QUEEN'S

704

ISSN 0316-5078

INSTITUTE FOR ECONOMIC RESEARCH

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SPATIAL COMPETITION AND THE CORE

Discussion Paper #704

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Dec. 1987

Acknowledgements: The authors thank B. Allen, J. Gabszewicz, R. Holzman, and M. Slade for helpful discussions and suggestions. Financial support from CORE, the Public Policy Research Center (University of Florida), F.N.R.S. (Belgium), N.S.F. (U.S.A.) and S.S.H.R.C. (Canada) is gratefully acknowledged.

ABSTRACT

Models of spatial competition have proven to be very useful in describing differentiated products markets. A serious problem is that nonexistence of Nash equilibria seems endemic. This problem is resolved by modelling the price formation process using the core. The equilibrium is the outcome of a two-stage process. In the first stage, two firms choose locations simultaneously, looking ahead to the second stage. The second stage has prices determined by an allocation in the core of a cooperative subgame allowing for coalitions of buyers and sellers. The price selection is the joint profit maximum for the duopolists. This selection exists for all location pairs and coincides with the pure strategy Nash equilibrium of duopoly competition when the latter exists. Furthermore, these prices approach the competitive level as the distance between the firms goes to zero, thus capturing the essence of duopoly rivalry. For this price selection, in the location game, the two firms establish themselves at the efficient locations--the first and third quartiles.

I. INTRODUCTION

Models of spatial competition have proven to be very useful in describing differentiated products markets. However, one major problem with these models is that nonexistence of equilibria seems endemic. In particular, in Hotelling's [1929] original model, there exists no Nash price equilibrium in pure strategies for a wide range of firms' locations. This is so because of the large gain in sales that a firm can obtain by undercutting its rival and, therefore, capturing the whole market when firms are not sufficiently far apart (see d'Aspremont et al., [1979]). One way to restore the existence of equilibrium in Hotelling model is to use mixed strategies in the price subgame as shown by Osborne and Pitchik [1985].

In this paper, we resolve this problem by modelling the price formation process using the core. The equilibrium is the outcome of a two-stage process. In the first stage, firms simultaneously choose their locations in the market. In the second stage, a core allocation between buyers and sellers who are free to form any coalition determines the terms of trade.

Our justification for using the core is that, once a firm has chosen its location, the actual process of price determination can be quite complicated. For example, individuals buying expensive durables commonly attempt to use price quotes from a seller to obtain a better deal from a competitor. As Milgrom [1987] has pointed out in the context of price discrimination, models of price determination should explicitly include the possibility that both buyers and sellers may behave strategically. To analyze such markets, a preferred research strategy might be to specify a bargaining game that is more realistic. However, this requires the definition of a specific game for each conceivable situation (see Bester [1986] for an example). The core solution has the advantage of not being sensitive to the rules of the game under consideration, except rules about possible coalitions. In our approach, the core is better viewed as a stability concept which defines the set of allocations which cannot be upset by bargaining. Although the core may seem restrictive in that it allows for any possible coalitions of agents, it will be seen that the only relevant coalitions are quite simple; they consist of one seller and its customers. A coalition therefore corresponds to the set of agents involved in a simple retail transaction. Through use of the core, we have a model sensitive to the power of all agents in the economy and that does not suffer the weakness of strategic models whose outcomes are sensitive to the rules of the game.

Though the core restricts the set of possible allocations, it does not yield a unique outcome. We resolve this by having the firms select the core outcome which maximizes joint profit. This assumption is consistent with views of the duopoly problem dating back to Bertrand. This approach has surprising consequences. If a Nash equilibrium for the price game exists, then it corresponds to the core allocation that maximizes any weighted sum of profits. This suggests that, in an oligopoly, the Nash equilibrium is biased in favor of producers. In section 2, we show that the core is never empty for any pair of firm locations. In section 3, we study the properties of the core allocation and its relationship to the Nash equilibrium for the price subgame.

