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"PREFERENCE REVERSAL" AND THE THEORY
OF CHOICE UNDER RISK+

by

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ABSTRACT

The paper shows that: (1) The preference reversal phenomenon is consistent with transitive preferences and constitutes a violation of the independence axiom. (2) The preference reversal phenomenon is accounted for by the interaction between the experimental design and respondents preferences and there is no experiment which can overcome this difficulty.

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1. INTRODUCTION

Preference reversal, first reported by Lichtenstein and Slovic [1971], describes experimental results which appear to indicate systematic violations of the transitivity of preferences axiom. In these experiments respondents were asked to express their preferences between, suitably chosen, pairs of lotteries. Having done so they were asked to state the lowest price they would be willing to accept in exchange for their right to participate in the lotteries. In many cases the respondents set a lower price for the preferred lottery. This phenomenon is more prevalent when the preferred lottery assigns high probability to winning a small sum of money and low probability for losing a small sum of money.

The original results of Lichtenstein and Slovic were replicated by Grether and Plott [1979] in experiments carefully designed to test for various explanations, including _inter alia_ misspecified incentives, strategic responses and decision and information processing costs. Experiments by Pommerehne, Schneider, and Zweifel [1982] and Reilly [1982] employing the method of Getner and Plott with variations in the design to increase the motivation and reduce the possible sources of confusion and misunderstandings produced a slight decline in the observed reversals. By and large, however, these studies reaffirmed the existence of the phenomenon.[1]

From the point of view of the theory of choice these results challenge one of the most fundamental tenets of rational behavior, namely the transitivity of preferences. Unlike the Allais paradox and other reported violations of the independence axiom which challenge the expected utility theory, the phenomenon of preference reversal, if true, contradicts any theory of choice under risk which is based on transitive preferences.[2]

In this paper we show that the aforementioned experimental evidence does not imply reversal of preferences. We present a theory of choice under risk based upon transitive preferences consistent with the experimental results. According to this theory, the preference reversal phenomenon is accounted for by the experimental design which solicits the respondents <u>reservation prices</u> of the lotteries presented to them. The reservation prices differ from the <u>certainty equivalents</u> of the lotteries. Furthermore, under the rules of the experiments it is quite possible that the reservation prices of some respondents are in reverse order to their certainty equivalents. Thus, <u>the experimental results indicate reservation price reversal rather than preference reversal</u>. In addition, the theory presented here postulates that the preferences are complete, continuous, monotonic (in the sense of first-order stochastic dominance), and satisfy the axiom of reduction of compounded lotteries. The preferences do not satisfy the von Neumann-Morgenstern independence axiom.

We also prove the futility of attempting to observe such preferences by experimental methods. There is no experiment which would reveal the certainty equivalents or will not produce reservation-price reversals for some preference orderings.

The formal presentation of the theory appears in Section 2. In Section 3 we show how, given the nature of the experiments, the theory accounts for the reservation price reversal phenomenon. The impossibility of observing preferences by experimental methods is proved in Section 4. Additional aspects of the theory are discussed in Section 5.

2. THE THEORY

Let X be a set of prizes. A finite lottery on X is a function from a finite subset of X to the half-open interval $(0,1]$ and the values taken by the function add up to one. The set of all finite lotteries on X is:

$$L = \{(x_1,p_1;x_2,p_2;\ldots;x_n,p_n) \in [X \times (0,1]]^n \mid \textstyle\sum_{i=1}^n p_i = 1,\ 1 \leqq n < \infty\}.$$

A binary relation $\succsim$ on L is said to be a preference relation if it is complete and transitive. We define the relation of strict preference, $\succ$, and indifference, $\sim$, as follows: For any $A, B \in L$, $A \succ B$ if and only if $A \succsim B$ and not $B \succsim A$, and $A \sim B$ if and only if both $A \succsim B$ and $B \succsim A$.

For every $x \in X$, we identify x with $(x,1) \in L$, thus X is ordered by preference relations on L. Given a preference relation $\succsim$ on L and any $A \in L$ we assume that $x_{i+1} \succsim x_i$ $i = 1,\ldots,n-1$. We assume further that $\succsim$ is represented on L by a function $V: L \to \mathbb{R}$ of the following form:

$$(2.1) \quad V(x_1,p_1;x_2,p_2;\ldots;x_n,p_n) = \textstyle\sum_{i=1}^n u(x_i)[f(\sum_{j=i}^n p_j) - f(\sum_{j=i+1}^n p_j)]$$

where $u(x_i) \equiv V(x_i,1)$ is a real valued continuous function on X and $f:[0,1] \to [0,1]$ is a continuous and monotonic increasing function satisfying $f(0) = 0$ and $f(1) = 1$. The function $u(\cdot)$ is referred to as a utility function and represents the preference relation $\succsim$ restricted to the subset of L which consists of the sure outcomes, (i.e., $n = 1$). The function $f(\cdot)$ is a probability transformation function. Both $u(\cdot)$ and $f(\cdot)$ are unique up to a positive linear transformation. For any $A, B \in L$, $A \succsim B$ if and only if $V(A) \geq V(B)$. We denote by Ω the set of all preference relations on L that are representable by $V(\cdot)$ given in (2.1).

The function $V(\cdot)$ may be thought of as a measure of the area above the distribution on $\{u(x_1), u(x_2),\ldots,u(x_n)\}$. When $f(\cdot)$ is the identity function $V(\cdot)$ is the expected utility functional.

