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Book

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RISK AVERSION FOR STATE-DEPENDENT UTILITY FUNCTIONS:
MEASUREMENT AND APPLICATIONS

by

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1. Introduction

To a large extent, advances in the theory of risk aversion in recent years were confined to the realm of state independent utility functions.^{1/} This theory has proved to be a powerful tool for the analysis of portfolio-selection and a large class of risk-sharing problems. No parallel advances were made in developing a theory of attitudes towards risk for state-dependent utility functions, i.e. where the utility assigned to any given level of wealth varies with the state of nature. The need for such a theory is manifested by the relatively few substantive results in the analysis of insurance problems involving irreplaceable objects such as life insurance, personal injury insurance as well as insurance of heirlooms.^{2/} The differentiating characteristic of these problems which necessitates a special theory is that random variations in the object of interest, or the state of nature, may not be reduced through insurance. Thus, for instance, theft insurance for heirlooms provides for a compensating variation in wealth rather than for the restoration of the stolen heirloom.

The main purpose of this paper is to fill this gap by developing measures of risk aversion which would permit the partial ordering of state-dependent utility functions according to their aversion to risk. The characterization of risk aversion using these measures is shown to be useful for the analysis of optimal risk-sharing

problems in the context of state-dependent preferences. Furthermore, the criterion for comparability of attitudes towards risk, as well as the measures developed to describe them, are analogous to the corresponding notions in the theory of risk aversion for state-independent utility functions. This resemblance suggests that our results constitute a proper generalization of the existing theory.

The plan of the paper is as follows: the condition for comparability of risk aversion among decision makers is discussed in Section II. Section III develops the notion of a risk premium as a formal measure of individual risk aversion. The meaning of this measure and its relation to the properties of the utility function is explored in Section IV. In Section V, we illustrate the usefulness of the concepts developed within the context of an elementary life-insurance problem. In Section VI, we study the implications of our notion for a single utility function and develop the notion of decreasing risk aversion. Section VII presents the main results within the framework of the state preference approach to decision making under uncertainty and provides an axiomatic treatment of the notion of decreasing absolute risk aversion. A brief summary is contained in the concluding section.

II. Preferences and Comparability

Let θ and $\bar{\theta}$ denote a pair of mutually exclusive and jointly exhaustive events describing the set of "states of nature".^{3/} Let the probability of the occurrence of θ and $\bar{\theta}$ be denoted p and $(1-p)$ respectively. Following Yaari [1969] we define a gamble on the event θ . as a real valued function on the set $\{\theta, \bar{\theta}\}$, i.e.

$$G: \{\theta, \bar{\theta}\} \rightarrow \langle w_1, w_2 \rangle$$

where w_1 and w_2 denote the respective payoffs to the decision maker in the events θ and $\bar{\theta}$.

We shall assume that for a given gamble the decision maker's preferences over the ordered pairs $\langle w_1, w_2 \rangle$ are representable by a continuous, monotone increasing, quasi-concave, thrice continuously differentiable, utility function, and that this utility representation reflects expected utility maximizing behavior. More specifically, let $U(w, \theta)$ and $U(w, \bar{\theta})$ denote the utility assigned to w in the two states of nature. Then the utility of a given gamble on θ is given by:

$$U(G) = pU(w_1, \theta) + (1-p) U(w_2, \bar{\theta}). \quad (1)$$

Consider next the issue of comparability of risk aversion among decision makers. The criterion for comparability suggested by Pratt [1964] for the case of state-independent preferences is that of risk premium. Thus, one utility function is said to be more risk averse than another, in a global sense, if its risk premium is higher for every risk and every level of wealth. The problem of comparability in this case is simplified significantly as a result of the existence of natural reference points. These points have the following three properties: (i) the total payoff, (ii) the total utility of wealth and (iii) the marginal utility of wealth, are the same across states of nature. No such points exist when the preference ordering is state dependent. Equal payoff in any two states of nature no longer implies either equal levels of utility or equal marginal utility of wealth. Thus, the first question that must concern us is: Which of the three properties mentioned above should be preserved in determining the reference points for the definition of risk premium?

