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ON THE CORRESPONDENCE BETWEEN MULTIVARIATE RISK AVERSION AND RISK AVERSION WITH STATE-DEPENDENT PREFERENCES

by

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I. INTRODUCTION

Advances in the analysis of risk bearing in situations involving univariate state-independent utility functions, made possible with the introduction of risk aversion measures by Arrow [1971] and Pratt [1964], inspired the search for similar measures for other utility functions.[1] Research of this subject has been pursued in two directions: (i) the development of risk aversion measures for utility functions with many commodities and multivariate risks (Stiglitz [1969], Kihlstrom and Mirman [1974], Duncan [1977], Karni [1979]) and (ii) the development of risk aversion measures for state-dependent utility functions (Karni [1981]).[2]

Pursuing the first line of research, Kihlstrom and Mirman [1974] observed that a necessary prerequisite for the comparison of attitudes towards multivariate risks is that the decision-makers being compared have identical ordinal preferences over the commodity space. Restricting comparability in this manner permits the development of risk aversion measures which capture those cardinal properties of the utility functions relevant for decision-making under uncertainty. Not surprisingly, these measures were found to be the same measures suggested by Arrow and Pratt. The restriction on the ordinal preferences is not explicitly mentioned in the discussions of Arrow [1971] and Pratt [1964] since, unlike the multivariate case, all univariate utility functions represent the same ordinal preferences.

Pursuing the second line of research, the present author (Karni [1980]) observed that the measurement of risk aversion requires a definition of reference points in the domain of the utility functions where the measure of risk aversion can be properly defined. Univariate state-independent utility functions have "natural" reference points - the certainty points. These are points in the domain of the utility function, say wealth, which have the following three properties: (i) wealth is the same across states-of-nature; (ii) utility is the same across states-of-nature, and (iii) marginal utility of wealth is the same across states-of-nature. Moreover, this set of points, which we call the reference set, is the same for all state-independent univariate utility functions. Hence, in this sense all such functions are comparable.

State-dependent utility functions do not have reference sets which share the aforementioned attributes. Thus, the first question we must ask is: Which property is essential to retain in defining a reference set for state-dependent preferences? We find, perhaps not surprisingly, that the important attribute of the reference set is the equality of the marginal utility of wealth across states-of-nature. Defining the reference set in this manner, we see immediately that the reference sets of different decision-makers are not, in general, the same. Because of the indispensability of the reference set in the measurement of risk aversion, a necessary prerequisite for the comparison of aversion to risk among decision-makers when preferences are state-dependent is that the decision-makers have identical reference sets.

So far the discussion indicates the necessary limitations in trying to extend the Arrow-Pratt measures of risk aversion beyond the relatively simple case of univariate state-independent utility functions. In all of these extensions, the partial ordering given by the relation "more risk averse than..." must be restricted to subsets of decision-makers; those with identical preference ordering in the case of multivariate utility functions, and those with identical reference sets in the case of state-dependent utility functions.

In this paper, we study the correspondence between multivariate risk aversion and risk aversion with state-dependent preferences. In particular we show that these are not two distinct theories applicable to different problems but rather that the theory of multivariate risk aversion is a specific case of the more general theory of risk aversion for state-dependent preferences. Since the theory of risk aversion for state-independent utility functions may also be regarded as a particular case of the same theory, the theory of risk aversion for state-dependent utility functions appears to be a general theory of measurement of risk aversion.

In Section II below, we define the notion of a reference set for utility functions with many commodities and state the condition for comparability for these functions. Section III contains the main results of the paper in the form of two theorems.

II. THE REFERENCE SET FOR UTILITY FUNCTIONS WITH MANY COMMODITIES

Let $\underset{\sim}{x} \varepsilon R^n_+$ be a vector of commodities and $u(\underset{\sim}{x}) \varepsilon R^1$ be a utility representation of the preference ordering $\succsim_u$. Denote by y and $\underset{\sim}{p} \varepsilon R^n_+$ the money income and the n-dimensional vector of money prices, respectively, and define the indirect utility function, $U(y,\underset{\sim}{p}) \equiv u(\underset{\sim}{x}^*(y,\underset{\sim}{p}))$, were $x^*(y,\underset{\sim}{p})$ is the solution to:

$$\text{(P.1)} \qquad \underset{\{\underset{\sim}{x}\}}{\text{Max}} \; u(x) \qquad \text{s.t.} \; \underset{\sim}{x}\,\underset{\sim}{p}^T = y$$

