AN EXPECTED UTILITY THEORY FOR STATE-DEPENDENT PREFERENCES

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0. ABSTRACT

The theory presented here belongs to a class of models following Savage's expected utility theory, where both the subjective probabilities over states of nature and the utility of consequences are derived jointly from preferences over acts. In the present paper, subjective probability and utility are determined without restrictions on the relations among the preference orderings for distinct states of nature. The key element in the model is the decision maker's ability to adjust his preferences over acts when confronted with new information regarding the likely realization of states of nature. The prior and posterior preferences are linked by a consistency axiom.

1. INTRODUCTION

The standard problem of decision-making under uncertainty involves three basic elements: acts, states of nature and consequences. Each combination of an act and a state of nature determines a unique consequence. Uncertainty, captured by the notion of states of nature results from the lack of advanced knowledge of the exact consequence that follows from a given act. The decision-maker's purpose is to choose an act which results in the most desirable consequences. This requires a criterion for judging the desirability of acts. The expected utility is one such criterion.

The expected utility theory assumes that decision-makers have preferences over acts and postulates that these preferences have a structure that permits their representation via an expected utility index. The use of expectation operation presumes the existence of a utility index for each consequence and a subjective probability distribution over states of nature. This presumption is made plausible by Savage's theory, which builds upon the works of Ramsey, De Finetti, von Neumann-Morgenstern, and others. In what follows, we use a simplified version of Savage's model, in which a consequence is a lottery with objectively known probabilities (extraneous lottery) over a set of prizes.

There are circumstances, however, in which the evaluation of the prizes is not independent of the prevailing state of nature. These include a class of insurance problems involving irreplaceable objects such as life, health and heirlooms. Another example is criminal activity where one possible outcome is loss of freedom. The expected utility theory can be extended to include state-dependent preferences by assigning a utility index to each prize-state of nature pair. The utility of two such pairs may differ, even if the prize is the same in both.

Some implications of expected utility maximizing behavior with state dependent preferences for optimal insurance have been studied by Arrow [1974], Cook and Graham [1977] and Karni [1980]. Recognizing that the then existing expected utility theory did not apply to state dependent preferences, Arrow [1974, p.61]was careful to assume the existence of objective probabilities over states of nature. Furthermore, he claimed that with state dependent preferences it is impossible to separate tastes (as represented by a utility function) and beliefs (as represented by probabilities) in a unique way with the use of observations. This claim depends, of course, on the extent to which behavior can be reasonably restricted.

Fishburn [1973] suggests an axiomatization of preferences over conditional acts which can be represented by a unique subjective probability and state dependent utilities. For each event (non-empty subsets of the set of states of nature) he defines the set of acts conditioned on events and assumes the existence of a preference relation over all conditional acts. In addition to the standard Savage axioms (see Fishburn [1970]) Fishburn introduces a major structural restriction. This restriction requires that for every two disjoint events, not all of the consequences conditioned upon one event are preferred to all of the consequences conditioned upon the other event. This is irreconcilable with some applications (e.g. life insurance problems) that motivated our research (see Fishburn's own criticism [1974]).

In this paper we propose an alternative axiomatization of expected utility theory for state-dependent preferences. To facilitate the exposition we introduce the following notation:

Let the set of states of nature be denoted by S and the set of prizes by X. Each prize need not be available under all states of nature. Hence we introduce the notation $X(s)$ for the set of prizes available if state of nature s in S prevails. We assume that for all s in S, $X(s)$ is non-empty. Clearly, $X = \cup_{s \varepsilon S} X(s)$. We denote by Y the set of all available prize-state pairs, $Y = \{(x,s) \varepsilon X{\times}S \mid x \varepsilon X(s).\}$

An act assigns an extraneous lottery to each state of nature. In other words, in choosing an act, the decision-maker chooses a list of lottery tickets, one for each state of nature. The prizes of the lottery corresponding to each state s in S belong to $X(s)$. Formally, an act, say f, is a mapping from Y to the closed unit interval $[0,1]$, such that for all s in S, $\sum_{x \varepsilon X(s)} f(x,s) = 1$. Thus $f(x,s)$ is the extraneous probability of obtaining the prize x if state s prevails. The decision-maker is assumed to possess a preference relation over all acts which satisfies the standard von Neumann-Morgenstern axioms of weak-order, independence, and continuity.