As will become clear in the sequel, the important property to be preserved in choosing reference points is the equality of the marginal utility of wealth across states of nature. Thus, for a given gamble, G , let $\langle w_1^*(c), w_2^*(c) \rangle$ be the solution to the following problem:

$$\max_{\{w_1, w_2\}} [pU(w_1, \theta) + (1-p)U(w_2, \bar{\theta})]$$

subject to

$$pw_1 + (1-p)w_2 = c \text{ and } w_i \geq 0, \quad i = 1, 2.$$

Then, assuming an internal solution, we have

$$U_w(w_1^*, \theta) = U_w(w_2^*, \bar{\theta}) \quad (2)$$

where $U_w(\cdot)$ is the partial derivative of U with respect to w . We shall not concern ourselves with detailed analysis of corner solutions. None of the main results is affected if the reference points represent corner solution to the above problem.

Defining a reference point in this manner, we shall refer to the set of all such points as the reference set, and denote it by $RS(U)$, thus,

$$RS(U) = \{ \langle w_1^*, w_2^* \rangle \mid U_w(w_1^*, \theta) = U_w(w_2^*, \bar{\theta}) \}.$$

Utilizing these notions, we introduce the following condition for comparability:

Definition 1: Two utility functions are said to be (globally) comparable with respect to a given gamble on θ if and only if their reference sets are identical.

Finally, assuming that the state-dependent utility function displays risk aversion with respect to random variations in wealth, we define, for gambles on θ , the following function:

$$w_1^* = g(w_2^*) = U_w^{-1}[U_w(w_2^*, \bar{\theta}), \theta], \quad g' > 0 \quad (3)$$

III. Risk Premiums

Consider next the notion of a risk premium defined as the maximum amount of money a decision maker would pay at the reference point $\langle g(w_2^*), w_2^* \rangle$ rather than bear actuarially neutral risks. Let this amount be denoted $\pi(w_2^*, x, p)$. Then the risk premium is formally defined by the following equation:

$$pU(g(w_2^*) - \pi, \theta) + (1-p)U(w_2^* - \pi, \bar{\theta}) = pU(g(w_2^*) + x, \theta) + (1-p)U(w_2^* - \frac{p}{1-p}x, \bar{\theta}) \quad (4)$$

where $x = w_1 - w_1^* = w_1 - g(w_2^*)$.

This definition of risk premium is the state-dependent analogue of the notion introduced by Pratt [1964]. In essence it means that, facing a gamble on θ , a decision maker might improve his position if he could reallocate his payoff among the states of nature so as to attain the reference point with the same actuarial value. Hence, the risk premium is the monetary equivalent of this improvement, measured along the optimal trajectory of contingent payoffs. In the case of state-independent preferences, the optimal allocation, namely the reference point, as well as the certainty equivalent point, are attained when the wealth is the same in every state of nature. In the case of state-dependent preferences, however, the reference points do not coincide with the certainty equivalent points. Hence, to preserve its meaning, the definition of risk premium had to be modified accordingly.

Utilizing the notion of risk-premium, we introduce the following formal measure of risk aversion:

Definition 2: Let U and V be two comparable state-dependent utility functions.

Then, U is said to be more risk averse than V if and only if

$\pi_U(w_2^*, x, p) \geq \pi_V(w_2^*, x, p)$ for all w_2^*, x , and $0 \leq p \leq 1$, where π_U and π_V are the risk premium functions corresponding to U and V respectively.

In order to relate this measure of risk aversion to the properties of the utility function, we consider next the case of small risks, namely, of x close to zero. Taking the Taylor expansion on both sides of (4) we get:

$$pU(g(w_2^*), \theta) + (1-p)U(w_2^*, \bar{\theta}) \sim pU(g(w_2^*), \theta) + (1-p)U(w_2^*, \theta) - [pU_w(g(w_2^*), \theta)g'(w_2^*) + (1-p)U_w(w_2^*, \bar{\theta})]\pi. \quad (5)$$

and

$$pU(g(w_2^*) + x, \theta) + (1-p)U(w_2^* - \frac{p}{1-p}x, \bar{\theta}) \sim pU(g(w_2^*), \theta) + (1-p)U(w_2^*, \bar{\theta}) + p[U_w(g(w_2^*), \theta) - U_w(w_2^*, \theta)]x + \frac{1}{2} \frac{p}{1-p} x^2 [(1-p)U_{ww}(g(w_2^*), \theta) + pU_{ww}(w_2^*, \bar{\theta})]. \quad (6)$$

