Now the time has come to phrase:

3.3.4. Theorem. Let P1, P2*, P3*, P5*, P6* and P6** hold.

Then P7* holds iff

$$f \geq g \iff I(f) \geq I(g) \quad \text{for all } f, g \in F.$$

Proof: First assume that P7* holds. Denote

$$\bar{F} = \{f \in F \,/\, \exists\, \bar{x}, \underline{x} \in X,\ \bar{x} \geq f \geq \underline{x}\}$$

Note that $f \in \bar{F}$ iff there is an act $\bar{f} \in F^*$ such that $f \sim \bar{f}$. If $f \notin \bar{F}$, then either $f > x\ \forall x \in X$, in which case $f \geq g$ for all $g \in F$, or $f < x$ for all $x \in X$, and then $f \leq g\ \forall g \in F$. Following Savage we shall call the former kind of acts 'big', and the latter - 'small'. It is important to note that if f is big, $I(f) = \sup_{x\in X}\{u(x)\}$, and similarly $I(f) = \inf_{x\in X}\{u(x)\}$ for small f. (Surely $I(f) \leq \sup_{x\in X}\{u(x)\}$. If the inequality is strict, P2* is contradicted, as in the proof of Theorem 3.3.1.) Now let there be given $f, g \in F$. If both are in $\bar{F}$, lemma 3.3.3. concludes the proof. If both are big or small, then $f \sim g$ and indeed $I(f) = I(g)$. Now consider the case of f being big and g not. Then surely $f > g$. But for g there is an x^* such that $x^* \geq g$, and $I(g) \leq u(x^*)$. If $u(x^*) = \sup_{x\in X}\{u(x)\}$, $f \leq x^*$, which is false, so that $I(f) > I(g)$. If g is small and f is not, we have a similar proof, so that

$$f \geq g \iff I(f) \geq I(g) \quad \forall F, g \in F.$$

The fact that IR of $\geq$ implies P7* is trivial, so that the proof is complete.//

4. The necessity of the technical axioms

In subsection 1.4, discussing the axioms, we left P6* and P6** inadequately justified on intuitive grounds. It is the goal of this section to justify these axioms at least mathematically, i.e. to prove that they are not superfluous by counterexamples.

Let us start by considering P6*. The fact that without it, nonatomicity may not be satisfied by the measure v is not a striking assertion. We would like to justify not only the use of P6* per se, but also as opposed to Savage's P6. For this we have:

4.1. Claim P1, P2*, P3*, P5*, P6, P6**, and P7* may hold, and yet P6* may fail to hold.

Proof: Consider the following example:

$$S = [0,1) \times N \; ; \; X = \{0, 1/2, 1\} \; ; \; u(x) = x.$$

Define $\phi : [0,1] \to \{0,1\}$ as $\chi_{\{1\}}$:

$$\phi(x) = \begin{cases} 1 & x = 1 \\ 0 & x < 1. \end{cases}$$

Now let $A_i = \{x \,/\, (x,i) \in A\}$ for $A \subset S$, and define

$$v(A) = \phi(\limsup_{n \to \infty}(\frac{1}{n} \sum_{i=1}^{n} \phi(\sup A_i))).$$

Note that v is a measure for S.

Now define $\geq$ by the integral:

$$f \geq g \iff \int u(f)dv \geq \int u(g)dv \qquad \forall\, f, g \in F.$$

We wish to prove that $\geq$ thus defined is the required example.

P1, P2*, P3*, P5* are easily shown to hold. P7* holds since X is finite, and P6** holds because $\{\int u(f)dv\}_{f \in F}$ is finite (in fact it is $\{0,1/2,1\}$), so that archimedianity is satisfied.

The rub is, as expected, P6. To show that it is indeed satisfied, it suffices to prove that for any $A >\cdot B$ there is a finite partition of S, $(C_1,\ldots,C_k)$ such that

$$A-C_i >\cdot B \text{ and } A >\cdot B \cup C_i \quad \forall\, i \leq k.$$

So let $A >\cdot B$, or explicitly, $v(A) = 1$, $v(B) = 0$.