The price vector $\underset{\sim}{p}$ may be interpreted as a "state-of-nature". Then the set of states of nature is R^n_+. Uncertainty in this interpretation is the lack of advanced knowledge regarding the realization of $\underset{\sim}{p}$. Suppose that to every realization of $\underset{\sim}{p}$ corresponds an income level given by the function $\hat{y}(\underset{\sim}{p})$ and consider the following problem:

$$\text{(P.2)} \qquad \begin{aligned} &\underset{\{y(p)\}}{\text{Max}} \int_{R^n_+} U(y(\underset{\sim}{p}),\underset{\sim}{p})\,dF(\underset{\sim}{p}) \\ &\text{subject to} \quad \int_{R^n_+} [y(\underset{\sim}{p}) - \hat{y}(\underset{\sim}{p})]\,dF(\underset{\sim}{p}) \leq 0. \end{aligned}$$

where $F(\underset{\sim}{p})$ is the joint cummulative probability distribution of $\underset{\sim}{p}$. That is, find the function $y^*(\underset{\sim}{p})$ which, for a given $F(\underset{\sim}{p})$ and an initial income distribution $\hat{y}(\underset{\sim}{p})$ maximizes the expected indirect utility function provided that the actuarial value of $y^*(\underset{\sim}{p})$ does not exceed that of $\hat{y}(\underset{\sim}{p})$. In other words, $y^*(\underset{\sim}{p})$ represents the optimal distribution of income across states of nature for a given actuarial value of money income. Let c denote this actuarial value, i.e. $c = \int_{R^n_+} y(\underset{\sim}{p})\,dF(\underset{\sim}{p})$. Then the Euler conditions are:

(1) $$U_y(y^*(\underset{\sim}{p};c),\underset{\sim}{p}) = \lambda(c) \quad \text{for all} \quad \underset{\sim}{p} \in R^n_+$$

where $U_y(\cdot\cdot)$ denotes the partial derivative of $U(\cdot\cdot)$ with respect to y, and $\lambda(c)$ is the Euler-Lagrange multiplier. Clearly, $\lambda(c)$ is equal across states-of-nature.

Using this result we define the reference set as follows:

<u>Definition 1:</u> $RS(U) = \{y^*(\underset{\sim}{p},c) \,|\, U_y(y^*(\underset{\sim}{p},c),\underset{\sim}{p}) = \lambda(c),\ c \geq 0,\ \underset{\sim}{p} \in R^n_+\}$.

The reference set is the set of functions $y^*(\underset{\sim}{p},c)$ such that for every given $c \geq 0$, the marginal utility of income is equal across states of nature.[3] This notion of a reference set was introduced in Karni [1981] in the more general context of state-dependent utility functions.

Utilizing definition 1, we define two indirect utility functions, $U(y,\underset{\sim}{p})$ and $V(y,\underset{\sim}{p})$ to be comparable if they have identical reference sets. Formally,

<u>Definition 2</u>: Let $U(y,\underset{\sim}{p})$ and $V(y,\underset{\sim}{p})$ be two indirect utility functions, then $U(y,\underset{\sim}{p})$ and $V(y,\underset{\sim}{p})$ are said to be <u>comparable</u> if and only if $RS(U) = RS(V)$.

III. <u>THE MAIN RESULTS</u>

In this section we state two main results. The first is that $U(y,\underset{\sim}{p})$ and $V(y,\underset{\sim}{p})$ are comparable if and only if $u(\underset{\sim}{x})$ and $v(\underset{\sim}{x})$, the corresponding direct utility functions, represent the same ordinal preferences on the commodity space. The second result is that $U(y^*(\underset{\sim}{p},c),\underset{\sim}{p})$ is a concave transformation of $V(y^*(\underset{\sim}{p},c)\underset{\sim}{p})$ (that is $U(\cdot\cdot)$ is a concave transformation of $V(\cdot\cdot)$ on the reference set) if and only if $u(\underset{\sim}{x})$ is a concave transformation of $v(\underset{\sim}{x})$ on the commodity space. The latter claim ties together well-known results from the literature on risk aversion. These are summarized in Theorem 2 below.