Equating (5) and (6) as required by (4) and using (2) we get,

$$\pi \sim \frac{1}{2} \left[(1-p) \left(- \frac{U_{ww}(g(w_2^*), \theta)}{U_w(g(w_2^*), \theta)} \right) + p \left(- \frac{U_{ww}(w_2^*, \bar{\theta})}{U_w(w_2^*, \bar{\theta})} \right) \right] \frac{p}{1-p} x^2 [pg'(w_2^*) + (1-p)]^{-1} \quad (7)$$

The expression in the squared brackets, to be denoted $R(w_2^*, \theta, p)$, is another measure of local absolute risk aversion. Notice also that $\frac{p}{1-p} x^2$ is the mean of the squared deviations of the actual payoff from the reference point, i.e. since $x = w_1 - w_1^*$,

$$\frac{p}{1-p}(w_1 - w_1^*)^2 = p(w_1 - w_1^*)^2 + (1-p)(w_2 - w_2^*)^2.$$

Consequently, $R(w_1^*, \theta, p)$ is analogous to the Arrow-Pratt measure of risk aversion for state-independent utility functions in that it is proportional to the risk premium per unit of variation for small risks.

In the next section we explore the restrictions imposed by the criterion introduced in definition 2 on the global properties of the utility functions under comparison. In particular, we prove that this criterion is equivalent to $R_U(w_2^*, \theta, p) \geq R_V(w_2^*, \theta, p)$ for any given θ and all $w_2^* \geq 0$, and $0 \leq p \leq 1$.

IV. Comparative Risk Aversion

The comparison of decision makers' aversion to risk with state-dependent preferences requires measures of risk aversion analogous to those demanded by Arrow [1971] and Pratt [1964]. The nature of these measures is suggested by the discussion in the preceding section. The following theorem extends the results to the case of large risks, thus permitting comparisons of global attitudes towards risks.

Theorem 1: Let U and V be comparable, state-dependent utility functions. Then the following conditions are equivalent:

- (i) $R_U(w_2^*, \theta, p) \geq R_V(w_2^*, \theta, p)$ for all $w_2^* \geq 0$, $0 \leq p \leq 1$.
- (ii) For every $0 \leq p \leq 1$, there exist a transformation $T_p[]$ such that $pU(g(w_2^*), \theta) + (1-p)U(w_2^*, \bar{\theta}) = T_p[pV(g(w_2^*), \theta) + (1-p)V(w_2^*, \bar{\theta})]$
 $T_p' > 0$, $T_p'' < 0$, and

$$T_1'[V(g(w_2^*), \theta)] = T_0'[V(w_2^*, \bar{\theta})].$$

(iii) $\pi_U(w_2^*, x, p) \geq \pi_V(w_2^*, x, p)$ for all x , $w_2^* \geq 0$, and $0 \leq p \leq 1$.

Proof: (i) $\Rightarrow$ (ii).

For any given $0 \leq p \leq 1$ differentiating (ii) with respect to w_2^* we get:

$$T'_p = \frac{pU_w(g(w_2^*), \theta)g'(w_2^*) + (1-p)U_w(w_2^*, \bar{\theta})}{pV_w(g(w_2^*), \theta)g'(w_2^*) + (1-p)V_w(w_2^*, \bar{\theta})} = \frac{U_w(g(w_2^*), \theta)}{V_w(g(w_2^*), \theta)} = \frac{U_w(w_2^*, \bar{\theta})}{V_w(w_2^*, \bar{\theta})} > 0.$$

Thus, $T'_p[pV(g(w_2^*), \theta) + (1-p)V(w_2^*, \bar{\theta})]$ is independent of p . In particular,

$$T'_0[V(g(w_2^*), \theta)] = T'_1[V(w_2^*, \bar{\theta})] \quad (8)$$

And,

$$-2 \ln T'_p = - \ln \frac{U_w(g(w_2^*), \theta)U_w(w_2^*, \bar{\theta})}{V_w(g(w_2^*), \theta)V_w(w_2^*, \bar{\theta})}$$