Yaari [1982] shows the existence of a function like that of (2.1) where the utility function $u(\cdot)$ is linear. Segal [1984] shows the existence for general utility functions $u(\cdot)$. Both Yaari and Segal assume that $\succsim$ satisfy among other conditions: weak order, continuity, and monotonicity in the sense that for $A, B \in L$ such that A stochastically dominates B to the first order, $A \succsim B$.

3. RESERVATION PRICE REVERSALS AND TRANSITIVITY

In this section we show that if, when choosing among risky prospects in L, decision-makers maximize the value of the functional $V(\cdot)$ given in (2.1) then the results of the studies of Grether and Plott [1979], Pommerehne, Schneider, and Zweifel [1982]; and Reilly [1982] do not imply a violation of transitivity of the decision-maker's preferences. Rather, given the nature of the preferences, the design of the experiments and lotteries A and B in L such that $A \succsim B$ it may be optimal for the decision-maker to announce reservation prices for A and B, $P(A)$ and $P(B)$ respectively, such that $P(A) < P(B)$. The announced prices are not equal to the certainty equivalents $C(A)$ and $C(B)$ of the respective lotteries which satisfy $C(A) > C(B)$. Consequently the reservation price reversal does not imply preference reversal and a violation of transitivity.

(3.1.) Reductions of compounded lotteries – Anticipating the experimental evidence we shall assume henceforth that the prizes in X are sums of money. The real line is embedded in L by the identification of $x \in \mathbb{R}$ with $(x,1) \in L$.

A two-stage compounded lottery is a lottery which offers elements of L as prizes. Thus the vector $(A^1, q_1; A^2, q_2; \ldots; A^m, q_m) \in [L \times [0,1]]^m$, $1 \leq m < \infty$ represents a two-stage lottery if $\sum_{i=1}^m q_i = 1$. We assume that compounded lotteries are reducible to elements of L using the usual calculus of probabilities, (i.e., L is a mixture set). This assumption may be stated formally as the axiom of reduction of compounded lotteries:

If $A^i = (x_1^i, p_1^i; x_2^i, p_2^i; \ldots; x_{n_i}^i, p_{n_i}^i)$

then

$$(3.1) \qquad (A^1, q_1; A^2, q_2; \ldots; A^m, q_m) \quad (x_1^1, q_1 p_1^1; \; x_2^1, q_1 p_2^1; \ldots, x_{n_1}^1, q_1 p_{n_1}^1; \ldots; x_{n_m}^m, q_m p_{n_m}^m) .$$

Given the nature of X, another way of reducing compounded lotteries to elements of L is by replacing the lotteries A^i with their certainty equivalents, $C(A^i)$. This reduction is possible if the preference relations on L satisfy the independence axiom, namely, for every A, B and C in L and $\alpha \in (0,1)$ if $A \succ B$ then $(A,\alpha;C,1-\alpha) \succ (B,\alpha;C,1-\alpha)$. The independence axiom implies that:

$$(3.2) \quad (A^1, q_1; A^2, q_2; \ldots; A^m, q_m) \sim (C(A^1), q_1; C(A^2), q_2; \ldots; C(A^m), q_m)$$

where $C(A^i) \in X$ and $(C(A^i),1) \sim A^i$, $i = 1,\ldots,m$.

Each method of reducing compounded lotteries is consistent with the representation of preferences on L by a function $V(\cdot)$ of the type (2.1). Except in the case where $V(\cdot)$ is linear in the probabilities, however, (3.1) and (3.2) are not equivalent.[3] We choose to assume the reduction of compounded lotteries and abandon the independence axiom.[4] Thus in evaluating a non-degenerate compounded lottery we cannot replace a lottery in L by its certainty equivalent. This is stated formally in:

Lemma 3.1.: Suppose that the preference relation $\gtrsim$ on L is monotonic in the sense of satisfying first-order stochastic dominance and that the independence axiom does not hold. Then there exists $A, B \in L$ and $\alpha \in (0,1)$ such that $(A, \alpha; B, (1-\alpha)) \not\sim (C(A), \alpha; B, (1-\alpha))$ where $(C(A),1) \sim A$.

Proof: Assume by way of negation that for all $A, B \in L$ and all $\alpha \in (0,1)$ $(A,\alpha;B,(1-\alpha)) \sim (C(A), \alpha; B, (1-\alpha))$. Let $D \in L$, then, since $C(A)$, $C(B)$, $C(D)$ are in X,

$$(A, \alpha;B, (1-\alpha)) \sim (C(A), \alpha; C(B), (1-\alpha))$$

and

$$(D,\alpha;B,(1-\alpha)) \sim (C(D),\alpha; C(B), (1-\alpha)).$$

But $A \succ D \Rightarrow C(A) > C(D) \Rightarrow (C(A), \alpha; C(B), (1-\alpha)) \succ (C(D), \alpha; C(B), (1-\alpha)$, where the first implication follows from the monotonicity of $u(\cdot)$ and the second implication follows from the first-order stochastic dominance assumption. Thus, by transitivity, we have for all D, B and A in L and any $\alpha \in (0,1)$, $A \succ D \Rightarrow (A, \alpha; B, (1-\alpha)) \succ (D, \alpha; B, (1-\alpha))$ and the independence axiom holds. A contraction. //

Lemma 3.1. has far reaching consequences for the interpretation of the "preference reversal" phenomenon. To see this we must first describe the experiment used to produce this phenomenon.