Next consider the notion of prize-state lotteries, namely extraneous lotteries over the set , Y, of prize-state pairs. Our decision-maker is assumed to have preference relation over the set of prize-state lotteries in addition to his preference relation over acts. Whereas the existence of a preference relation over acts is an inherent primitive of the problem which requires no further elaboration, the preference relation over the prize-state lotteries does require explanation.

The role of the prize-state lotteries in our model is best explained within the neo-Bayesian framework. To begin with, suppose, as we shall indeed prove later, that the decision-maker is an expected utility maximizer and that his preferences over acts are induced by state-dependent utilities $u(x,s)$ and subjective probabilities $p(s)$ over S. Given an act f and the subjective probability p on S, define for each prize-state pair, (x,s), in Y, $\hat{f}(x,s) = p(s)\ f(x,s)$. Clearly $\sum_{(x,s) \varepsilon Y} \hat{f}(x,s) = 1$, thus $\hat{f}$ is a prize-state lottery. Next suppose that the decision-maker can experiment or otherwise obtain information pertinent to the eventual realization of the state of nature.

He will then obtain a new (posterior) subjective probability $q(s)$ over S. The new subjective probability induces a new preference order on the set of acts. Furthermore, for each act f, the product of q and f as defined previously, yields a new prize-state lottery. Applying this procedure to all conceivable posteriors and all acts, we obtain the set of prize-state lotteries.

We return now to the question of preference relation over prize-state lotteries. Consider a decision-maker who must choose between watching a football game in an open stadium and watching it at home on television. As he faces uncertainty with regard to the possibility of rain, we are clearly in the state-dependent preferences framework. Suppose that the decision-maker learns from a weather forecast that there is say, a fifteen percent chance of rain. This new information transforms the acts into two prize-state lotteries: "fifteen percent chance of watching the game in the stadium while it rains, eighty five percent chance of watching it in the stadium in nice weather" and "fifteen percent chance of watching the game on television while it rains outside, eighty-five percent chance of watching the game on television while the sun is shining". The assumption that the decision-maker has preferences over prize-state lotteries means that he can express preferences between these two lotteries prior to obtaining the weather forecast. For example, it is conceivable that our football fan may decide to watch the game on television unless the forecast is for less than a ten percent chance of rain. In other words, the requirement of comparability of any prize-state lotteries obtained from two acts under the same (posterior) probability is quite plausible.

The critical aspect of the preference relation over prize-state lotteries is the comparability of two lotteries based on different, hence conflicting, (posterior) probabilities over S. In particular this implies the comparability of degenerate lotteries, namely prize-state pairs for distinct states of nature. For instance, the decision-maker is supposed to rank the alternatives of watching the game in the stadium when it rains and watching the game at home when the sun is shining. The difficulty in accepting this supposition stems from the fact that it is hard to imagine an actual situation involving this kind of choice. Still it seems to us that such comparisons are conceivable. This institution is supported by the presence of moral hazard in many risk-sharing situations. Moral hazard describes the phenomenon where, by omission or by commission, the decision-maker changes the probability distribution over S, thereby revealing his preference for a given act under one probability distribution over the same act under a different probability distribution. More generally, situations in which the decision-maker, through his choice of acts, reveals his preference between lotteries that assign different probabilities to the same state of nature are quite common. The purchase of life insurance is an act whose consequences depend on the insured person's lifetime. When the insured person cancels a flight and instead travels by land because of bad weather, he changes the probability distribution of his lifetime. By doing so he reveals his preferences between two prize-state lotteries that involve different probabilities over S.