Though we assume that the terms of trade are determined using the core solution, we model the location choice in the first stage as a simultaneous move game. This is appropriate given the standard interpretation of location choice as a once-and-for-all decision not subject to readjustment, unlike prices. In section 4, we show that the equilibrium for the full two stage game is characterized by firms locating at the quartiles, which are the cost-minimizing locations. Even though, by assumption, firms choose locations noncooperatively, the equilibrium has efficient locations.

One of the first papers to apply the core notion to the problem of imperfect competition is that of Aumann [1973]. He shows that, for a monopolist facing a continuum of buyers, there may be allocations in the core that are worse for the monopolist than the competitive allocation. Shitovitz [1973] shows that, if there were at least two large identical traders, then the core and competitive allocations coincide. These results, as well as those of Gabszewicz and Mertens [1971] and Okuno, Postlewaite and Roberts [1980], suggest that the core does not adequately capture the essence of imperfect competition. Our results show that it is not the size of the sellers that is crucial, but their ability to commit to decisions, such as the degree of product differentiation, that allows them to extract surplus in a core allocation.

Our results also provide a simple solution to the generic non-existence problem in spatial models, as pointed out in d'Aspremont et al. [1979] and MacLeod [1985]. What causes non-existence in some of these models is the potential for undercutting in price and capturing all a rival's customers. A standard solution is to expand the strategy space to allow the use of mixed strategies, as done for the Hotelling model by Osborne and Pitchik [1985]. While this provides a formal solution to the existence problem, there are many difficulties of interpretation and stability of equilibria in mixed strategies (see Luce and Raiffa [1958, p. 74-76]). By using the core to model trade, we have a model that depends only on the fundamental characteristics and is not sensitive to the rules of the game. The work of Kreps and Scheinkman [1983] and Davidson and Deneckere [1986] shows that the equilibria are sensitive to the rules for rationing output in short supply across buyers. Recent literature on durable goods monopoly shows that the lack of commitment in output leads to the competitive allocation in a dynamic game. The use of the core is consistent with this latter result, while avoiding much of the analytical difficulty of dynamic games.

II. THE MODEL AND SOME PRELIMINARY RESULTS

Consider an economy with two firms, $i = 1, 2$, selling a homogeneous product to a continuum of consumers uniformly distributed at unit density along a line of unit length. Each consumer derives utility from consumption of a numeraire and at most one indivisible unit of product sold by the firms. Firms produce the product at zero marginal cost up to a fixed capacity. Transportation costs for the product are linear in distance and weight.

We model Hotelling's spatial competition as a two-stage game. In the first stage, firms 1 and 2 simultaneously choose their locations at respective distances a and b from the endpoints of the market (with $a + b \leq 1$). In the second stage, trade between firms and consumers takes place according to a process described as a cooperative game.

Each consumer is identified by his location $x \in [0,1]$ and has a utility function of the form

$$u_x(c, m) = \begin{cases} \alpha + m & \text{if } c = 1 \\ m & \text{if } c = 0 \end{cases}$$

where c denotes consumption of the firm's product, m is consumption of the numeraire and α is the reservation price of the product by the consumer. Except for location, consumers are identical and have an initial endowment $\bar{m}$ of the numeraire and zero of the product. Denote consumer x's endowment by $\omega(x) = (\bar{m}, 0)$. Let $z(x) = (m(x), c(x))$ be x's consumption at an allocation Z.

Apart from locations, the firms are identical and each have an endowment $\bar{c}$ of the product given by its capacity and zero of the numeraire. Denote firm i's endowment $\omega(i) = (0, \bar{c})$. In order to allow each firm to serve the entire set of consumers so that all coalitions are possible, we assume that $\bar{c} \geq 1$. The firms' payoffs are the amounts of numeraire received from consumers in the trading process, m(i), so $u_i(c, m) = m(i)$. Let $z(i) = (m(i), c(i))$ denote the firm's allocation at Z.