(3.2.) <u>The experimental design</u> - To obtain the certainty equivalents of the lotteries which were used in their experiments Grether and Plott [1979] employed a method which was suggested by Becker, DeGroot, and Marschak [1964].[5] According to this method participants in the experiments were given the right to participate in some lotteries. For each lottery $A \in L$ each participant was asked to state the smallest price, $P(A)$ say, for which he would sell this right. The participants were informed that after they have set the prices a random sum would be chosen uniformly from $\{\$0.00, \$0.01, \$0.02, \ldots, \$9.98, \$9.99\}$. If the sum of money selected in this way exceeds the price set by the participant then he is paid the money and foregoes the right to participate in the lottery. If the sum of money selected randomly falls short of the price set by the participant, then the lottery is played out and the participant is paid the prize according to the outcome. The participants in the experiments were told that it is in their best interest to be accurate. If a participant sets a price which exceeds his reservation price and the outcome of the random draw is an amount between the stated price and the reservation price, the participant is forced to participate in the lottery while he would rather take the sum of money that was drawn. Similarly, if he states a price lower than his reservation price and the outcome of the random drawing is between the two prices, then he is forced to forego the opportunity to play the lottery when he would rather participate in it. This elaborate experiment is designed to motivate the participants to be as accurate as possible and to eliminate strategic responses which may cause preference reversal.

We claim that while this method reveals the reservation price of the participants for the given lotteries, it does not reveal the certainty equivalents of the lotteries. Furthermore, it is quite possible, as we show below, that the reservation prices which are affected by the experimental design will rank lotteries in reverse order to their certainty equivalents. To grasp the argument, suppose that the respondent declared a reservation price of \$5.00 for a lottery A, (i.e., $P(A) = \$5.00$). Then, by the rules of the experiment he faces the following two-stage compounded lottery

$$(A, \tfrac{1}{2}; \$5.00, \tfrac{1}{1000}; \$5.01, \tfrac{1}{1000}; \ldots; \$9.99, \tfrac{1}{1000}).$$

Any other choice of $P(A)$ results in a different two-stage compounded lottery. Presumably the rational participant is looking for the response which yields the most preferred among these lotteries, and declares the corresponding $P(A)$ to be his price for A. Given the design of the experiment, this is indeed the reservation price for A. However, except when the participants preferences satisfy the independence axiom, the reservation price obtained in this manner is not the certainty equivalent of A. Next we show that given there exist $\succsim \in \Omega$ and $A, B \in L$ such that $A \succ B$ then $C(A) > C(B)$ but, given the experimental design, $P(B) > P(A)$.

3.3. Reservation price reversal - We shall now show that reservation price reversal of the kind obtained in the experiments described above do not contradict the transitivity of the preference relation $\succsim \in \Omega$.

THEOREM 3.1: There exist $\succsim$ in Ω and $A, B \in L$ such that $A \succ B$ (and $C(A) > C(B)$) and $P(A) < P(B)$.

Proof: Since $V(A) > V(B)$ and $u(C(A)) = V(A)$, $U(C(B) = V(B)$, $C(A) > C(B)$ by monotonicity of u. To show that $P(A) < P(B)$ we use a pair of lotteries that were used in the experiments of Grether and Plott [1979]. Let

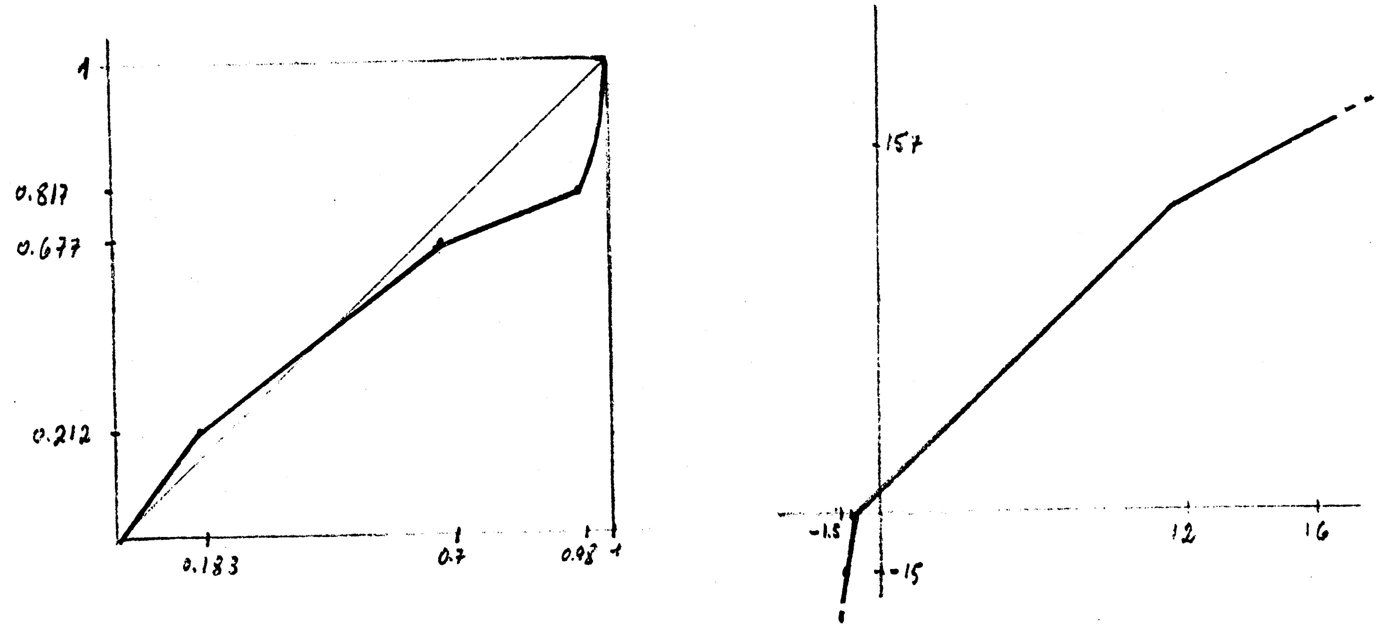
$$A = (-1, \tfrac{1}{36}; 4, \tfrac{35}{36}) \text{ and } B = (-1.5, \tfrac{25}{36}; 16, \tfrac{11}{36}).$$ [6]