<u>Theorem 1</u>. Let $U(y,\underset{\sim}{p}) \equiv u(\underset{\sim}{x}^*(y,\underset{\sim}{p}))$ and $V(y,\underset{\sim}{p}) \equiv v(\underset{\sim}{x}^*(y,\underset{\sim}{p}))$ be two indirect utility functions, then $U(y,\underset{\sim}{p})$ and $V(y,\underset{\sim}{p})$ are comparable (in the sense of Definition 2) if and only if $u(\underset{\sim}{x})$ and $v(\underset{\sim}{x})$ represent the same ordinal preferences, $\succsim$, on R^n_+.

<u>Proof</u>: (a) <u>Necessity</u>. Suppose that $u(\underset{\sim}{x})$ and $v(\underset{\sim}{x})$ are two representations of $\succsim$ on R^n_+, then there exists $H[v(x)]$, $H'(\ ) > 0$ such that $u(\underset{\sim}{x}) = H[v(\underset{\sim}{x})]$. Let $y^*(\underset{\sim}{p},c)$ be the solution to (P.2) above.

But,

$$U(y,\underset{\sim}{p}) = H[V(y,\underset{\sim}{p})], \quad H' > 0.$$

Hence, $y^*(\underset{\sim}{p},c)$ is the solution to:

$$\text{(P.3)} \qquad \underset{\{y(\underset{\sim}{p})\}}{\text{Max}} \int_{R^n_+} V(y(\underset{\sim}{p}),\underset{\sim}{p})\,dF(\underset{\sim}{p})$$

$$\text{subject to} \int_{R^n_+} y(\underset{\sim}{p})\,dF(\underset{\sim}{p}) = c.$$

Thus $y^*(\underset{\sim}{p},c)$ is in both $RS(U)$ and $RS(V)$. Since this is true for all $\underset{\sim}{p} \in R^n_+$ and all $c \geq 0$, it follows that $RS(U) = RS(V)$ and U and V are comparable.

(b) <u>Sufficiency</u>. Let $U(y,\underset{\sim}{p})$ and $V(y,\underset{\sim}{p})$ be comparable and let the solution of (P.2) and (P.3) be $y^*(\underset{\sim}{p},c)$. Then there exists a transformation G, such that

$$U(y^*(\underset{\sim}{p},c),\underset{\sim}{p}) = G[V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})], \quad G' > 0 .$$

The existence of this transformation can be shown as follows. Let

$$t = V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})$$

Then,

$$\{y^*(\underset{\sim}{p},c),\underset{\sim}{p}) = V^{-1}(t) .$$

Thus, for all $c \geq 0$ and $\underset{\sim}{p} \in R^n_+$,

$$U(y^*(\underset{\sim}{p},c),\underset{\sim}{p}) = U[V^{-1}(t)] = G[V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})]$$

The monotonicity of G follows from the monotonicity of U and V in $y^*(\underset{\sim}{p},c)$.

But $\{y^*(\underset{\sim}{p},c),\underset{\sim}{p} \mid c \geq 0, \ \underset{\sim}{p} \in R^n_+\}$ covers the domain of U and V. Hence,

$$U(y,\underset{\sim}{p}) = G[V(y,\underset{\sim}{p})]$$

This implies that $u(\underset{\sim}{x})$ and $v(\underset{\sim}{x})$ represent the same ordinal preferences over R^n_+.

Q.E.D.

The following notation and definitions facilitate the statement of the second result. Let $L(\underset{\sim}{z})$ be a joint probability distribution over R^n, and $E\{\underset{\sim}{z}\} = \underset{\sim}{0}$ where E denotes the expectations operator. Following Paroush [1975] we define a vector-valued risk premium function $\underset{\sim}{\Pi}^u(\underset{\sim}{x};L)$ by the equation

(2) $$U(\underset{\sim}{x} - \underset{\sim}{\Pi}) = E\{u(\underset{\sim}{x} + \underset{\sim}{z})\}$$

where the expectations on the right-hand-side of (2) is assumed to exist. The risk premium defined above is a vector of commodities. Next we define a risk premium, $\Pi_U(y(\tilde{p}),F)$, in terms of income on the reference set by the following equation,

$$(3) \qquad \int_{R_+^n} U[y^*(\tilde{p},c-\Pi(y(\tilde{p}),F)),\tilde{p}]dF(\tilde{p}) = \int_{R_+^n} U(y(\tilde{p},c),\tilde{p})dF(\tilde{p})$$

where

$$\int_{R_+^n} [y^*(\tilde{p},c) - y(p,c)]dF(p) = 0 \ .$$

Finally, $u_i = {}^{\partial u}/\partial x_i$ and $u_{ii} = {}^{\partial^2 u}/\partial x_i^2$. Using these definitions we are in a position to state Theorem 2.