For any n there is a partition (A_n^{I}, A_n^{II}) of $[0,1)$, such that

If $\sup A_n = 1$, then $\sup(A_n^{I} \cap A_n) = \sup(A_n^{II} \cap A_n) = 1$.

Since $v(B) = 0$, there is an $M \in N$ such that

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(\sup B_i)\right) + \frac{1}{M} < 1.$$

Now we may define the partition of S into 2M events: for j, $1 \leq j \leq M$, let

$$C_j^{I} = \bigcup_{n\equiv j \bmod M} A_n^{I} \;;\; C_j^{II} = \bigcup_{n\equiv j \bmod M} A_n^{II}.$$

Let $A' = A - C_j^{I(II)}$ for any $j \leq M$. Since for every $n \geq 1$,

$$\phi(\sup A_n) = \phi(\sup A_n'), \quad v(A') = v(A) = 1.$$

Next consider $B' = B \cup C_j^{I(II)}$. Obviously,

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(\sup B_i')\right) \leq \limsup_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(\sup B_i)\right) + \frac{1}{M} < 1.$$

– whence $v(B') = v(B) = 0$, and P6 holds.

It is clear that P6* is not satisfied, and, moreover, this example does not conform with any intuitive concept of continuity, since $v[2^S]$ is not a convex set.

Next, consider P6**. Its counterpart in Savage's theory is again P6, which serves both for non-atomicity (and the convexity of the range of the measure) – and for archimedianity. (Note that P6* does not involve any kind of partition of S, and therefore sets no restriction on its 'size'.) Since P6 was renounced in favor of P6*, we now wish to prove:

4.2. Claim P1, P2*, P3*, P5*, P6*, p7* may hold where P6** fails to hold.

Proof: Take $S = R_+$, $X = \{0,1/2,1\}$.

For any $f \in F$ define $A_f \subset R$ as follows:

$$A_f = \{x \;/\; f(x) \geq 1/2\} \cup \{x \;/\; f(-x) \geq 1\}.$$

Denote $\mathcal{Q} = \{A_f\}_{f \in F}$.

We shall now define $\geq' \subset \mathcal{Q} \times \mathcal{Q}$, and $\geq \subset F \times F$ will be the induced preference order. To this end, let λ be some (finitely additive) extension of the Lebesgue measure to all 2^R. We are interested in the family of 1-1 correspondences which are λ-preserving:

$$\mathcal{C} = \{\psi: R \to R \;/\; \exists\, \psi^{-1};\ \forall A \subset R \quad \lambda(A) = \lambda(\psi[A])\}.$$

Note that $\mathcal{C}$ is a group (w.r.t. functions superposition).

Now define a binary relation $\approx$ over 2^R: for $A, B \subset \mathbf{R}$, $A \approx B$ iff there is a $\psi \in \mathcal{C}$ such that $\lambda(A - \psi[B]) + \lambda(\psi[B] - A) < \infty$.

It is easily seen that $\approx$ is an equivalence relation.

Next define a binary relation $\succ$ over $2^R/_{\approx}$: Let $\mathcal{D}, \mathcal{E} \in 2^R/_{\approx}$.

If there is a $D \in \mathcal{D}$ and an $E \in \mathcal{E}$ such that $D \subset E$, define $\mathcal{D} \prec \mathcal{E}$. It can be seen, though the proof is not immediate, that $\succ$ is a strict order for $2^R/_{\approx}$. (I.E. it is irreflexive, transitive and complete in the following sense: for any $\mathcal{D} \neq \mathcal{E} \in 2^R/_{\approx}$, either $\mathcal{D} \succ \mathcal{E}$ or $\mathcal{D} \prec \mathcal{E}$.)