In this paper we are not concerned with the issue of moral hazard or with the decision-maker's acts involving changes in the probability distributions over S. For a discussion of these questions see Dreze [1961]. The above examples, however, illustrate that the assumption that prize-state lotteries are comparable is not far fetched. Obviously there are instances where it is difficult to conceive of an experiment that would reveal the decision-maker's preferences among certain pairs of prize-state lotteries. The same can be said about comparison between acts. Savage's constant acts were criticized on these grounds. Note also that Fishburn [1973] assumes that any two conditional acts are comparable. A prize-state pair is a special case of Fishburn's conditional acts.

The preference relation over prize-state lotteries, $\hat{\succsim}$ is also assumed to satisfy the aforementioned three axioms of von Neumann-Morgenstern utility theory. This assumption implies the existence of a state-dependent utility function u such that for any two prize-state lotteries, $\hat{f}$ and $\hat{g}$,

$$\text{(i)} \quad \hat{f} \mathrel{\hat{\succsim}} \hat{g} \quad \text{iff} \quad \textstyle\sum_{s\varepsilon S}\sum_{x\varepsilon X(s)} u(x,s)[\hat{f}(x,s) - \hat{g}(x,s)] \geqq 0$$

Similarly the assumptions on the preference relation, $\succsim$, over acts implies the existence of an evaluation function w such that for any two acts, f and g:

$$\text{(ii)} \quad f \succsim g \quad \text{iff} \quad \textstyle\sum_{s\varepsilon S}\sum_{s\varepsilon X(s)} w(x,s)[f(x,s) - g(x,s)] \geqq 0$$

We refer to w as an evaluation function rather than a utility function, since implicit in w is the decision-maker's subjective probability on S. By this we mean that w(x,s) in (ii) can be replaced by the product p(s)v(x,s) where p(s) is the subjective probability of s and v(x,s) is a state-dependent utility function. However, under the assumptions on the preference relation $\succsim_p$ is not unique. Furthermore, the introduction of the preference relation $\hat{\succsim}$ over prize-state lotteries does not in itself solve the problem unless the two preference relations $\succsim$ and $\hat{\succsim}$ are tied together through a consistency axiom. The nature of this axiom is taken up next.

As we have said already every prize-state lottery, $\hat{f}$, is the product of an act, f, and a probability distribution over S. Consider two prize-state lotteries $\hat{f}$ and $\hat{g}$ and two acts f and g used in the construction of these lotteries respectively with the same $p(\cdot)$. Assume further that f and g are identical except for a unique state of nature, say s. This implies in turn that $\hat{f}$ and $\hat{g}$ also coincide over all states of nature except s. Loosely speaking, the consistency axiom states that $f \succ g$ if and only if $\hat{f} \hat{\succ} \hat{g}$. To be more precise this implications must be qualified in two ways. First, $\hat{f}$ and $\hat{g}$ are restricted to be non-degenerate in the sense that $p(\cdot)$ is positive for each state in S. Second, and more importantly, the state s is non-null. The meaning of a null state in the present context is somewhat different from the usual usage of this term, and thus requires some elaboration.

In the state-independent framework a state is null if and only if the decision-maker is indifferent between any two acts whose consequences differ only in this state. In the state-dependent context, however, the decision-maker may be indifferent between any two acts whose consequences differ in one particular state for two different reasons: One is that his implicit subjective probability for this state is zero. The other is that he is indifferent between any two prizes when this state prevails. We refer to a state as null if the first reason holds and the second does not. Further discussion of the null state, including an example, is deferred to the last section.

In Section 2 we state in detail the von Neumann-Morgenstern expected utility theory as applied to the preference relation over acts and over prize-state lotteries. Our main result is stated and proved in Section 3.