Differentiating the last expression with respect to w_2^* , we obtain:

$$\begin{aligned} - \frac{T''_p}{T'_p} &= g'(w_2^*) \left(- \frac{U_{ww}(g(w_2^*), \theta)}{U_w(g(w_2^*), \theta)} \right) + \left(- \frac{U_{ww}(w_2^*, \bar{\theta})}{U_w(w_2^*, \bar{\theta})} \right) \\ &\quad - g'(w_2^*) \left(- \frac{V_{ww}(g(w_2^*), \theta)}{V_w(g(w_2^*), \theta)} \right) + \left(- \frac{V_{ww}(w_2^*, \bar{\theta})}{V_w(w_2^*, \bar{\theta})} \right) \end{aligned} \quad (9)$$

From (i), for all $0 \leq p \leq 1$,

$$\frac{(1-p)}{p} \left(- \frac{U_{ww}(g(w_2^*), \theta)}{U_w(g(w_2^*), \theta)} \right) + \left(- \frac{U_{ww}(w_2^*, \bar{\theta})}{U_w(w_2^*, \bar{\theta})} \right) \geq \frac{(1-p)}{p} \left(- \frac{V_{ww}(g(w_2^*), \theta)}{V_w(g(w_2^*), \theta)} \right) + \left(- \frac{V_{ww}(w_2^*, \bar{\theta})}{V_w(w_2^*, \bar{\theta})} \right)$$

(10)

In particular, let $0 \leq p(w_2^*) \leq 1$ in (10) be such that:

$$\frac{1-p(w_2^*)}{p(w_2^*)} = g'(w_2^*)$$

From (9) and (10) it follows that:

$$- \frac{T''}{T'_p}(\cdot) \geq 0$$

Hence $T''_p[\cdot] \leq 0$, $0 \leq p \leq 1$.

Notice also that $p = 1$ implies

$$U(w_1, \theta) = T_1[V(w_1, \theta)] \quad w_1 \geq 0.$$

and $p = 0$ implies

$$U(w_2, \bar{\theta}) = T_0[V(w_2, \bar{\theta})] \quad w_2 \geq 0.$$

(ii) $\Rightarrow$ (iii)

For any given $0 < p < 1$,

$$\begin{aligned} pU(g(w_2^* - \pi_U), \theta) + (1-p)U(w_2^* - \pi_U, \bar{\theta}) &= pU(g(w_2^*) + x, \theta) + (1-p)U(w_2^* - \frac{p}{1-p}x, \bar{\theta}) \\ &= pT[V(g(w_2^*) + x, \theta)] + (1-p)T[V(w_2^* - \frac{p}{1-p}x, \bar{\theta})] \\ &\leq T[pV(g(w_2^*) + x, \theta) + (1-p)V(w_2^* - \frac{p}{1-p}x, \bar{\theta})] \\ &= T[pV(g(w_2^* - \pi_V), \theta) + (1-p)V(w_2^* - \pi_V, \bar{\theta})] \\ &= pU(g(w_2^* - \pi_V), \theta) + (1-p)U(w_2^* - \pi_V, \bar{\theta}) \end{aligned}$$

where use has been made of the definition of risk premiums and Jensen's inequality.

Since U is monotone it follows that:

$$\pi_U(w_2^*, x, p) \geq \pi_V(w_2^*, x, p) \quad \forall x, w_2^* \geq 0 \text{ and } 0 < p < 1.$$

For $p = 1$ and $p = 0$, $\pi_U = \pi_V = 0$.

(iii) $\Rightarrow$ (i).

For any given w_2^* , and a utility function h ,

$$\frac{d\pi_h}{dx} = \frac{-p[h_w(g(w_2^*) + x, \theta) - h_w(w_2^* - \frac{p}{1-p}x, \bar{\theta})]}{ph_w(g(w_2^* - \pi_h), \theta)g'(w_2^* - \pi_h) + (1-p)h_w(w_2^* - \pi_h, \bar{\theta})}$$

Thus, from (2),

$$\left. \frac{d\pi_h}{dx} \right|_{x=0} = 0.$$

Hence, for (iii) to hold for all $x \geq 0$, we must have;

$$\left. \frac{d^2\pi_U}{dx^2} \right|_{x=0} \geq \left. \frac{d^2\pi_V}{dx^2} \right|_{x=0} \quad (11)$$