At long last we turn to define $\geq\cdot$ over $\mathcal{Q}$: let D, E be elements of $\mathcal{Q}$. If $D \in \mathcal{D}$, $E \in \mathcal{E}$, and $\mathcal{D} \prec \mathcal{E}$, take $D \cdot \leq E$ (and similarly, $\mathcal{D} \succ \mathcal{E} \Rightarrow D \geq\cdot E$.) If $\mathcal{D} = \mathcal{E}$ (i.e., $D \approx E$), take a $\psi \in \mathcal{C}$ such that

$$(*) \qquad \lambda(D-\psi[E]) + \lambda(\psi[E]-D) < \infty,$$

and define $D \geq\cdot E$ iff

$$(**) \qquad \lambda(D-\psi[E]) - \lambda(\psi[E]-D) \geq 0.$$

(Note that for all $\psi \in \mathcal{C}$ satisfying (*), the expression on the left-hand-side of (**) is the same.)

Intuitively speaking, $\geq\cdot$ is simply governed by λ, only that events with infinite measures may still be distinguished.

The definition of $\geq$ by $f \geq g \iff A_f \geq\cdot A_g$ is, in a way, a definition by the integral of the identity utility, w.r.t. some extension of λ. Hence it is clear that $\geq$ satisfies P1, P2* and P3*. P5* is obviously satisfied as well. P6* follows easily from the non-atomicity of λ, and P7* is a direct consequence of the fact that $/ X / < \infty$. However, it is evident that P6** does not hold: consider the equivalence class of ϕ, which contains events with finite but unbounded measure. //

5. Continuity and quasi-continuity

The theory developed so far is, like Savage's, general enough to include strange preference relations, inducing exotic measures. Although we would like the basic model to be as general as possible, we are interested in the characterizations of restrictive conditions imposed on it.*

In this section we deal with two basic concepts of continuity, to be named 'continuity' and 'quasi-continuity'. A subsection will be devoted to each, but first we need:

Definitions

1. A measure v will be said to be upper(lower) quasicontinuous at $A \subset S$, iff for any sequence $\{A_n\}_{n \geq 1}$ such that $A_n \subset A_{n+1}$ ($A_n \supset A_{n+1}$) and $\bigcup_{n \geq 1} A_n = A$ ($\bigcap_{n \geq 1} A_n = A$), if $v(A_n) = \alpha$ for all $n \geq 1$, then $v(A) = \alpha$.

2. A measure v will be called upper(lower) quasicontinuous iff it is upper (lower) quasicontinuous at any $A \subset S$.

3. A measure v is quasicontinuous iff it is both upper- and lower- quasi-continuous.

4. A measure v is continuous iff for any $\{A_n\}_{n \geq 1}$ satisfying $A_n \subset A_{n+1}$, $v(\bigcup_n A_n) = \sup_n v(A_n)$ and for any $\{A_n\}_{n \geq 1}$ such that $A_n \supset A_{n+1}$, it is true that $v(\bigcap_n A_n) = \inf_n v(A_n)$.

5. A measure v is nonatomic iff for all $A \subset S$ and all $s \in S$, $v(A \cup \{s\}) = v(A)$.

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* It should be pointed out that the results in the sequel are also valid in Savage's model.

5.1. Observations

1. If v is finitely additive, and upper (lower) quasicontinuous at A, then it is lower (upper) quasicontinuous at A^c.

2. If v is continuous, it is quasi-continuous, but the converse does not have to hold.

3. If v is finitely additive, continuity coincides with the usual definition and with σ-additivity.

To motivate the study of continuity, let us investigate an anomalous preference relation, which satisfies all the axioms, but, as it turns out, induces a measure which is not continuous. Consider the following (famous) example:

$$S = N \;;\; X = R_+ \;;\; v(A) = \limsup_{n\to\infty}[\tfrac{1}{n}\textstyle\sum_{i=1}^{n} 1_A(i)] \;;$$

$$u(x) = x \quad , \quad \text{and} \;\; \geq \;\; \text{defined by} \;\; \int u(\cdot)dv.$$

Take $f(n) = \frac{1}{n}$, so that $f(n) > 0$, for all n, but $\int u(f)dv = 0$.

This example and its variants cause some uneasiness, because the act f is sure to yield some consequence which is strictly better than zero, and yet it is considered equivalent to the zero act. It is obvious that the measure v is not continuous. However, continuity is too strong a condition on v to exclude this example. Subsection 5.2 is devoted to the study of preference relations immune to the discussed anomaly, and 5.3 to the study of continuity.