Consider the following functions $f(\cdot)$ and $u(\cdot)$ respectively:

$$f(p) = \begin{cases} 1.1564p & 0 \leqq p \leqq 0.1833 \\ 0.9p + 0.047 & 0.1833 \leqq p \leqq 0.7 \\ 0.5p + 0.327 & 0.7 \leqq p \leqq 0.98 \\ p^{10} & 0.98 \leq p \leqq 1 \end{cases}$$

$$u(x) = \begin{cases} 30x + 30 & x \leqq -1 \\ 10x + 10 & -1 \leqq x \leqq 12 \\ 6.75x + 49 & 12 \leqq x \end{cases}$$

These functions are depicted in Figure 1 panels (a) and (b) respectively:[7]



(A) The probability transformation function (b) The Utility function

FIGURE 1

For simplicity we suppose that the random drawing is from the set of five numbers $\{1, 2, 3, 4, 5\}$ and the probability of drawing any one of these numbers is 1/5.[8] For $i < P(A) \leq i+1$, $i = 0,\ldots,4$, the value of V is given by:

$$V(A|P(a)) \equiv V(A,\frac{i}{5};\ i+1,\ \frac{1}{5};\ldots;5,\frac{1}{5}) = V(-1,\ \frac{i}{5}\cdot\frac{1}{36};4,\frac{i}{5}\cdot\frac{35}{36};i+1,\frac{1}{5},\ldots,5,\frac{1}{5}).$$

Similarly for B. The values of V for different choices of $P(k)$, $k = A$, B are summarized in Table 1 below. (Notice that if $P(A) \leq 1$ then the respondent sells the right to participate in A with certainty, thus, for $k = A, B$

$$V(k \mid P(k) \leq 1) = V(1,\frac{1}{5};2,\frac{1}{5};3,\frac{1}{5};4;\frac{1}{5};5,\frac{1}{5}).)$$

First we note from the first line that $V(A) > V(B)$, thus $A \succ B$ as hypothesized. Second, Max $V(A \mid P(A)) = 45.25$.

TABLE 1

	$V(A \mid P(A))$	$V(B \mid P(B))$	Value of $P(k)$
1)	$40.65 = V(A)$	$40.38 = V(B)$	$5 < P(k)$
2)	43.06	44.42	$4 < P(R) \leq 5$
3)	44.53	46.66	$3 < P(k) \leq 4$
4)	45.25	45.06	$2 < P(k) \leq 3$
5)	43.70	39.61	$1 < P(k) \leq 2$
6)	39.48	39.48	$P(k) \leq 1$
7)	$3.065 = C(A)$	$3.038 = C(B)$	

(line 4) and thus $2 < P(A) \leq 3$ while Max $V(B \mid P(B)) = 46.66$ (line 3) and hence $3 < P(B) \leq 4$ and the reservation price reversal is established. //

We thus proved that reservation price reversal do not imply intransitivity of preferences. (As is shown in line (7) $C(A) > C(B)$). The term preference reversal is misleading and should be replaced by the term reservation price reversal as a description of the phenomenon under consideration.

Next we show that, given the nature of the experiments, the results are not independent of the range of the offer prices, i.e., the range of the sums of money used to establish whether the individual participates in the lottery. In other words, by changing the range we can obtain reversal of the reservation price reversal. This observation is of immediate relevance for the interpretation of the results of Reilly [1982], who modified the original

Grether and Plott [1979] experimental design in several ways, including an extension of the range of the offer prices. This extension in itself may account for the reduction in the reservation price reversal.

THEOREM 3.2: There exist preference relation $\gtrsim \, \varepsilon \, \Omega$, lotteries A, B in L and offer price ranges I_1, I_2 contained in $\mathbb{R}$ such that $A \succ B$, $P(A|I_1) < P(B|I_1)$, $P(A|I_2) > P(B|I_2)$, where $P(k|I_i)$, $k = A, B$, $i = 1,2$ is the reservation price of k conditional on the offer price range I_i.

Proof: Consider again the lotteries A and B as in the proof of Theorem 3.1. Let $I_1 = \{1,2,3,4\}$ and $I_2 = \{1,2,3,4,5\}$. Let $f(p) = p^2$ and suppose that $u(x)$ takes the following values:

u(x)	0	10	20	75	120	200	210	1000
x	-15	-1	1	2	3	4	5	16

(Since these are the only relevant values for the present purpose $u(x)$ can be thought of as consisting of linear segments). Tables 2 and 3 below summarize the values of $V(A|P(A))$ and $V(B|P(B))$ for I_1 and I_2 respectively. Clearly $V(A) > V(B)$. Given the offer price range I_1, Max $V(A|P(A)) = 192.21$ and consequently the conditional reservation price $p(A|I_1)$ satisfies $3 < P(A\ I_1) \leq 4$. Max $V(B|P(B)) = V(B) = 93.4$ consequently the conditional reservation price $P(B|I_1) > 4$. Hence $P(A|I_1) < P(B|I_1)$, a reservation price reversal.