But

$$\left. \frac{d^2\pi_h}{dx^2} \right|_{x=0} = \frac{-\frac{p}{1-p}[h_{ww}(g(w_2^*), \theta)(1-p) + ph_{ww}(w_2^*, \bar{\theta})]}{[pg'(w_2^*) + (1-p)]h_w(w_2^*, \bar{\theta})}$$

where use has been made of (2). Consequently, (11) implies

$$R_U(w_2^*, \theta, p) \equiv (1-p) \left(- \frac{U_{ww}(g(w_2^*), \theta)}{U_w(g(w_2^*), \theta)} \right) + p \left(- \frac{U_{ww}(w_2^*, \bar{\theta})}{U_w(w_2^*, \bar{\theta})} \right) \geq$$

$$(1-p) \left(- \frac{V_{ww}(g(w_2^*), \theta)}{V_w(g(w_2^*), \theta)} \right) + p \left(- \frac{V_{ww}(w_2^*, \bar{\theta})}{V_w(w_2^*, \bar{\theta})} \right) \equiv R_V(w_2^*, \theta, p)$$

for all $w_2^* \geq 0$, $0 \leq p \leq 1$.

Q.E.D.

It is worth noting at this point that one implication of Theorem 1 is that if U and V are comparable in the sense of definition 1, then U is more risk averse than V in the sense of Definition 2, if, and only if, U is more risk averse than V (with regard to variations in wealth) in the sense of Arrow-Pratt at each and every state of nature. Consequently, if U is more risk averse than V in the sense of Ross [1979] at each and every state of nature, then it is more risk averse than V in the sense of Definition 2.

V. Application: An Elementary Life Insurance Problem

Consider the following problem. A passenger about to board a plane is offered a life insurance policy for the duration of the flight. Given the terms of the policy, how does the passenger's attitude towards risk affect his desired coverage?

In this particular instance, nature admits the two states, "life" and "death", which shall be denoted θ and $\bar{\theta}$ respectively with the corresponding probabilities p and $(1-p)$. Let I denote the indemnity payment in case of death and let $\ell(I)$ be the insurance premium.

Suppose that the insurer is risk neutral and that the policy is designed to cover a proportional loading factor, bI . Thus ℓ is given by the equation:

$$\ell - I(1-p) = bI \quad 0 \leq b < p, \quad (12)$$

where the restriction on b is imposed to assure that $\ell < I$.

Thus, upon buying insurance a decision maker with initial wealth position $\langle w_1, w_2 \rangle$ in the two states of nature can attain a terminal position (for the duration of the insurance coverage) $\langle Y_1, Y_2 \rangle$ where

$$Y_1 = w_1 - \ell = w_1 - [(1-p) + b]I \quad (a)$$

(13)

$$Y_2 = w_2 - \ell + I = w_2 + (p-b)I. \quad (b)$$

The decision maker's objective is to choose a level of coverage, I^* , so as to maximize the expected utility, $\bar{U}$, where:

$$\bar{U} = pU(Y_1, \theta) + (1-p)U(Y_2, \bar{\theta}). \quad (14)$$

Theorem 2: Let U and V be comparable, state-dependent utility functions. If U is more risk averse than V in the sense of definition 2, then for every initial wealth position $\langle w_1, w_2 \rangle$ and for any given terms of insurance,

$$I_U^* \geq I_V^*$$

where I_U^* and I_V^* denote the optimal coverage of U and V respectively.

Proof: Let Y_{1j}^* and Y_{2j}^* $j = U, V$ denote the terminal wealth corresponding to I_j^* . Then, the first order optimality condition implies:

$$\begin{aligned} \frac{\partial \bar{U}}{\partial I} &= -p(1-p+b)U_w(Y_{1U}^*, \theta) + (1-p)(p-b)U_w(Y_{2U}^*, \bar{\theta}) \\ &= p(1-p)[U_w(Y_{2U}^*, \bar{\theta}) - U_w(Y_{1U}^*, \theta)] - b[pU_w(Y_{1U}^*, \theta) + (1-p)U_w(Y_{2U}^*, \bar{\theta})] \\ &= 0 \end{aligned}$$

But $b \geq 0$, hence

$$U_w(Y_{2U}^*, \bar{\theta}) \geq U_w(Y_{1U}^*, \theta)$$

thus,

$$Y_{1U}^* \geq g(Y_{2U}^*) \quad (15)$$

where $g(\cdot)$ is defined in (3) above.