5.2. Quasicontinuity

One can easily formulate an axiom that will rule out anomalies as that exhibited by the example:

P7**. Let f be an act and A an f-convex event.

Then there are $s^*, s_* \in A$ such that

$$f \,/_A^{f(s_*)} \leq f \leq f \,/_A^{f(s^*)} \quad .$$

To verify that P7** does indeed exclude the preceding example, we have

5.2.1. Lemma. Let P7** hold. Assume that $f = g$ on A^c, where A is both f- and g- convex. Then

$$f \underset{(<)}{>} g \,/\, \underset{A}{g(s)} \quad \forall\, s \in A \Rightarrow f \underset{(<)}{>} g.$$

Proof: Assume the contrary, i.e., $g \underset{(\leq)}{\geq} f \underset{(\geq)}{(>)} g \,/\, \underset{A}{g(s)}$ $\forall\, s \in A$, which is an obvious contradiction to P7**.//

We now wish to prove, first of all, that P7** is adequate for the existence of an IR of $\geq$ (in the absence of P7*). (Throughout subsection 5.2 we assume P1–P6** to hold.)

5.2.2. Lemma. If P7** holds, u is bounded.

Proof: The proof is quite similar to that of theorem 3.3.1. In fact it is even easier, since once an act f was found such that $f > x$ for all $x \in X$, P7** is contradicted.//

Following the proof in Section 3.3, we now have:

5.2.3. Lemma. Let P7** hold, and suppose that $\phi = B_0 \subset B_1 \subset \ldots \subset B_n = S$ is a chain of events such that $B_i - B_{i-1}$ is f-convex for some $f \in F$. Denote $\underline{u}_i = \inf_{s \in B_i - B_{i-1}} \{u(f(s))\}$, $\bar{u}_i = \sup_{s \in B_i - B_{i-1}} \{u(f(s))\}$, and assume that $\bar{f} \in F^*$ satisfies $\bar{f} \sim f$. Then $\sum_{i=1}^{n} (\underline{u}_i - \underline{u}_{i+1}) v(B_i) \leq$ $\leq I(\bar{f}) \leq \sum_{i=1}^{n} (\bar{u}_i - \bar{u}_{i+1}) v(B_i)$, where $\bar{u}_{n+1} \equiv \underline{u}_{n+1} \equiv 0$, and u is normalized 0-1.

Proof: Again the proof turns out to be easier than that of the Lemma's P7* - counterpart. For any $A_i \equiv B_i - B_{i-1}$, there are $s_i^*, s_{*i} \in A_i$ such that $f \,/\, {}^{f(s_i^*)}_{A_i} \geq f \geq f / {}^{f(s_{*i})}_{A_i}$.

Consequently there is an event $C_i \subset A_i$ such that

$$(f \,/\, {}^{f(s_i^*)}_{C_i}) \,/\, {}^{f(s_{*i})}_{A_i - C_i} \sim f.$$

Using this fact inductively, one results with an $\tilde{f} \in F^*$ such that $\tilde{f} \sim f \sim \bar{f}$, and $\underline{u}_i \leq u(\tilde{f}(s)) \leq \bar{u}_i$ for all $s \in A_i$, and all $i \leq n$, which completes the proof.//

Now we have:

5.2.4. Theorem. If P7** holds, then $\geq$ is integral-represented by u and v.

Proof: The proof is a simplifying adaptation of that of theorem 3.3.4, since lemma 3.3.3. holds, and P7** implies the existence of an act $\bar{f} \in F^*$ satisfying $\bar{f} \sim f$, for every $f \in F$. (There are no 'big' or 'small' acts.)//

Before stating the characterization theorem, we need another

Definition. $\{x_n\}_{n \geq 1} \subset X$ is an ascending (descending) sequence iff $x_n < x_{n+1}$ ($x_n > x_{n+1}$) for all $n \geq 1$. It is said to be a bounded ascending (descending) sequence if, in addition to the previous condition, there is an $x \in X$ such that $x_n < x$ ($x_n > x$) for all $n \geq 1$.