Theorem 2. Let $U(y,\tilde{p}) \equiv u(x^*(y,\tilde{p}))$ and $V(y,p) \equiv u(x^*(y,\tilde{p}))$ be comparable (in the sense of Definition 2), suppose that $V(y,\tilde{p})$ is concave in y for every $\tilde{p}$ in R_+^n, then the following conditions are equivalent,

(a) $$- \frac{u_{11}(\tilde{x})}{u_1(\tilde{x})} \geqq - \frac{v_{11}(\tilde{x})}{v_1(\tilde{x})} \quad \text{for all } \tilde{x} \text{ in } R^n.$$

(b) There exists a transformation T such that $u(\tilde{x}) = T[v(\tilde{x})]$
$T' > 0$, $T'' \leq 0$.

(c) For every $\Pi^v(\tilde{x},\tilde{L})$ there exists $\Pi^u(\tilde{x},\tilde{L})$ such that $\Pi^u(\tilde{x},\tilde{L}) \geqq \Pi^v(\tilde{x},\tilde{L})$ for all joint probability distributions $L(\tilde{z})$ on R^n and all $\tilde{x}$ in R_+^n.

(d) $$-\frac{U_{yy}(y^*(\underset{\sim}{p},c),\underset{\sim}{p})}{U_y(y^*(\underset{\sim}{p},c),\underset{\sim}{p})} \geqq -\frac{V_{yy}(y^*(\underset{\sim}{p},c),\underset{\sim}{p})}{V_y(y^*(\underset{\sim}{p},c),\underset{\sim}{p})} \quad \text{for all } \underset{\sim}{p} \text{ in } R^n_+$$

and $c \geqq 0$.

(e) For every joint probability distribution $F(\underset{\sim}{p})$ on R^n_+ there exists a transformation H_F such that

$$\int_{R^n_+} U(y^*(\underset{\sim}{p},c),\underset{\sim}{p})\,dF(\underset{\sim}{p}) = H_F\left[\int_{R^n_+} V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})\,dF(\underset{\sim}{p})\right]$$

$H_F' > 0$, $H_F'' \leqq 0$, and H_F' is independent of F.

(f) $\Pi_U(y(\underset{\sim}{p}),F) \geqq \Pi_V(y(\underset{\sim}{p}),F)$ for all $y(\underset{\sim}{p})$ and all $F(\underset{\sim}{p})$ on R^n_+.

To prove Theorem 2 we need to establish the following result:

<u>Lemma 1</u>: For any given $F(\underset{\sim}{p})$ and all $h \cdot w(\underset{\sim}{p},c)$ such that $\int_{R^n_+} w(\underset{\sim}{p},c)\,dF(\underset{\sim}{p}) = 0$, $\int_{R^n_+} H_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})]\,dF(\underset{\sim}{p}) \leqq H_F\left[\int_{R^n_+} V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})\,dF(\underset{\sim}{p})\right]$, where $H_{\underset{\sim}{p}}(\quad)$ is H_F for F such that $Pr\{\underset{\sim}{p}\} = 1$.

The proof of Lemma 1 is provided in the Appendix.

<u>Proof of Theorem 2</u>: (a) <=> (b) <=> (c) follows from Kihlstrom & Mirman (1974) and Paroush (1975).

We shall prove the equivalence of (d), (e) and (f), and then the equivalence of (b) and (e).

(d) => (e). Differentiating with respect to c, and using (1) we get:

$\lambda(c) \int_{R^n_+} \frac{\partial y^*(\underset{\sim}{p},c)}{\partial c}\,dF(\underset{\sim}{p}) = H_F'[\quad]\,\delta(c) \int_{R^n_+} \frac{\partial y^*(\underset{\sim}{p},c)}{\partial c}\,dF(\underset{\sim}{p})$ where $\delta(c)$ is the Euler-Lagrange multiplier corresponding to (P.3).