Consider next the offer price range I_2. Max $V(A|P(A)) = 194$ and the corresponding conditional reservation price $P(A|I_2)$ satisfies $3 < P(A|I_2) \leq 4$. Max $V(B|P(B) = 97.69$ and the corresponding conditional reservation price $P(B|I_2)$ satisfies $2 < P(B|I_2) \leq 3$. Thus, $P(B|I_2) < P(A|I_2)$. //

TABLE 2: OFFER PRICE RANGE I_1

$V(A\|P(A))$	$V(B\|P(B))$	Value of $P(k)$
189.59 = V(A)	93.4 = V(B)	$4 < P(k)$
192.21	88	$3 < P(k) \leq 4$
159.9	82.48	$2 < P(k) \leq 3$
127.45	79	$1 < P(k) \leq 2$
66.8	66.8	$P(k) \leq 1$

TABLE 3: OFFER PRICE RANGE I_2

$V(A\|P(A))$	$V(B\|P(B))$	Value of $P(k)$
189.59 = V(A)	93.4 = V(B)	$5 < P(k)$
192	87.3	$4 < P(k) \leq 5$
194	96.73	$3 < P(k) \leq 4$
168.6	97.69	$2 < P(k) \leq 3$
131.1	95.25	$1 < P(k) \leq 2$
84.6	84.6	$P(k) \leq 1$

4. NON-OBSERVABLE PREFERENCES

What Grether and Plott and others tried and, as our discussion indicates, failed to do is to identify by experiments the elements of equivalence classes of given lotteries in L that belong to a pre-specified subset of L, namely the real line. Their failure is exacerbated by the fact that the elements in $\mathbb{R}$ identified by the experiments do not preserve the preference ordering on the lotteries that were used in the experiments. This raises the question of whether there exist experiments, which, for any given element of L, identify a corresponding element in a subset M of L such that for every preference ordering on L which belong to a prespecified set, these elements belong to the same equivalence class? Failing that, is there an experiment that reveals the respondents' preference relation on L by making him assign elements of L to elements of M which, for every preference relation in a given set preserve the order on L? To answer these questions, we must first define the set of admissible experiments and the set of preference relations to be considered. We assume that $M = \{(x,1) \in L \mid x \in \mathbb{R}\}$.

(4.1) Preliminaries - If the preference relations to be considered satisfy the von Neuman-Morgenstern axioms of weak order, independence, and continuity, then obviously the experiments of Grether and Plott [1979] would reveal the certainty equivalents and the answer to both questions raised here would be positive. The results of these experiments, however, imply that either the independence axiom or the transitivity axiom must be dropped. We shall consider preference relations that do not satisfy the independence axiom. Thus let Ω' be the set of all the preference relations on L which are complete, transitive, continuous, and satisfy first-order stochastic dominance.

To define the set of admissible experiments we begin by considering the strategies of the respondents and of the experimenter respectively.

Let $(\succsim, A) \in \Omega' \times L$. A strategy of a respondent whose preference relation is $\succsim$ and is endowed with a right to participate in A is to announce a real number p say which has the interpretation of a reservation price. An experiment E is a function which assigns to every pair (p,A) an element $E(p,A) \in L$. The set of <u>admissible experiments,</u> $\mathcal{E}$, ie the set of all $E: \mathbb{R} \times L \to L$[9].

For all E in $\mathcal{E}$ we define

$$E(\succsim, A) = \{E(r\ ,A) \mid E(r,A) \succsim E(P,A) \text{ for all } P \in \mathbb{R}\}$$

$$R(E, \succsim, A) = \{r \in \mathbb{R} \mid E(r,A) \in E(\succsim, A)\}.$$

$E(\succsim, A)$ is the set of $\succsim$ -maximizers subject to $\{E(P,A)\}_{p \in \mathbb{R}}$. $R(E, \succsim, A)$ is the set of reservation prices that give elements of $E(\succsim, A)$. If $E(\succsim, A) = \phi$ we define $R(E, \succsim, A) = \{\infty\}$.

Notice that $\{E(P,A)\}_{P \in \mathbb{R}}$ depends on A but not on $\succsim$ since the experimenter does not know the preference relation $\succsim$.

Define the certainty equivalence function $C: \Omega' \times L \to \mathbb{R}$ by the number $C(\succsim, A)$ such that $(C(\succsim, A), 1) \sim A$.

<u>Definition 4.1.</u> The set Ω' is <u>observable at C under $\mathcal{E}$</u> if there exists E in $\mathcal{E}$ such that

$$R(E, \succsim, A) = \{C(\succsim, A)\} \text{ for all } (\succsim, A) \in \Omega' \times L.$$

Ω' is <u>non-observable at C under $\mathcal{E}$</u> if for every E in $\mathcal{E}$ there is $(\succsim, A) \in \Omega' \times L$ such that

$$R(E, \succsim, A) \neq \{C(\succsim, A)\}$$

In other words, Ω' is observable at C under $\mathcal{E}$ if there is an experiment in $\mathcal{E}$ such that, given the rules of that experiment, for every lottery A in L, every participant with preference relation in Ω' will announce his certainty equivalent of A.

<u>Definition 4.2.</u> The set Ω' is <u>non-observable under $\mathcal{E}$</u> if there is no E in $\mathcal{E}$ such that for all A, B in L and $\succsim$ in Ω':

$A \succ B$ iff $r_1 > r_2$ for all $r_1 \in R(E, \succsim, A)$ and $r_2 \in R(E, \succsim, B)$.