We may now formulate

5.2.5. Theorem. Suppose P1–P6** hold, and let u, v be the utility and the measure attached to $\geq$. Then the following two statements are equivalent:

(1) P7** holds;

(2) (i) (u,v) are an IR of $\geq$, and u is bounded;

(ii) If X contains a bounded ascending (descending) sequence, then v is lower (upper) quasicontinuous;

(iii) If X contains an ascending (descending) sequence, then v is is lower (upper) quasicontinuous at $\phi(S)$.

Proof: First suppose that P7** holds. We already know, by 5.2.2. and 5.2.4. that (2)(i) is true. Now suppose X contains a bounded descending sequence : $u(x_n) > u(x_{n+1})$; $u(x_n) > u(x_*)$. We wish to prove v to be upper quasicontinuous. Assume, therefore, that $\{A_n\}_{n\geq 1}$ is a sequence of events such that $A_n \subset A_{n+1}$, and $v(A_n) = \alpha \;\; \forall\, n \geq 1$.

Denote $A \equiv \bigcup_{n\geq 1} A_n$, and suppose that $v(A) = \beta > \alpha$. Define

$$f(s) = \begin{cases} x_1 & s \in A_1 \\ x_n & s \in A_n - A_{n-1} \quad n>1 \\ x_* & s \in A^c \end{cases} \quad , \text{ and } f_n = f \,/\, \overset{x_n}{B} \text{ for } B \equiv A - A_1.$$

It is easily seen that $I(f_n) = I(f) + [u(x_n) - u(x_*)]\,(\beta-\alpha)$, where $I(\cdot) \equiv \int u(\cdot)dv$. Hence $f < f \,/\, \overset{f(s)}{B} \;\; \forall\, s \in B$, in contradiction to P7**.

Note that if $A = S$, the existence of x_* is immaterial (since $A^c = \phi$). Therefore we also conclude that if X contains a descending sequence (not necessarily bounded) v is upper quasicontinuous at S.

This proves that P7** implies a half of (2)(ii) and a half of (2)(iii). The other two halves are proved similarly.

Now assume that (2) is true, but that P7** is not. More specifically, let there be an act f and an event A, which is f-convex, such that $f < f \,/\, {}^{f(s)}_{A}$ $\forall\, s \in A$. (The other case, namely, that $f > f \,/\, {}^{f(s)}_{A}$ $\forall\, s \in A$, is dealt with in the very same way.)

Denote $C = \{s \in S \,/\, f(s) \geq f(t) \quad \forall\, t \in A\} - A$;

$$u_* = \inf_{s \in A}\{u(f(s))\} \quad ; \quad u^* = \sup_{s \in A}\{u(f(s))\} \; ;$$
$$\alpha = v(C) \quad ; \quad \beta = v(C \cup A).$$

Define $a, b : S \to R$ as follows:

$$a(s) = \begin{cases} u(f(s)) & s \in A^c \\ u_* & s \in A \end{cases} \quad ; \quad b(s) = \begin{cases} u(f(s)) & s \in A^c \\ u^* & s \in A \end{cases} .$$

Surely $\int a dv \leq I(f) \leq \int b dv$ and $\int a dv \leq I(f \,/\, {}^{f(s)}_{A}) \leq \int b dv$ for all $s \in A$. Note that $\int b dv - \int a dv = (u^* - u_*)(\beta - \alpha)$, so that $u^* > u_*$ and $\beta > \alpha$. (Otherwise, in view of (2)(i), it is impossible that $f < f \,/\, {}^{f(s)}_{A}$ for any $s \in A$.)

We know that there is no $s \in A$ for which $u(f(s)) = u_*$, since for such an s we would have $f \geq f \,/\, {}^{f(s)}_{A}$. Therefore there is a sequence $\{s_n\}_{n \geq 1} \subset A$ such that $\{f(s_n)\}_{n \geq 1}$ is a descending sequence, and $u(f(s_n)) \underset{n \to \infty}{\to} u_*$.