2. NOTATIONS, DEFINITIONS AND SOME KNOWN RESULTS.

Let S be a finite nonempty set whose elements are referred to as states of nature. For each state s in S let $X(s)$ be a finite non-empty set of prizes available under s and denote $X = \cup_{s \varepsilon S} X(s)$. The set of prize-state of nature pairs is $Y = \{(x,s) \varepsilon X \times S \mid x \varepsilon X(s)\}$. Denoting the non-negative numbers by R_+ we define the set of acts and the set of prize-state lotteries respectively as follows:

$$L = \{f \varepsilon R_+^Y \mid \text{for all } s \text{ in } S \ \textstyle\sum_{x \varepsilon X(s)} f(x,s) = 1\}$$

$$\hat{L} = \{\hat{f} \varepsilon R_+^Y \mid \textstyle\sum_{(x,s)\varepsilon Y} \hat{f}(x,s) = 1\}$$

Corresponding to the two sets of lotteries L and $\hat{L}$ are two binary relations denoted $\gtrsim$ and $\hat{\gtrsim}$ respectively. We deduce the relation $\succ$, $\sim$ by the following implications. For all f, g in L: $f \succ g$ if and only if $f \gtrsim g$ and not $g \gtrsim f$, and $f \sim g$ if and only if $f \gtrsim g$ and $g \gtrsim f$. The relations, $\hat{\succ}$ $\hat{\sim}$ are defined analogously.

The binary relation $\gtrsim$ on L is assumed to satisfy the classical von Neumann-Morgenstern axioms:

(A.1) (Completeness and transitivity) (a) for all f and g in L: $f \gtrsim g$ or $g \gtrsim f$. (b) For all f, g and h in L; if $f \gtrsim g$ and $g \gtrsim h$ then $f \gtrsim h$.

(A.2) (Independence) For all f, g and h in L, and for all $a \varepsilon R_+$, $0 < a < 1$: if $f \succ g$ then $af + (1-a)h \succ ag + (1-a)h$.

(A.3) <u>(Continuity)</u> For f, g and h in L: if $f \succ g$ and $g \succ h$ then there exist a, b in $(0,1)$ such that $af + (1-a)h \succ g$ and $g \succ bf + (1-b)h$.

<u>THEOREM 1:</u> Suppose that the binary relation $\succsim$ on L satisfies (A.1), (A.2) and (A.3). Then:

(a) There exists a function $w \in R^Y$ such that for all f and g in L, $f \succsim g$ if and only if $\sum_{s\in S}\sum_{x\in X(s)} w(x,s)f(x,s) \geq \sum_{s\in S}\sum_{x\in X(s)} w(x,s)g(x,s)$.

(b) For $v \in R^Y$ the relation: "For all f and g in L, $f \succsim g$ if and only if $\sum_{s\in S}\sum_{x\in X(s)} v(x,s)[f(x,s) - g(x,s)] \geq 0$" holds if and only if there is $c > 0$ and $d \in R^S$ such that for all $x \in X(s)$ and s in S, $v(x,s) = cw(x,s) + d(s)$.

This Theorem is (Theorem (13.1) in Fishburn [1970],) is a minor extension of the original von Neumann-Morgenstern result.and makes use of the fact that L is a mixture set. (For definition and references see Fishburn [1970]).

The binary relation $\hat{\succsim}$ is also assumed to satisfy the von Neumann-Morgenstern utility theory assumptions.

(A.4) <u>(Completeness and Transitivity)</u> (a) for all $\hat{f}$ and $\hat{g}$ in $\hat{L}$, $\hat{f} \hat{\succsim} \hat{g}$ or $\hat{g} \hat{\succsim} \hat{f}$. (b) For all $\hat{f}$, $\hat{g}$ and $\hat{h}$ in $\hat{L}$, if $\hat{f} \hat{\succsim} \hat{g}$ and $\hat{g} \hat{\succsim} \hat{h}$ then $\hat{f} \hat{\succsim} \hat{h}$.