In other words, for every E in $\mathcal{E}$ there are lotteries A, B in L and a participant with preference relation in Ω' such that

$$A \succ B \text{ and } r_1 \leq r_2 \text{ for some } r_1 \in R(E, \succsim, A) \text{ and } r_2 \in R(E, \succsim, B).$$

4.2. <u>Non-observability of Ω'.</u> We now prove that Ω' is non-observable under $\mathcal{E}$, i.e., there is no admissible experiment which does not result in reservation price reversal for some $\succsim$ in Ω'. A corollary to this result is that Ω' is non-observable at C (under $\mathcal{E}$).[10]

<u>THEOREM 4.1.</u> Ω' is non-observable under $\mathcal{E}$.

<u>Proof</u>: Let E be an experiment in $\mathcal{E}$.

If there is $\succsim$ in Ω' and A in L such that $R(E, \succsim, A) = \{\infty\}$ take a lottery B which first-order stochastically dominates A. We have A B and $r_1 \geq r_2$ for all $r_1 \in R(E, \succsim, A)$ and $r_2 \in R(E, \succsim, B)$. Thus, by definition 4.2 Ω' is non-observable under $\mathcal{E}$.

Suppose that $E(\succsim, A) \neq \phi$ for all $(\succsim, A) \in \Omega' \times L$. We now show that it is possible to have $\succsim$ in Ω' and A, B in L such that $A \neq B$, $A \sim B$, and $B \notin E(\succsim, A)$. Take some $(\succsim, A)$ in $\Omega' \times L$. If $B \sim A$ implies $B \in E(\succsim, A)$ (which means $E(\succsim, A) = \{B \in L | B \sim A\}$) then change $\succsim$ slightly to $\succsim'$ in Ω' for which there exists B such that $A \sim' B$ and $A \succ B$ (see Figure 2).

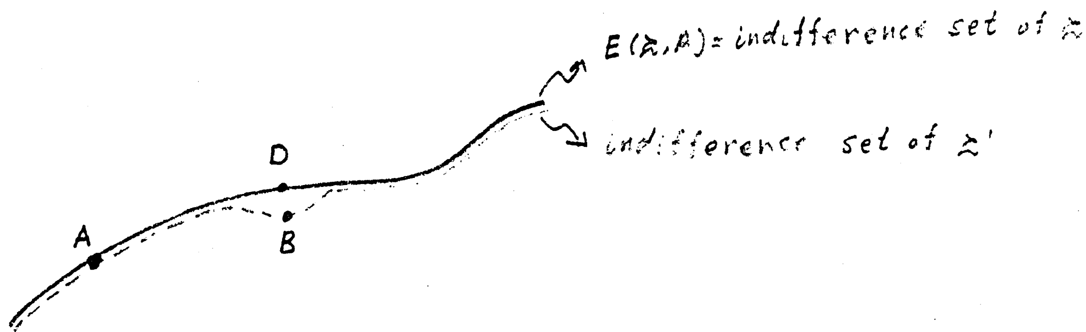


FIGURE 2

Since $\{E(P,A)\}_{P \in \mathbb{R}}$ does not change and there exists D such that $D \in E(\succsim, A)$ (which satisfies $D \sim A$) and $D \succ' B$, we get for Ω' the existence of $B \neq A$ such that $B \sim' A$ and $B \notin E(\succsim, A)$. Without loss of generality we denote $\succsim'$ by $\succsim$.

Henceforth let A, B in L be such that $A \neq B$, $A \sim B$ and $B \in E(\succsim, A)$ and consider the following cases:

(a) There exists $D \in E(\succsim,B)$ such that $D \neq B$. Let r_2 be such that $D = E(r_2,B)$. If $r_2 \geq r_1$ for some $r_1 \in R(E,\succsim,A)$ then we change slightly to $\succsim'$ in a small neighborhood of B such that $\succsim'$ is in Ω', $A \succ' B$, $R(E,\succsim',A) = R(E,\succsim,A)$ and r_2 is still in $R(E,\succsim',B)$. We get

$$A \succ' B \text{ and } r_1 \leq r_2 \text{ for } r_1 \in R(E,\succsim',A) \text{ and } r_2 \in R(E,\succsim',B).$$

Thus, by definition 4.1. Ω' is non-observable under $\mathcal{E}$.

If $r_2 < r_1$ for some $r_1 \in R(E,\succsim,A)$ then by similar argument we get B such that

$$A \prec' B \text{ and } r_1 > r_2,$$

and again Ω' is non-observable under $\mathcal{E}$.

(b) Suppose that $E(\succsim,B) = \{B\}$. In this case $A \notin E(\succsim,B)$. Hence if there exists $D \in E(\succsim,A)$ such that $D \neq A$ then by the same arguments as above, Ω' is non observable under $\mathcal{E}$.

(c) Suppose that $E(\succsim,B) = \{B\}$ and $E(\succsim,A) = \{A\}$. Without loss of generality assume the existence of $r_1 \in R(E,\succsim,A)$ and $r_2 \in R(E,\succsim,A)$ such that $r_1 \leq r_2$. We now change $\succsim$ slightly to $\succsim''$ near A such that $\succsim''$ is in Ω', $A \succ'' B$ and still $\{A\} = E(\succsim'',B)$ and $\{B\} = E(\succsim'',B)$ (see Figure 3).