Let us define $f_n = f \,/\, {}^{f(s_n)}_{A}$, so that

$$I(f_n) = \int a dv + [u(f(s_n)) - u_*]\beta \underset{n \to \infty}{\to} \int a dv.$$

Since $I(f) < I(f_n)$ for all $n \geq 1$, $I(f) = \int a dv$.

Now define $B_n = \{s \in S \,/\, f(s) \geq f(s_n)\}$ (satisfying $C \subset B_n \subset C \cup A$.).

Let $b_n : S \to R$ be defined by

$$b_n(s) = \begin{cases} u(f(s)) & s \in A^c \\ u(f(s_n)) & s \in B_n - C \\ u_* & s \in A - B_n \end{cases},$$

so that $u(f(s)) \geq b_n(s)$ for all $s \in S$ and $n \geq 1$. It is easily verifiable that

$$\int b_n dv = \int a dv + [u(f(s_n)) - u_*](v(B_n) - v(C)).$$

Since $I(f) \geq \int b_n dv$, and $I(f) = \int a dv$, we are lead to the conclusion that $v(B_n) = v(c) = \alpha$ for $n \geq 1$. Note that $\bigcup_{n \geq 1} B_n = C \cup A$, whence $v(\bigcup_{n \geq 1} B_n) = \beta > \alpha$. To conclude, note that if the sequence $\{f(s_n)\}_{n \geq 1}$ is bounded, (2)(ii) is contradicted; if, however, it is not, then necessarily $\bigcup_{n \geq 1} B_n = S$, in which case (2)(iii) is contradicted. //

5.2.6. Corollary. If either $X/\sim$ is finite or v is quasicontinuous, then P7** holds.

5.3. Continuity

The next problem we would like to consider is the characterization of preference orders which induce continuous measures. We begin with some

Definitions (1) $>^* \subset S \times S$ is a strict order on S iff it satisfies:

(i) for $s \neq t$, $s >^* t$ or $s ^*< t$;

(ii) for every s, t, it is false that $s >^* t$ and $t >^* s$;

(iii) if $s >^* t$ and $t >^* r$, then $s >^* r$.

(2) An act f is said to be comonotonic with $>^*$ iff

$f(s) > f(t) \Rightarrow s >^* t, \ \forall s, t \in S$.

(3) A class of acts $\mathcal{C} \subset F$ is maximally comonotonic iff it is a maximal class of p.c. acts.*

It is evident that there is a 1-1 correspondence between the class of strict orders on S and the class of all maximally comonotonic classes of acts. (To each $>^*$ on S, attach the class of acts which are comonotonic with it.)

We extend the definition of convexity in an obvious manner: an event $A \subset S$ is $>^*$-convex iff

$$s >^* r >^* t, \quad s, t \in A \Rightarrow r \in A, \quad \text{for all } s, r, t \in S.$$

Similarly, A is comonotonic with $>^*$ iff it is comonotonic with all the acts, which are comonotonic with $>^*$.

Now we will formulate an axiom which resembles Savage's P6, but is, in fact, much stronger:

P6***. Let f, g be acts, x a consequence, and $>^*$ a strict order on S.

Suppose that $f \underset{(<)}{>} g$, and that f is comonotonic with $>^*$. Then there exists a finite partition of S into $>^*$-convex events $(B_1, \ldots, B_n)$ such that $f \,/\, {}^{x}_{B_i} \underset{(<)}{>} g$ for all i such that $f \,/\, {}^{x}_{B_i}$ is commonotonic with f.

*In the sequel, the Axiom of Choice will be used freely.

Now we have:

5.3.1. Lemma. In the presence of P1–P5*, P6*** implies P6*.

Proof: Suppose that $f \,/\, \overset{x}{A} > g > f \,/\, \overset{y}{A}$ where $f \,/\, \overset{x}{A}$ and $f \,/\, \overset{y}{A}$ are comonotonic, and let $>^*$ be a strict order on S, comonotonic with both. (There exists such an order, since $f \,/\, \overset{x}{A}$ and $f \,/\, \overset{y}{A}$ belong to some maximally comonotonic class.)