Since U and V are comparable it follows that:

$$(4) \qquad H_F'\Big[\int_{R_+^n} V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p})\Big] = \frac{\lambda(c)}{\delta(c)} > 0$$

and H_F' is independent of $F(\underset{\sim}{p})$. Differentiating $-\ell n\, H_F'$ with respect to c we get

$$-\frac{H_F''}{H_F'}\Big[\int_{R_+^n} V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p})\Big]\delta(c)\int_{R_+^n}\frac{\partial y^*(\underset{\sim}{p},c)}{\partial c} dF(\underset{\sim}{p}) = \Big[-\frac{\lambda'(c)}{\lambda(c)}\Big]-\Big[-\frac{\delta'(c)}{\delta(c)}\Big]$$

But from the concavity of $V(y,\underset{\sim}{p})$ in y it follows that $\partial y^*(p,c)/\partial c > 0$. Hence, $H_F''[\quad] \leqq 0 \iff [-\frac{\lambda'(c)}{\lambda(c)}] \geqq [-\frac{\delta'(c)}{\delta(c)}]$. From 1, $-\frac{\lambda'(c)}{\lambda(c)} = -\frac{U_{yy}(y^*(\underset{\sim}{p},c),\underset{\sim}{p})}{U_y(y^*(\underset{\sim}{p},c),\underset{\sim}{p})} \cdot \Big[\frac{\partial y^*(\underset{\sim}{p},c)}{\partial c}\Big]$

and $-\frac{\delta'(c)}{\delta(c)} = -\frac{V_{yy}(y^*(\underset{\sim}{p},c),\underset{\sim}{p})}{V_y(y^*(\underset{\sim}{p},c)\underset{\sim}{p})}\Big[\frac{\partial y^*(\underset{\sim}{p},c)}{\partial c}\Big]$. Therefore, (d) implies (e). Notice also that in the degenerate case, (e) implies $U(y(c),\underset{\sim}{p}) = H_{\underset{\sim}{p}}[V(y(c),\underset{\sim}{p})]$ for all $c \geqq 0$, and hence for all $y \geqq 0$, and all $\underset{\sim}{p} \in R_+^n$.

(e) ⇒ (f).

Let $x(\underset{\sim}{p},c) \equiv y^*(\underset{\sim}{p},c) - y(\underset{\sim}{p},c)$, then from (3) we have

$$\int_{R_+^n} U(y^*(\underset{\sim}{p},c-\Pi_U),\underset{\sim}{p})\, dF(\underset{\sim}{p}) = \int_{R_+^n} U(y^*(\underset{\sim}{p},c) - x(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p}) = \int_{R_+^n} H_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c)-x(\underset{\sim}{p},c),\underset{\sim}{p})]\, dF(\underset{\sim}{p})$$

$$\leqq H_F\Big[\int_{R_+^n} V(y^*(\underset{\sim}{p},c) - x(\underset{\sim}{p},c),\ \underset{\sim}{p})\, dF(\underset{\sim}{p})\Big] = H_F\Big[\int_{R_+^n} V(y^*(\underset{\sim}{p},c-\Pi_V),\underset{\sim}{p})\, dF(\underset{\sim}{p})\Big]$$

$$= \int_{R_+^n} U(y^*(\underset{\sim}{p},c-\Pi_V),\underset{\sim}{p})\, dF(\underset{\sim}{p}).$$

where use has been made of the definition of risk premium and Lemma 1.

Since U is monotonic increasing in c, it follows that

$$\Pi_U(y(\underset{\sim}{p}),F) \geqq \Pi_V(y(\underset{\sim}{p}),F) \ .$$

(f) => (d).

Let $x(\underset{\sim}{p},c) \equiv h.w(\underset{\sim}{p},c)$. Differentiating Π_V with respect to h we obtain:

$$\frac{d\Pi_U}{dh} = \frac{\int_{R^n_+}[U_y(y^*(\underset{\sim}{p},c)-hw(\underset{\sim}{p},c),\underset{\sim}{p})w(\underset{\sim}{p},c)dF(\underset{\sim}{p})}{\lambda(c-\Pi_U)\int_{R^n_+}\frac{\partial y^*(\underset{\sim}{p},c-\Pi_U)}{\partial c}dF(\underset{\sim}{p})}$$

Since $\int_{R^n_+} w(\underset{\sim}{p},c)dF(\underset{\sim}{p}) = 0$, and U_y is constant on RS(U),

$$\left.\frac{d\Pi_U}{dh}\right|_{h=0} = 0.$$

The same result obtains for V. Hence, (f) implies that

$$\left.\frac{d^2\Pi_U}{dh^2}\right|_{h=0} \geqq \left.\frac{d^2\Pi_V}{dh^2}\right|_{h=0} \ .$$