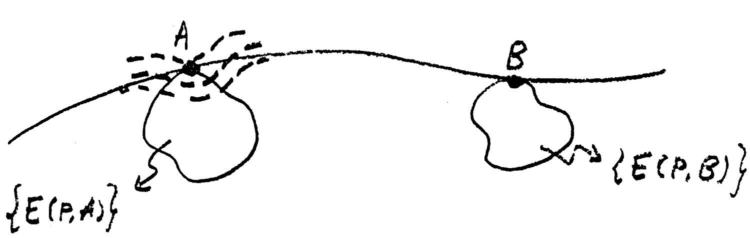


FIGURE 3

We get

$$A \succ'' B \text{ and } r_1 \leq r_2 \text{ for some } r_1 \in R(E, \succsim'', A) \text{ and } r_2 \in R(E, \succsim'', B),$$

a violation of the condition of definition (4.2). Thus Ω' is non-observable under $\mathcal{E}$.

<u>Corollary 4.1.</u> Ω' is non-observable at C under $\mathcal{E}$.

<u>Proof</u>. Immediately from Theorem 4.1., since for every experiment E in $\mathcal{E}$ the reservation price reversal imply the non-observability at C (the function $C(\cdot,\cdot)$ is monotonic in the second argument). //

Note that the reservation price reversal phenomenon is "open" in Ω', i.e. if it is satisfied at $\succsim$ for some A and B then it is still satisfied for many others $\succsim'$ in Ω' which are close enough to $\succsim$.

<u>Remark</u>. If we replace the real line $\mathbb{R}$ by more general set M (which is ordered by the first-order stochastic dominance relation), then similar results still hold.

<u>5. CONCLUDING REMARKS</u>

We conclude with three additional observations:

(5.1.) <u>Reservation price reversal and Machina's non-expected utility theory</u> – In a recent contribution Machina [1982] presented a theory of decision-making under risk which is capable of accounting for several reported violations of the independence axiom. The essence of this theory is the replacement of the independence axiom by a weaker assumption that the decision-maker's preferences over the set of lotteries L are representable by a Frechet differentiable preference functional.[11] Since, according to Machina's theory preferences on L are complete, transitive, and satisfy first-order stochastic dominance and are not inconsistent with the reduction of compounded lotteries axiom, it may seem that it includes (2.1) as a special case, at least when $X \equiv [0,M] \subset \mathcal{R}$. This impression, however, is wrong. For $V(\cdot)$ in (2.1) to be Frechet differentiable the probability transformation function $f(\cdot)$ must satisfy certain requirements which are not an intrinsic property of the model (2.1). Thus, for continuous probability distribution $f(\cdot)$ must be twice continuously differentiable. For discrete probability distributions it must be continuously differentiable. Since the latter case is relevant for the interpretation of the experimental evidence we present here a formal proof of this claim.

Let $T = \{x_1,\ldots,x_n\}$ be a given set of prizes in a closed interval on the real line such that $x_1 \leq x_2 \leq, \ldots, \leq x_n$. A probability distribution on T is an n-dimensional vector $(p_1, p_2,\ldots,p_n)$ such that $p_i \geq 0 \quad i = 1,\ldots,n$ and $\sum_{i=1}^{n} p_i = 1$.
Let P denote the set of all probability distributions on T.

<u>Lemma 5.1.</u> The preference functional $V(\cdot)$ in (2.1) restricted to P L is Frechet differentiable, if and only if $f'(\cdot)$ in (2.1) is continuous.

<u>Proof</u>: For $p \in P$, $V(p) = U(x_1) + [u(x_2) - u(x_1)]f(\sum_{i=2}^{n} p_i)$

$$+ [u(x_3) - u(x_2)]f(\textstyle\sum_{i=3}^{n} p_i) + \ldots + [u(x_n) - u(x_{n-1})]f(p_n)$$

Hence, $\frac{\partial V}{\partial p_1} = 0$, and for $j > 1$ $\frac{\partial V}{\partial P_j} = \sum_{k=2}^{j} [u(x_k) - u(x_{k-1})]f'(\sum_{i=k}^{n} p_i)$. For $p' \in P$ let $h = p - p'$, then $\frac{d}{d\alpha}V(p + \alpha h)\Big|_{\alpha=0} = \sum_{j=2}^{n} \frac{\partial V}{\partial P_j} h_j \equiv \delta V(p;h)$. Hence $\delta V(p;h)$ is linear and continuous in h. It can now be shown that the function $\delta V(p;h)$ is the Fréchet differential of $V(\cdot)$ if and only if $\frac{\partial V}{\partial p_j}$ is continuous for all j. (See Luenberger [1969]). This holds if and only if $f'(\cdot)$ is continuous.//

In the proof of Theorem (3.1) we assumed $f(p) = p^2$. Thus, $V(\cdot)$ was Frechet differentiable. Consequently Machina's theory is consistent with the reservation-price reversal phenomenon and thus accounts for the experimental results of Grether and Plott [1979] and others.[12]

(5.2) Reservation-price reversals and the reduction of compounded lotteries by the independence axiom - As already mentioned, an alternative way of reducing compounded lotteries is by substituting the certainty equivalent for each of the lotteries that appear in each stage after the first. This method was advocated by Segal [1984], who thus accepts the independence axiom and rejects the reduction of compounded lotteries axiom. Segal's approach, however, is inconsistent with the experimental evidence concerning reservation-price reversals. To see this, suppose that a participant in the experiments of Grether and Plott is given a lottery A and is asked to announce the smallest price, $P(A)$ which he would be willing to accept for A. Since the rules set by the experimenters imply that the respondent participates in a two-stage compounded lottery, he will substitute $C(A)$ for A in evaluating the optimal responses. Upon doing so, however, his optimal response is to set $P(A) = C(A)$. (If he sets $P(A) \neq C(A)$ then he subscribed to a lottery which is stochastically dominated by the lottery where $p(A) = C(A)$. This can be shown by the following consideration: If the realization of the random variable drawn to determine his position in the first stage is between $P(A)$ and $C(A)$ and $P(A) > C(A)$ he loses by having to accept $C(A)$ instead of the realized sum of money If $P(A) < C(A)$ he is forced to accept a sum of money smaller than $C(A)$ instead of $C(A)$ and thus loses again). In this case, therefore, reservation-price reversal is indeed preference reversal and the transitivity of preferences is violated.