Now let $\mathcal{B} = \{B \subset A \,/\, (f \,/\, \overset{x}{A-B}) \,/\, \overset{y}{B} < g,\ (f \,/\, \overset{x}{A-B}) / \overset{y}{B}$ comonotonic with $>^* \}$, and define $\overline{B} = \bigcup_{B \in \mathcal{B}} B$.

Obviously P6*** cannot hold unless $(f \,/\, \overset{x}{A-\overline{B}}) \,/\, \overset{y}{\overline{B}} \sim g$, which implies P6*.//

Note that we have proved a slightly stronger assertion: for a given $>^*$, comonotonic with $f \,/\, \overset{x}{A}$ and $f \,/\, \overset{y}{A}$, there can be found an act $(f \,/\, \overset{x}{A-\overline{B}}) \,/\, \overset{y}{\overline{B}}$ as required, which is also comonotonic with $>^*$. We shall use this fact in the sequel.

We wish to show that P6*** is strong enough to replace both P6* and P6** in the derivation of IR of $\underline{>}$. To this end we still have to prove that P6** is implied by P6***. However, it suffices to prove that assertion for a three-consequence world, since only in that stage of the proof was P6** required. (That is, in the presence of the other axioms, the two versions of P6** are equivalent). Therefore we prove only:

5.3.2. Lemma. If $X = \{x^*, x, x_*\}$ with $x^* > x > x_*$, and P1–P3* hold, P6*** implies P6**.

Proof: In this context we use the notations of section 2, and suppress the least-preferred consequence of any (simple) act. Suppose $f_n(s) \leq y_* \ \forall s \in S, \ \forall n \geq 1$, and $f_n / \overset{y^*}{A} \sim f_{n+1}$, where $y^* > y_*$. If $y_* = x_*$, the conclusion $A \sim \cdot \ \phi$ is immediate. Therefore let $y_* = x$, whence $y^* = x^*$. The act f_n may consequently be written as (x, B_n) for some $B_n \subset S$. In view of Lemma 5.3.1., one may assume w.l.o.g. that $B_n \subset B_{n+1}$, whence $\{f_n\}_{n \geq 1}$ are p.c.. Let $B = \bigcup_n B_n$ and $f = (x,B)$. Note that f is comonotonic with f_n, for all $n \geq 1$. Take $>^*$ to be a strict order on S, which is comonotonic with $\{f_n\}_n$ and f. Again by 5.3.1., let $C \subset B$ be such that $(x,C) / \overset{x^*}{A} \sim f$, and C is comonotonic with $>^*$.

Now: if for all n, $B_n \dot{\subset} C$, then $B = C$ and $f \sim f / \overset{x^*}{A}$, which, by P3*, implies $A \sim \cdot \ \phi$. Otherwise there is an n for which $C \subset B_n$. (Note that $\{D \subset S \ / \ (x,D)$ is comonotonic with $>^*\}$ is a chain.) But that means that $(x,B_n) / \overset{x^*}{A} \geq (x,C) / \overset{x^*}{A}$, whence $f_{n+1} \geq f \geq f_{n+2}$, or $f_{n+1} \sim f_{n+1} / \overset{x^*}{A}$, and, by 1.4.2, $A \sim \cdot \ \phi$. //

So we have:

5.3.3. Conclusion. If P1, P2*, P3*, P5*, P6*** and P7* hold, there is an IR of $\geq$ by some convex-ranged measure v and a bounded utility u.

In the light of this result, we can continue towards the characterization of continuous measures:

5.3.4. Theorem. The following two statements are equivalent:

(1) $\geq$ satisfies P1, P2, P3*, P5*, P6*** and P7*.

(2) (i) $\geq$ is IR by u and v, where u is bounded;

(ii) v is a continuous non-atomic measure.

Proof: First we shall prove that (1) implies (2). The fact that (2)(i) is implied by (1) is stated in 5.3.3. To prove the continuity of v, assume the contrary, e.g. there is a sequence $\{A_n\}_{n\geq 1}$, $A_n \subset A_{n+1}$ with $A = \bigcup_n A_n$, and $v(A) \neq \sup_n v(A_n)$. (Since $A_n \subset A$, this means that $v(A) > \sup_n v(A_n)$). Now let $>^*$ be a strict order on S, which, in the notations of section 2 again, $\{(x,A_n)\}_n$, (x,A) are all comonotonic with. Take $f = (x,A)$ and g to be some act such that $I(f) > I(g) > I((x,A_n))$ for all $n \geq 1$.