But

$$\left.\frac{d^2\Pi_U}{dh^2}\right|_{h=0} = \frac{-\int_{R^n_+}U_{yy}(y^*(\underset{\sim}{p},c),\underset{\sim}{p})[w(\underset{\sim}{p},c)]^2dF(\underset{\sim}{p})}{\lambda(c)\int_{R^n_+}\frac{\partial y^*(\underset{\sim}{p},c)}{\partial c}\,dF(\underset{\sim}{p})}$$

Since $\lambda(c) = U_y(y^*(\underset{\sim}{p},c),\underset{\sim}{p})$ it follows that (f) implies

(5) $$\int_{R^n_+}\left[-\frac{U_{yy}}{U_y}(y^*(\tilde{p},c),\tilde{p}) + \frac{V_{yy}}{V_y}(y^*(\tilde{p},c)\tilde{p})\right][w(\tilde{p},c)]^2 dF(\tilde{p}) \geqq 0$$

for all $w(\tilde{p},c)$ such that $\int_{R^n_+} w(\tilde{p},c)\,dF(\tilde{p}) = 0.$

Thus, $-\frac{U_{yy}}{U_y}(y^*(\tilde{p},c)\tilde{p}) \geqq -\frac{V_{yy}}{V_y}(y^*(\tilde{p},c)\tilde{p})$, otherwise we can choose $w(\tilde{p},c)$ so as to reverse the inequality (5).

To complete the proof it remains to be shown that (b) <=> (e). From (4) we have $H_F' = \frac{\lambda(c)}{\delta(c)}$ and the equivalence of (d) and (e) means that $H_F'' \leqq 0$ if and only if $-\frac{\lambda'(c)}{\lambda(c)} \geqq -\frac{\delta'(c)}{\delta(c)}$. But

$$\lambda(c) = \sum_{i=1}^{n} u_i[\tilde{x}^*(y^*(\tilde{p},c)\tilde{p})]\frac{\partial x_i^*}{\partial y^*} =$$

$$\sum_{i=1}^{n} T'[v(x^*(y^*(\tilde{p},c),\tilde{p})]v_i(x^*(y^*(\tilde{p},c)\tilde{p})\frac{\partial x_i^*}{\partial y^*} = T'[v(x^*)]\delta(c).$$

Hence, $H_F' = T'$ for all $F(\tilde{p})$ on R^n_+. Furthermore, differentiating $-\ell n\, T'[v(x^*(y^*(\tilde{p},c),\tilde{p})]$ with respect to c we obtain

$$-\frac{T''}{T'}[v(x^*)]\cdot\delta(c)\frac{\partial y^*(\tilde{p},c)}{\partial c} = -\frac{\lambda'(c)}{\lambda(c)} + \frac{\delta'(c)}{\delta(c)} .$$

Consequently, $T''[v(x^*0] \leqq 0$ if and only if $-\frac{\lambda'(c)}{\lambda(c)} \geqq -\frac{\delta'(c)}{\delta(c)}$. Thus $T'' \leqq 0$ if and only if $H_F'' \leqq 0$, and (b) <=> (e).

Q.E.D.

APPENDIX

Lemma 1 For any given $F(\underset{\sim}{p})$ and all $h.w(\underset{\sim}{p},c)$ such that

$$\int_{R_+^n} w(\underset{\sim}{p},c)\,dF(\underset{\sim}{p}) = 0, \tag{A.1}$$

$$\int_{R_+^n} H_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c)-hw(\underset{\sim}{p},c),\underset{\sim}{p})]\,dF(\underset{\sim}{p}) \leq H_F[\int_{R_+^n} V(y^*(\underset{\sim}{p},c)-hw(\underset{\sim}{p},c),\underset{\sim}{p})\,dF(\underset{\sim}{p})]$$

Proof: For a given $w(\underset{\sim}{p},c)$ such that $\int_{R_+^n} w(\underset{\sim}{p},c)\,dF(\underset{\sim}{p}) = 0$ define the following two functions:

$$I(h) = \int_{R_+^n} H_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})]\,dF(\underset{\sim}{p}) \tag{A.2}$$

$$J(h) = H_F[\int_{R_+^n} V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})\,dF(\underset{\sim}{p})] \tag{A.3}$$