(A.5) <u>(Independence)</u> For all $\hat{f}$, $\hat{g}$ and $\hat{h}$ in $\hat{L}$, and for all $a \in R_+$, $0 < a \leq 1$: If $\hat{f} \hat{\succ} \hat{g}$ then $a\hat{f} + (1-a)\hat{h} \hat{\succ} a\hat{g} + (1-a)\hat{h}$.

(A.6) <u>(Continuity)</u> For $\hat{f}$, $\hat{g}$ and $\hat{h}$ in $\hat{L}$, if $\hat{f} \hat{\succ} \hat{g}$ and $\hat{g} \hat{\succ} \hat{h}$ then there exists a and b in $(0,1)$ such that $a\hat{f} + (1-a)\hat{h} \hat{\succ} \hat{g}$ and $\hat{g} \hat{\succ} b\hat{f} + (1-b)\hat{h}$.

THEOREM 2: The relation $\hat{\succsim}$ on $\hat{L}$ satisfies axioms (A.4), (A.5) and (A.6) if and only if there exists a function $u \in R^Y$ such that for all $\hat{f}$ and $\hat{g}$ in $\hat{L}$: $\hat{f} \hat{\succsim} \hat{g}$ if and only if $\sum_{s \in S} \sum_{x \in X(s)} u(x,s)[\hat{f}(x,s) - \hat{g}(x,s)] \geq 0$. Moreover for $v \in R^Y$ the relation "for all $\hat{f}$ and $\hat{g}$ in $\hat{L}$, $\hat{f} \hat{\succsim} \hat{g}$ if and only if $\sum_{s \in S} \sum_{x \in X(s)} v(x,s)[\hat{f}(x,s) - \hat{g}(x,s)] \geq 0$" holds if and only if there is $c > 0$ and $d \in R$ such that for all x in X and s in S $v(x,s) = cu(x,s) + d$.

This version of the von Neumann-Morgenstern theorem appears in Fishburn [1970] as theorem (8.2).

3. THE MAIN RESULTS

We now formulate a consistency condition between the two binary relations, $\succsim$ and $\hat{\succsim}$. Given $\hat{f}$ in $\hat{L}$, $\hat{f}$ is said to be non-degenerate if and only if for every s in S $\sum_{x \in X(s)} f(x,s) > 0$.

Denote by H a function from nontrivial lotteries in $\hat{L}$ to L such that $H(\hat{f}(x,s)) = \hat{f}(x,s)/\sum_{y \in X(s)} \hat{f}(y,s)$ (for all (x,s) in Y and nontrivial $\hat{f}$ in $\hat{L}$). Given f and g in L and s in S, we use the self-explanatory term f equals g outside s if for all $t \neq s$ and for all x in $X(t), f(x,t) = g(x,t)$. Likewise for $\hat{f}$ and $\hat{g}$ in $\hat{L}$.

Finally a state of nature s is said to be null if: (i) For all f and g in L such that f equals g outside s, $f \sim g$, and (ii) There exist $\hat{f}$ and $\hat{g}$ in $\hat{L}$ such that $\hat{f}$ equals $\hat{g}$ outside s and $\hat{f} \hat{\succ} \hat{g}$.

(A.7) (Consistency) For all s in S and for all non-degenerate $\hat{f}$ and $\hat{g}$ in $\hat{L}$, if $\hat{f}$ equals $\hat{g}$ outside s and $H(\hat{f}) \succ H(\hat{g})$ then $\hat{f} \hat{\succ} \hat{g}$. Moreover, if s is non-null, then for all non-degenerate $\hat{f}$ and $\hat{g}$ in $\hat{L}$ with $\hat{f}$ equals $\hat{g}$ outside s: $\hat{f} \hat{\succ} \hat{g}$ implies $H(\hat{f}) \succ H(\hat{g})$.

(A.8) (Nontriviality of $\succsim$) There exist f^* and g^* in L such that $f^* \succ g^*$.

THEOREM 3: Suppose that binary relations $\succsim$ on L and $\hat{\succsim}$ on $\hat{L}$ satisfy (A.1) through (A.8).