(By 5.3.3. v is convex-ranged.) Obviously, any finite partition, each member of which is $>^*$-convex, will contain an event B_i, such that $f \,/\, {}^{x_*}_{B_i}$ is comonotonic with f, but for some n, $I(f \,/\, {}^{x_*}_{B_i}) \leq I((x,A_n))$, so that P6*** is contradicted.

For the case $A = \bigcap_n A_n$, $A_n \supset A_{n+1}$, but $v(A) < \inf_n v(A_n)$, we will have a symmetric proof. The proof that v is non-atomic is very similar and will be omitted.*

Now we turn to the other half of the theorem, namely, that (2) implies (1). All the axioms but P6*** are immediately satisfied by an

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* It can also be deduced from the convex-rangedness of v, assured by 5.3.3.

integral-represented $\geq$ (with a bounded utility).

Consider, therefore, P6***. To prove it we will need

5.3.4.1. Lemma. Let A, B be events comonotonic with $>^*$, such that $B \subset A$, and let $\beta \in [0,1]$. Then there is an event C, $B \subset C \subset A$ comonotonic with $>^*$, such that

$$v(C) = \beta v(B) + (1-\beta)v(A).$$

Proof: Let $\mathcal{C} = \{C \subset S \,/\, B \subset C \subset A,\ C \text{ is comonotonic with } >^*\}$.

Denote $\alpha = \beta v(B) + (1-\beta)v(A)$,

$$\underline{C} = \bigcup_{\substack{C \in \mathcal{C} \\ v(C) < \alpha}} C \quad ; \quad \bar{C} = \bigcap_{\substack{C \in \mathcal{C} \\ v(C) > \alpha}} C \quad .$$

Note that $\underline{C}, \bar{C} \in \mathcal{C}$.

By the continuity of v, $v(\underline{C}) \leq \alpha \leq v(\bar{C})$. Suppose that both inequalities are strict, for otherwise the proof is complete. Surely there is an $s \in \bar{C} - \underline{C}$. (By definition $\underline{C} \subset \bar{C}$. However, the converse cannot hold since $v(\bar{C}) > v(\underline{C})$.) Define

$$D_s = B \cup \{t \in A - B \,/\, t >^* s\} \quad \text{and} \quad D^s = D_s \cup \{s\}.$$

Since $\underline{C}$ is $>^*$-comonotonic, and $s \notin \underline{C}$, we have $\underline{C} \subset D_s$. However, if $v(D_s) > \alpha$, then $s \in \bar{C}$ should have been an element of D_s. Therefore $v(D_s) \leq \alpha$. If equality holds, we need say no more. Otherwise $D_s \subset \underline{C}$, whence $\underline{C} = D_s$.

Now consider D^s. Since $s \in D^s - \underline{C}$, that is, $D^s \not\subset \underline{C}$, $v(D^s) \geq \alpha$. If the inequality is not strict, we have completed the proof. Otherwise $D^s \supset \bar{C}$. But $s \in \bar{C}$, so that $D^s \subset \bar{C}$, and $D^s = \bar{C}$.

Combining the two equalities with the definition of D_s, D^s, one gets $\bar{C} = \underline{C} \cup \{s\}$, while $v(\bar{C}) > v(\underline{C})$, in contradiction to the non atomicity of v. //

We now turn back to prove that ((2) assumed), P6*** does indeed hold. Suppose f is comonotonic with $>^*$ and satisfies $f > g$. By repetitive applications of the lemma just proved, one may find a $>^*$-convex partition of S $(B_1,\ldots,B_{2^n})$ such that $v(\bigcup_{1 \leq j \leq i} B_j) = i/2^n$ $(1 \geq i \leq 2^n)$, for all $n \geq 1$. Since u is bounded, P6*** is satisfied and the proof is complete. //

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