This discussion shows that within the context of the theory of section 2, the reservation price reversals phenomenon is a violation of the independence axiom, rather than the transitivity axiom.

(5.3.) Reservation price reversal and other violations of the independence axiom - In view of the fact that the reservation price reversal phenomenon is a violation of the independence axiom, is it consistent with other reported violation of this axiom. Yaari [1984] and Segal [1984] demonstrate that with appropriate restrictions on $u(\cdot)$ and $f(\cdot)$ preference relations in are consistent with choice patterns, e.g., the Allais paradox, that violate the independence axiom. These restrictions are not inconsistent with reservation price reversals. For instance Segal shows that to obtain the Allais paradox $f(\cdot)$ must be convex and

$$\frac{f(1) - f(0.99) + f(0.1)}{f(0.1)} > \frac{u(5M)}{u(1M)} > \frac{f(0.11)}{f(0.1)} ,$$

where iM stands for i million dollars. Let $u(5M) = 1{,}500{,}000$ and $u(1M) = 1{,}000{,}000$ be the extension of the utility function in the proof of Theorem 3.2. and suppose that $f(p) = p^2$. A decision-maker whose preferences are representable by a functional combining these functions will display pattern of choice that violate the inddpendence axiom when facing the choices in

Alllais paradox and reservation price reversals. Generally speaking, reservation price reversals are not inconsistent with reported violations of the independence axiom.

FOOTNOTES

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1. For an evaluation of the results of Pommerehne, Schneider and Zweifel [1982] see Grether and Plott [1982]. For a broader perspective see Slovic and Lichtenstein [1983]. A discussion of the significance of these results appears in Arrow [1983] and Machina [1983].

2. A detailed discussion of the violations of the independence axiom will take us too far afield. For excellent reviews of this evidence see MacCrimmon and Larsson [1979] and Machina [1982]. Looms and Sugden's [1983] explanation of the preference reversal phenomenon using regret theory does not solve the intransitivity problem since regret theory does not assume transitivity.

3. That (3.1) and (3.2) are equivalent under the expected utility theory follows from the fact that (3.1) implies that L is a mixture set and since $V(\cdot)$ is a continuous linear representation of the binary relation on L it satisfies the axioms of weak order and continuity. By the von-Neumann-Morgenstern expected utility theory it must satisfy

the independence axiom and consequently (3.2). Conversely, since $V(\cdot)$ has the properties mentioned above and satisfies the independence axiom, it follows from the expected utility theorem that L is a mixture set and (3.1) must hold.

4. Segal [1984] took the alternative approach accepting the independence axiom and foregoing the reduction of compounded lotteries axiom. Segal's approach is an attempt to explain the Ellsberg [1961] paradox.
5. Pommerehne et-al. [1982] and Reilly [1982] employed the same method.
6. In the terminology used by Grether and Plott [1979] lottery A is a P bet and lottery B is a $ bet. The majority of respondents who preferred P bets over $ bets in these experiments set higher prices for the $ bets than for the P bets, thus creating reservation price reversals.
7. The probability transformation function is similar in shape to the weighting function of Kahneman and Tversky [1979]. Notice, however, that except for the case of two-price lotteries the probability transformation function enter the preference functional $V(\cdot)$ differently from the weighting function in prospect theory. Indeed prospect theory is inconsistent with preference reversals.
8. Notice that the range includes the expected monetary value of the two lotteries. The results for the case of the two numbers that were used in the Grether and Plott experiments and for $f(\cdot)$ and $u(\cdot)$ as in the proof of Theorem 3.1 are $V(A) = 40.65557$, $P(A) = 3.43$, $V(B) = 40{,}384$, $P(B) = 4.33$. Thus, $V(A) > V(B)$ and $P(B) > P(A)$. By the monotonicity of $u(\cdot)$ $C(A) > C(B)$. We would like to thank Lawrence Lessig for doing the calculations.
9. Notice that the experiment of Grether and Plott is an element of $\mathcal{E}$. Notice also that an experiment where the participants are offered to sell their right to participate in a lottery $A \in L$ for a given sum of money is also in $\mathcal{E}$. Let this sum of money be s then $E(P,A) = (s,1) \in L$ if $p \geq s$ and $E(P,A) = A$ if $p < s$.

10. The weak version of reservation price reversal is enough for breaking the transitivity when $r(E,\cdot,\cdot) = C(\cdot,\cdot)$ since transitivity implies $[A \succ B \iff C(\succsim, A) > C(\succsim, B)]$.

11. The set of lotteries in Machina's analysis is defined on $[0,M] \in \mathbb{R}$.

12. Machina didn't realize this fact. He regards the experimental evidence produced by Grether and Plott as incompatible with any fixed preference ranking over the set of lotteries (see Machina [1982] p.308).

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