Clearly $I(0) = J(0)$. Furthermore, since V is concave in y,

$$V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p}) - V(y^*(\underset{\sim}{p},c),\underset{\sim}{p}) \leq \tag{A.4}$$

$$- hV'(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})w(\underset{\sim}{p},c) \equiv -h.\beta(\underset{\sim}{p},c)$$

Similarly,

$$V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p}) - V(y^*(\underset{\sim}{p},c),\underset{\sim}{p}) \geq$$

$$- hV^{-1}(y^*(\underset{\sim}{p},c),\underset{\sim}{p})w(\underset{\sim}{p},c) \equiv -h.\alpha(\underset{\sim}{p},c) \tag{A.5}$$

and

$$\beta(\underset{\sim}{p},c) = V'(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})w(\underset{\sim}{p},c) = V'(y^*(\underset{\sim}{p},c),\underset{\sim}{p})w(\underset{\sim}{p},c) +$$

$$- V''(y^*(\underset{\sim}{p},c) - \theta w(\underset{\sim}{p},c),\underset{\sim}{p})[w(\underset{\sim}{p},c)]^2 \geqq V'(y^*(\underset{\sim}{p},c),\underset{\sim}{p})w(\underset{\sim}{p},c) = \alpha(\underset{\sim}{p},c), \quad 0 \leq \theta \leq h.$$

Hence, $\beta(\underset{\sim}{p},c) < 0$ implies $\alpha(\underset{\sim}{p},c) < 0$, and by (A.5) and concavity of $H'_{\underset{\sim}{p}}$ it also implies:

$$H'_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})] < H'_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})] \qquad (A.6)$$

Similarly, using (A.4), $\beta(\underset{\sim}{p},c) \geqq 0$ implies:

$$H'_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})] \geqq H'_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})] \quad . \qquad (A.7)$$

Differentiating $I(h)$ with respect to h we obtain,

$$I'(h) = - \int_{R^n_+} H'_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})]V'(y^*(\underset{\sim}{p},c)-hw(\underset{\sim}{p},c),\underset{\sim}{p})\ w(\underset{\sim}{p},c)dF(\underset{\sim}{p})$$

$$= - \int_{R^n_+} H'_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})\beta(\underset{\sim}{p},c)dF(\underset{\sim}{p})$$

$$\leqq - \int_{R^n_+} H'_{\underset{\sim}{p}}[V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})]\beta(\underset{\sim}{p},c)dF(\underset{\sim}{p}) \ ,$$

where the last inequality uses (A.6) and (A.7). Since $y^*(\underset{\sim}{p},c)$ is in $RS(V)$,

$$I'(h) \leqq - \int_{R^n_+} H'_F[\int_{R^n_+} V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p})]\,\beta(\underset{\sim}{p},c)\, dF(\underset{\sim}{p})$$

$$\leqq - \int_{R^n_+} H'_F[\int_{R^n_+} V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p})]\,\beta(\underset{\sim}{p},c)\, dF(\underset{\sim}{p}) \;.$$

The last inequality makes use of the fact that by definition of RS (V)

$$\int_{R^n_+} V(y^*(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p}) \geqq \int_{R^n_+} V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p}) \;.$$

Now differentiating $J'(h)$ with respect to h, we get

$$J'(h) = -H'_F[\int_{R^n_+} V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p})] \cdot \int_{R^n_+} V'(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})\, w(\underset{\sim}{p},c)\, dF(\underset{\sim}{p})$$

$$= - H'_F[\int_{R^n_+} V(y^*(\underset{\sim}{p},c) - hw(\underset{\sim}{p},c),\underset{\sim}{p})\, dF(\underset{\sim}{p})] \int_{R^n_+} \beta(\underset{\sim}{p},c)\, dF(\underset{\sim}{p})$$

Hence, $J'(h) \geqq I'(h)$ for all $h > 0$. Therefore $J(h) \geqq I(h)$ for all h.

Q.E.D.

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F O O T N O T E S

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1. Stronger measures of risk aversion for univariate state-independent utility functions were developed in a recent paper by Ross [1979]. These measures resolved some difficulties in the theories of portfolio selection and optimal insurance.

2. The references are to studies that emphasize the measurement of attitudes towards risk. Numerous authors have dealt with other aspects of the theory of risk bearing with state-dependent utility functions and utility functions with many commodities. See, for example Arrow [1974], Cook and Graham [1977], Paroush [1975].

3. Notice that in this problem the reference set is independent of $F(\underset{\sim}{p})$.