From Theorem 1 we have:

$$\frac{\partial \bar{U}}{\partial I} = -p(1-p+b)T'_1[V(Y_1, \theta)]V_w(Y_1, \theta) + (1-p)(p-b)T'_0[V(Y_2, \bar{\theta})]V_w(Y_2, \bar{\theta}). \quad (16)$$

Evaluating (16) at (Y_{1U}^*, Y_{2U}^*) and using (15) we observe that:

$$T'_1[V(Y_{1U}^*, \theta)] \leq T'_1[V(g(Y_{2U}^*), \theta)] = T'_0[V(Y_{2U}^*, \bar{\theta})]$$

where the last equality follows from Theorem 1, equation (8).

Thus,

$$0 = \frac{\partial \bar{U}}{\partial I} \geq T'_0[V(Y_{2U}^*, \bar{\theta})][-p(1-p+b)V_w(Y_{1U}^*, \theta) + (1-p)(p-b)V_w(Y_{2U}^*, \bar{\theta})],$$

Consequently,

$$-p(1-p+b)V_w(Y_{1U}^*, \theta) + (1-p)(p-b)V_w(Y_{2U}^*, \bar{\theta}) \leq 0$$

By concavity of V , this implies

$$I_U^* \geq I_V^*$$

Q.E.D.

It is easy to show that if the "loading factor" is a fixed amount, say C , then the decision to insure has the nature of "all or nothing". That is, if the decision maker decides to insure, he will buy coverage so as to equate the marginal utility of wealth in the two states of nature; otherwise, he will abstain from insuring altogether. Hence, if $C < \pi_V \leq \pi_U$, then both decision makers buy insurance, and since U and V are comparable, $I_U^* = I_V^*$ in this case. If, $\pi_V < C < \pi_U$, then the more risk averse decision maker, U , insures, whereas the less risk averse decision maker, V does not insure. Finally, if $\pi_V \leq \pi_U < C$, both decision makers will abstain from insuring.

VI. Decreasing Risk Aversion

To study the implications of our notion of risk aversion for the theory of individual behavior under uncertainty, we need to apply the concept of "more risk averse than" to a single, state-dependent, utility function. This requires a precise definition of decreasing (increasing, constant) absolute risk aversion. Hence,

Definition 3: A state dependent utility function U , is said to display decreasing absolute risk aversion if and only if $\forall(w_2^*, a > 0)$ and for any given x , and p

$$\pi(w_2^*, x, p) \geq \pi(w_2^* + a, x, p)$$

and U is said to display increasing absolute risk aversion if and only if $\forall(w_2^*, a > 0)$ and for any given x and p ,

$$\pi(w_2^*, x, p) \leq \pi(w_2^* + a, x, p)$$

where π is the risk premium

Notice that the definition compares the risk premium for identical risks at points along the reference path described by the function $g(w_2^*) = w_1^*$.

Equivalent and analytically useful characterizations of the notion of decreasing absolute risk aversion are given in the following:

Theorem 3: Let U be a state dependent utility function, then the following conditions are equivalent.

(i) $\pi(w_2^*, x, p) \geq \pi(w_2^* + a, x, p) \forall(w_2^*, a > 0)$ and all x and $0 \leq p \leq 1$.

(ii) $R(w_2^*, \theta, p)$ is decreasing in w_2^* for all $w_2^* > 0$, and every given gamble on θ .

(iii) For every $0 < p < 1$, there exists a transformation $T_p[\cdot]$ such that for every $w_2^* \geq 0$, and a real number $a > 0$ and all x ,

$$pU(g(w_2^*), \theta) + (1-p)U(w_2^*, \bar{\theta}) = T_p[pU(g(w_2^* + a), \theta) + (1-p)U(w_2^* + a, \bar{\theta})]$$

and $T_p' > 0$, $T_p'' < 0$. T_p' is independent of p .

Proof: These results follow upon application of Theorem 1 to $U(g(w_2^* + a) + x, \theta) = V(g(w_2^*) + x, \theta)$ and $U(w_2^* + a + x, \bar{\theta}) = V(w_2^* + x, \bar{\theta})$ for arbitrary w_2^* , and $a > 0$.