(a) There exists u in R^Y and a (subjective) probability p on S (i.e. $p \in R_+^S$ and $\sum_{s\in S} p(s) = 1$) such that for all f and g in L:

(1) $f \succsim g$ iff $\sum_{s\in S}\sum_{x\in X(s)} p(s)u(x,s)[f(x,s) - g(x,s)] \geqq 0$,

and for all $\hat{f}$ and $\hat{g}$ in $\hat{L}$:

(2) $\hat{f} \hat{\succsim} \hat{g}$ iff $\sum_{s\in S}\sum_{x\in X(s)} u(x,s)[\hat{f}(x,s) - \hat{g}(x,s)] \geqq 0$

(b) The u of part (a) is unique up to a positive linear transformation as in and by Theorem 2.

(c) For s null, $p(s) = 0$, and if there exist f_s and g_s in L such that f_s equals g_s outside a non-null state s and $f_s \succ g_s$, then $p(s) > 0$. Moreover, if for each state s there exist $\hat{f}_s$ and $\hat{g}_s$ in $\hat{L}$ such that $\hat{f}_s$ equals $\hat{g}_s$ outside s and $\hat{f}_s \hat{\succ} \hat{g}_s$, then the probability p of part a is unique.

Before proving the theorem we introduce two Lemmas. The first Lemma, which relates to Theorem 1, is included as part of Fishburn [1970] theorem (13.1). However, since we define null state differently from Fishburn we state it separately.

Lemma 1: Suppose that the relations $\succsim$ on L and $\hat{\succsim}$ on $\hat{L}$ exist and $\succsim$ satisfies (A.1), (A.2), (A.3), and let w denote an evaluation function from Theorem 1. If s is a null state then for all x and y in $X(s)$, $w(x,s) = w(y,s)$.

Proof: Assume, by way of negation, that for some x and y in $X(s)$, $w(x,s) > w(y,s)$. Define two functions f and g in L such that f equals g outside s and $f(x,s) = g(y,s) = 1$. Then $\sum_{t\in S}\sum_{z\in X(s)} w(z,t)[f(z,t) - g(z,t)] > 0$ which in turn implies $f \succ g$ - a contradiction to nullity of s. Q.E.D.

Lemma 2: Suppose that the relations $\succsim$ on L and $\hat{\succsim}$ on $\hat{L}$ exist and $\succsim$ satisfies (A.1), (A.2), (A.3) and (A.8), then there exist non-null s in S.

Proof: Again, by way of negation, suppose that every s in S is null. Then, by Lemma 1, for all s in S and all x and y in $X(s)$, $w(x,s) = w(y,s)$. Hence, for any f and g in L; $\sum_{s \in S} \sum_{x \in X(s)} w(x,s)[f(x,s) - g(x,s)] = 0$ i.e. $f \sim g$. A contradiction to (A.8). Q.E.D.

Proof of the Theorem: (a) Let w and u be the functions obtained from Theorems 1 and 2 respectively. Choose an arbitrary nontrivial $\hat{f}$ in $\hat{L}$ and let $f = H(\hat{f})$, f in L. For any non-null s in S define the sets

$$L_s = \{g \in L \mid g \text{ equals } f \text{ outside } s\}$$

and

$$\hat{L}_s = \{\hat{g} \in \hat{L} \mid \hat{g} \text{ equals } \hat{f} \text{ outside } s\}$$