A trade between a firm and a consumer consists of a transfer of numeraire to the firm for a transfer of one unit of the good to the consumer and transportation of the good by the consumer to his location. Let t be the unit transport cost, so that a consumer at x who obtains the good at location l_i uses up $t|x - l_i|$ of the numeraire in transportation cost. We assume that the firms and other consumers are unable to identify a consumer's location. Since consumers obtain only zero or one unit of the good, each firm will transfer the good to all consumers for the same quantity of numeraire. In addition, consumers will not make any side payments with each other, since they are identical except for locations. Rather than imposing these informational and demand constraints as self-selection constraints, we implicitly include them below in our definition of a feasible allocation.

Let $N = [0, 1] \cup \{1, 2\}$ be the set of agents. We now proceed by defining a feasible allocation with respect to a coalition S N.

<u>Definition 1</u>: An allocation Z is feasible for S iff

(a) When $S \cap \{1, 2\} = \emptyset$, $z(x) = \omega(x)\ \forall\ x \in S$;

(b) When $S \cap \{1, 2\} = i$ and $S \cap [0, 1] = M$, there is a scalar p_i, which is the amount of numeraire transferred to the firm, and a function $c\colon M \to \{0, 1\}$ such that

(i) $m(i) = p_i \int_M c(x)dx$;

(ii) $m(x) = \bar{m} - (p_i + t|x - l_i|)c(x)$

where $l_i = a(b)$ if $i = 1(2)$;

(c) When $S \cap \{1, 2\} = \{1, 2\}$ and $S \cap [0, 1] = M$, there is a pair of scalars, p_1 and p_2, and functions, $c_1\colon M \to \{0, 1\}$ and $c_2\colon M \to \{0, 1\}$, such that:

(i) $c(x) = c_1(x) + c_2(x) \leq 1$ for $x \in M$;

(ii) $m(x) = \bar{m} - \sum_{i=1}^{2} (p_i + t|x - l_i|c_i(x))$ for $x \in M$;

(iii) $m(i) = p_i \int_M c_i(x)dx$, for $i = 1, 2$.

Let A(S) be the set of feasible allocations for S. We can now define the set of core allocations.

Definition 2: An allocation $Z \in A(N)$ is in the core iff there does not exist a coalition $S \subseteq N$ and an allocation $Z' \in A(S)$ such that $u_n(z'(n)) > u_n(z(n))$ for all $n \in S$.

Core allocations can be completely described by a pair of mill prices (p_1, p_2) which are the scalars describing the amounts of numeraire transferred by consumers to firms. Clearly, in a core allocation, there is no nonnull set M^* of consumers who do not buy from the firm offering them the lower full price, $\bar{p}(x) = \min\{p_1 + t|a - x|, p_2 + t|1 - b - x|\}$ for $x \in M^*$. Otherwise, these consumers could form a blocking coalition with the other firm and its customers and make everybody in the coalition strictly better off by choosing this firm's mill price appropriately.

Given this rule we may define the firms' profit functions as follows:

$$\Pi_i(p_1, p_2) = m(i) = p_i \int_{X_i} c(x)dx$$

where $X_i = \{x \in [0, 1];\ p_i + t|x - l_i| = \bar{p}(x)\}$

and $c(x) = 1$ iff $\bar{p}(x) \leq \alpha$.

Assume that α, the reservation price, is large enough (relative to t) for all consumers to buy at prices p_1 and p_2. Denote by $\bar{x}$ the market boundary between firms 1 and 2. If $|p_1 - p_2| \leq t(1 - a - b)$, then $\bar{x}$ is the location of consumer indifferent between purchasing the product from either firm:

$$\bar{x} = \frac{p_2 - p_1}{2t} + \frac{1 + a - b}{2} .$$

If $p_1 < p_2 - t(1 - a - b)$, then all consumers choose to buy from firm 1 and $\bar{x}$ is the right endpoint of the market:

$$\bar{x} = 1 .$$

Finally, if $p_1 > p_2 + t(1 - a - b)$, then all consumers purchase from firm 2 and $\bar{x}$ is the left endpoint of the market:

$$\bar{x} = 0 .$$

Since each consumer consumes a single unit of the product the demands to firms 1 and 2 are