One implication of Theorem 3 is that if an individual displays decreasing absolute risk aversion in the sense of Arrow [1971] in each and every state of nature, then for a given risk, he will buy less life insurance the larger is his wealth.

VII. The State Preference Approach

The notion of risk aversion for state-dependent utility functions can be presented within the context of the state-preference approach in a manner analogous to Yaari's [1969] treatment of risk aversion for state-independent utility functions. In particular, it is possible to show that the definition of decreasing absolute risk aversion is equivalent to a slightly modified version of Yaari's Axiom V'.^{5/} The modification is required to account for the possibility that the reference set in the case of state-dependent utility functions deviates from the 45° line, which is the reference set in the state-independent case.

To explore this analogy, let $\succsim_\theta$ be a decision maker's preference ordering of the set of all gambles on θ and $\bar{\theta}$. Assume that $\succsim_\theta$ satisfies the following axioms.^{6/}

- I. $\succsim_\theta$ is a complete weak ordering.
- II. $\succsim_\theta$ is monotone.
- III. $\succsim_\theta$ is continuous.
- IV. $\succsim_\theta$ is convex.

Given a reference position, $w^* = \langle g(w_2^*), w_2^* \rangle$ a decision maker will accept a gamble $w = \langle w_1, w_2 \rangle$ if and only if $w \succsim_\theta w^*$. Thus, the set $S(w^*) = \{w | w \succsim_\theta w^*\}$ is the decision maker's acceptable set. From the above axioms it follows that $(\forall w^*) S(w^*)$ is convex and is bounded below by a graph of a continuous, non-increasing function, passing through the reference point w^* . We shall refer to this function as the decision maker's acceptance frontier and denote it by $f(w^*, t)$.^{7/}

It can be shown that two state-dependent utility functions are comparable in the partial ordering "globally more risk averse than ..." relative to gambles on θ if their reference sets are identical. Assume that f is differentiable at the reference point w^* , since, by definition $U_w(w_1^*, \theta) = U_w(w_2^*, \bar{\theta})$, the quantity

$$p(w^*) = \frac{f_t(w^*, 0)}{f_t(w^*, 0) - 1}$$

may be referred to as the decision maker's subjective probability for the event θ . Consequently, two utility functions are comparable in the partial ordering "is locally more risk averse than..." if the set of subjective probabilities of the one is contained in that of the other. Furthermore, if U is more risk averse than V relative to gambles on θ and $\bar{\theta}$, then

$$f_{tt}^U(w^*, 0) \geq f_{tt}^V(w^*, 0) \quad (\forall w^*).$$

It can also be shown that for a function U given in (1), ^{8/}

$$f_{tt}^U(w^*, 0) = R_U(w^*, \theta, p) \frac{p}{(1-p)^2}.$$

Finally, to introduce the analogue to Yaari's Axiom V' , we define an actually neutral gamble as $\hat{x} = \langle x, \frac{-p}{1-p} x \rangle$. If $x > 0$ then we shall say that $\hat{x}$ is a bet on θ , and if $x < 0$ then we shall say that $\hat{x}$ is a bet on $\bar{\theta}$.

Using these definitions we state the following axiom, which captures the notion of decreasing absolute risk aversion: Axiom V - Let $\hat{x}$ and $\hat{y}$ be gambles, and let a be a nonnegative, real number. If $\hat{x}$ and $\hat{y}$ are bets on the same event, and $w^* + \hat{x} + \hat{y} \succ_{\theta} w^* + \hat{x}$, then $w^{**} + \hat{x} + \hat{y} \succ_{\theta} w^{**} + \hat{x}$ where $w^* = \langle g(w_2^*)w_2^* \rangle$ and $w^{**} = \langle g(w_2^* + a), w_2^* + a \rangle$.

The interpretation of Axiom V is that as we move up parallel to the reference set, RS, the slope of the acceptance frontier (indifference curve) does not increase. This situation is depicted in Figure 1, where the indifference curves "open up" as we move up along the reference set.

FIGURE 1:

PREFERENCE ORDERING DISPLAYING DECREASING ABSOLUTE RISK AVERSION

To establish the correspondence between Axiom V and the conditions of Theorem 3, we prove the following:

Proposition 1: Let U be a state-dependent utility representation of the preference ordering $\succsim_\theta$, and assume that for a given $0 \leq p \leq 1$, U has the form

$$pU(w_1, \theta) + (1-p)U(w_2, \bar{\theta})$$

than U displays decreasing absolute risk aversion if and only if λ_{θ} satisfies Axiom V.