Note that the mapping H restricted to $\hat{L}_s$ is one-to-one, and we can identify L_s with $\hat{L}_s$. Since s is non-null, the relations $\succsim$ and $\hat{\succsim}$ restricted to L_s or $\hat{L}_s$ are identical. For the restricted relation $\succsim$, however, the function $w(\cdot,s)$ constitutes a von Neumann-Morgenstern utility function on $X(s)$. Similarly, for the restricted relation $\hat{\succsim}$ the function $u(\cdot,s)$ is also a von Neumann-Morgenstern utility function on $X(s)$. Hence $w(\cdot,s)$ is a positive linear transformation of $u(\cdot,s)$, i.e. there is $c(s) > 0$ and $d(s)$ such that for all x in X, $w(x,s) = c(s)u(x,s) + d(s)$. Invoking Theorem 1(b), we can rescale w so that for each such s, we subtract from $w(\cdot,s)$ the constant $d(s)$. To avoid complicating the notations we also denote the rescaled evaluation function by w. Therefore, for each non-null s and each x in $X(s)$ we have $w(x,s) = c(s)u(x,s)$.

Using Lemma 1 and Theorem 1(b) we assume, without loss of generality, that for all x in $X(s)$ and for all null s, $w(x,s) = 0$. Defining for each null s in S $c(s) = 0$, we have the equality $w(x,s) = c(s)u(x,s)$ for all (x,s) in R^Y.

By Theorem 1(a) we have for all g and h in L: $g \succsim h$ if and only if

$$\sum_{s\varepsilon S} \sum_{x\varepsilon X(s)} w(x,s)[g(x,s) - h(x,s)] \geq 0.$$

which in turn is equivalent to; $g \succsim h$ if and only if

$$\sum_{s\varepsilon S} \sum_{x\varepsilon X(s)} c(s)u(x,s)[g(x,s) - h(x,s)] \geq 0.$$

By Lemma 2 there exists a non-null s in S, so $\sum_{s\varepsilon S} c(s) > 0$, and we can define, for all s in S, $p(s) = c(s)/\sum_{t\varepsilon S} c(t)$. In this case, the earlier inequality may be rewritten as $g \succsim h$ iff

$$\sum_{s\varepsilon S} \sum_{x\varepsilon X(s)} p(s)u(x,s)[g(x,s) - h(x,s)] \geq 0.$$

p is the desired (subjective) probability distribution on S, and $p(s) = 0$ for null s.

(c) Let s be a state for which the relation $\hat{\succ}$ restricted to s is non-empty, i.e., there are $\hat{f}_s$ and $\hat{g}_s$ in $\hat{L}$ such that $\hat{f}_s$ equals $\hat{g}_s$ outside s and $\hat{f}_s \hat{\succ} \hat{g}_s$. Let p and u satisfy (1) and (2) of part (a), then $u(\cdot,s)$ is not constant on $X(s)$. Thus there exist $\overline{x}_s$ and $\underline{x}_s$ in $X(s)$ such that

$$u(\overline{x}_s,s) > u(\underline{x}_s,s).$$

If s is null then, by definition $\hat{\succ}$ restricted to s is nonempty. Define g and h in L_s by $g(\overline{x}_s,s) = h(\underline{x}_s,s) = 1$. Then $p(s) > 0$ implies $p(s)[u(\overline{x}_s,s) - u(\underline{x}_s,s)] > 0$, which in turn implies $g \succ h$, a contradiction of the definition of nullity of s. Hence for s null $p(s) = 0$.

If s is non-null, and f_s and g_s are as in part (c) above, then by (1) of part (a): $\sum_{x\varepsilon X(s)} p(s)u(x,s)[f_s(x,s) - g_s(x,s)] > 0$. Thus, $p(s) > 0$.

Next we shall prove that if for all s the relation $\hat{\succ}$ is nonempty, then the probability p of part (a) is unique. By way of negation, suppose that there exists a probability distribution over S, p' not equal to p. Using part (b), we may now write condition (1) of part (a) as:

$$g \succsim h \text{ iff } \sum_{s\varepsilon S}\sum_{x\varepsilon X(s)} p'(s)u(x,s)[g(x,s) - h(x,s)] \geq 0$$

iff $\sum_{s\varepsilon S} \sum_{x\varepsilon X(s)} p(s)[g(x,s) - h(x,s)] \geq 0$. We define lotteries in L such that the above inequalities contradict the assumption that for some t and s in S, $p(s) > p'(s)$ and $p(t) < p'(t)$.

For r in $[0,1]$ define:

$$g_r(\overline{x}_s,s) = h_r(\underline{x}_t,t) = r$$

$$g_r(\underline{x}_s,s) = h_r(\overline{x}_t,t) = 1-r$$

$$g_r(\underline{x}_t,t) = h_r(\underline{x}_s,s) = 1$$

and outside s and t, g_r and h_r coincide.

Using condition (1) of part (a) as above

$$rp(s)[u(\overline{x}_s,s) - u(\underline{x}_s,s)] + (1-r)p(t)[u(\underline{x}_t,t) - u(\overline{x}_t,t)] \geqq 0$$

if and only if

$$rp'(s)[u(\overline{x}_s,s) - u(\underline{x}_s,s)] + (1-r)p'(t)[u(\underline{x}_t,t) - u(\overline{x}_t,t)] \geqq 0.$$

The difference between the values of u in the first square brackets is positive and in the second square brackets is negative. The inequalities between the values of p and p' imply that $p(s)$ and $p'(t)$ are positive. Thus there exist $\overline{r}$ which turn the first inequality into an equality. Clearly for the same r the second inequality is false, and the required contradiction has been obtained. Q.E.D.

4. DISCUSSION AND CONCLUDING REMARKS

(1) The formal definition of a null state introduced in the preceding section does not capture the intuitive notion of a null state, that is, a state whose prior (subjective) probability is zero. To see this, it is useful to classify the various situations that may arise while restricting the preference relations $\succsim$ and $\hat{\succsim}$ to a state s in S as in Table 1.

$\succ$ \ $\hat{\succ}$	$\hat{\succ}_s \neq \phi$	$\hat{\succ}_s = \phi$
$\succ_s \neq \phi$	$p(s) > 0$	ϕ
$\succ_s = \phi$	$p(s) = 0$	$p(s) \in [0,1)$

TABLE 1

The notations $\succ_s$ and $\hat{\succ}_s$ denote the corresponding strict preference relations restricted to the state s, as in Theorem 3(c).

The right top entry of Table 1 is empty by (A.7). The left bottom entry corresponds to our definition of a null state and $p(s) = 0$ by Theorem 3(c). The inequality $p(s) > 0$ in the left top entry is also implied by Theorem 3(c). The configuration $(\succ_s = \phi, \hat{\succ}_s = \phi)$ which corresponds to the right bottom entry does not permit us to differentiate between null and non-null s in the intuitive sense. This is illustrated by the example in Table 2.

S \ X	s	t
x	3	1
y	0	1
P	p(s) > 0	p(t)=1-p(s)

TABLE 2

The four entries in Table 2 represent utilities, on the basis of which we construct a unique well-defined preference relation, $\hat{\succsim}$ on $\hat{L}$. For any probability vector p satisfying the restriction p(s) > 0 of Table 2 we construct a unique well-defined preference relation, $\succsim$ on L. Moreover, all relations $\succsim$ on L constructed in this manner are identical. Therefore using Theorem 3(a) to deduce the probability p from these relations, we show that p(t) can be any number in the interval [0,1). The state t corresponds to the right bottom entry of Table 1. Although formally defined as a non-null state it may be assigned probability zero. This example illustrates that although a state may be null in an intuitive sense, it may not be possible to deduce this property from (the observations) $\hat{\succsim}$ and $\succsim$. Notice that the example in Table II can be easily extended to any finite number of states and prizes.

(2) To avoid the indeterminancy of p as described in (1) above, the condition for uniqueness of p was stated in Theorem 3(c) as: for all s in S, the relation $\hat{\succ}_s$ is nonempty. However, we proved a somewhat stronger result. Denote by T the set of states s such that $\hat{\succ}_s \neq \phi$. Then the probability p from Theorem 3(a) conditioned on T is unique.

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