Proof: From Theorem 3 Condition (iii) we have:

$$\begin{aligned} - \frac{U_w(g(w_2^* + a) + x, \theta)}{U_w(w_2^* - \frac{p}{1-p}x, \bar{\theta})} &= - \frac{T_1'[U(g(w_2^* + a) + x, \theta)]U_w(g(w_2^* + a) + x, \theta)}{T_0'[U(w_2^* + a - \frac{p}{1-p}x, \bar{\theta})]U_w(w_2^* + a - \frac{p}{1-p}x, \bar{\theta})} \\ &\geq - \frac{U_w(g(w_2^* + a) + x, \theta)}{U_w(w_2^* + a - \frac{p}{1-p}x, \bar{\theta})} \end{aligned} \quad (17)$$

where the last inequality follows from the fact that for $x > 0$,

$$T_1'[U(g(w_2^* + a) + x, \theta)] \leq T_0'[U(w_2^* + a - \frac{p}{1-p}x, \bar{\theta})] \quad (18)$$

When $x < 0$, the inequalities in (17) and (18) are reversed.

If Axiom V does not hold for some x , say $x > 0$, then $(g(w_2^* + a) + x, \theta)$ such that the inequality (17) is reversed.

Thus, the inequality (18) must be reversed. Consequently, either $T_p'' > 0$ or $T_1'[V(g(w_2^* + a), \theta)] > T_0'[V(w_2^* + a, \bar{\theta})]$ and condition (iii) of Theorem 3 is negated.

Q.E.D.

VIII. Summary and Conclusions

This paper develops measures of risk aversion for state-dependent utility functions. These measures permit the partial ordering of decision makers with respect to the relation "is more risk averse than..." over gambles involving "ethically nonneutral" events.^{9/} In this sense, it is an extension of the Arrow-Pratt theory of risk aversion.

The usefulness of the notions developed here is illustrated within the context of an elementary life insurance problem, where the intuitive result that a more risk averse individual (with respect to a gamble on "life") will always buy at least as much life insurance coverage as a comparable, less risk averse individual, is formally proved. This example serves to illustrate but one of the wide range of problems that can be analyzed with the use of our theory.

The relation "is more risk averse than..." is also applied to a given individual at different levels of wealth. This, in conjunction with the hypothesis of decreasing absolute risk aversion, may provide the basis for a theory of risk-bearing in situations involving "ethically nonneutral risks", such as health insurance and insurance of irreplaceable property, such as heirlooms.^{10/}

Finally, the axiomatic characterization of the notion of decreasing absolute risk aversion is discussed within the framework of the state preference approach to decision making under uncertainty.

FOOTNOTES

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1. See Arrow [1971], Pratt [1964], Yaari [1969], Ross [1979], Diamond and Stiglitz [1974] for a discussion of univariate risk aversion, and Kihlstrom and Mirman [1974], Duncan [1978], Karni [1979] for extension of the theory to the multivariate case.
2. For a discussion of this problem and some interesting results see Cook and Graham [1977]. This general issue is also discussed in a recent survey by Hirschleifer and Riley [1979].
3. In simple life insurance problems θ and $\bar{\theta}$ may describe the states "life" and "death". See Section V below for further concrete discussion.
4. That such a transformation exists follows from the fact that for any given p , $V(w_2^*) = pV(g(w_2^*), \theta) + (1-p)V(w_2^*, \bar{\theta})$ is continuous and monotonic increasing in w_2^* . Let $V(w_2^*) = y$, then $w_2^* = V^{-1}(y)$, and $U(w_2^*) = U(V^{-1}(y))$, which is the desired transformation.
5. See Mayshar [1975] for another statement of this axiom.
6. For more detailed discussion see Yaari [1969].
7. The acceptance frontier is defined by the equation

$$pU(w_1^* + t, \theta) + (1-p)U(w_2^* + f(w_2^*, t), \bar{\theta}) = \text{Const.}$$

8. This result is analogous to Yaari [1969] p.318.
9. This terminology is due to Ramsey [1950].
10. See Cook and Graham [1977].

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