$$D_1(p_1, p_2) = \bar{x}$$

and

$$D_2(p_1, p_2) = 1 - \bar{x} .$$

For given locations a and b, the profit functions for the two firms are

$$\begin{aligned} \Pi_1(p_1, p_2) &= \frac{p_1(p_2 - p_1)}{2t} + \frac{p_1(1 + a - b)}{2} &&, \text{ if } |p_1 - p_2| \leq t(1 - a - b) \\ &= p_1 &&, \text{ if } p_1 < p_2 - t(1 - a - b) \\ &= 0 &&, \text{ if } p_1 > p_2 + t(1 - a - b) \\ \Pi_2(p_1, p_2) &= \frac{p_2(p_1 - p_2)}{2t} + \frac{p_2(1 - a + b)}{2} &&, \text{ if } |p_1 - p_2| \leq t(1 - a - b) \\ &= p_2 &&, \text{ if } p_2 < p_1 - t(1 - a - b) \\ &= 0 &&, \text{ if } p_2 > p_1 + t(1 - a - b) . \end{aligned}$$

Under the restrictions on α, t and $\bar{c}$ such that the whole market is served, a pair of prices is sufficient to identify an allocation. All consumers in $[0, \bar{x}]$ purchase the good from firm 1 and receive utility $u_x(c, m) = \alpha + \bar{m} - p_1 - t|a - x|$, while all consumers in $[\bar{x}, 1]$ purchase from firm 2 and receive utility $u_x(c, m) = \alpha + \bar{m} - p_2 - t|1 - b - x|$. For firms, payoffs are simply given by profits, $u_i(c, m) = \Pi_i(p_1, p_2)$.

Let $A^*(p_1, p_2)$ be the set of feasible allocations corresponding to prices p_1 and p_2. The following result characterizes the core allocations.

Proposition 1: An allocation $Z \in A^*(\hat{p}_1, \hat{p}_2)$ with corresponding prices, $\hat{p}_1$ and $\hat{p}_2$, is in the core iff:

(1) $\quad \Pi_1(\hat{p}_1, \hat{p}_2) \geq \Pi_1(p_1, \hat{p}_2), \quad \forall\, p_1 \leq \hat{p}_1$

(2) $\quad \Pi_2(\hat{p}_1, p_2) \geq \Pi_2(\hat{p}_1, p_2), \quad \forall\, p_2 \leq \hat{p}_2$.

Proof: (i) Let $Z \in A^*(p_1, p_2)$. Suppose that (1) does not hold; then by lowering p_1, Π_1 increases. Thus there exists a blocking coalition since firm 1's customers face lower prices. A similar argument applies if (2) does not hold. (ii) Assume now that conditions (1) and (2) hold. Clearly, all blocking coalitions must include at least one firm. No blocking coalition can be formed in which one firm increases its price, because its customers are made worse off. If one firm lowers its price, then conditions (1) and (2) ensure that the firm does not gain. Thus, no blocking coalition can be formed with one firm decreasing its price. Similarly, the two firms cannot gain by jointly lowering prices since one firm's market will not be larger than with $(\hat{p}_1, \hat{p}_2)$, so this firm is made worse off. Q.E.D.

A direct consequence of the result is:

Proposition 2: The core of the price subgame is non-empty for any location pair (a, b) with $a + b \leq 1$.

Proof: Consider $\hat{p}_1 = \hat{p}_2 = 0$. Conditions (1) and (2) are trivially satisfied, so that any $Z \in A^*(0, 0)$ belongs to the core. Q.E.D.

Intuitively, when an allocation is such that both firms charge zero prices, no consumer wants to join a blocking coalition with a firm charging a positive price. An allocation in $A^*(0, 0)$ is competitive and belongs to the core.

As in many cooperative games, the core may be large so that there exist many price pairs associated with core allocations. As suggested by the proof of Proposition 2, some of these pairs may be unreasonable outcomes from the firms' point of view. Given that in this game, firms have more market power than consumers, it seems natural to focus on allocations favorable to the firms. More specifically, we choose prices that are Pareto-optimal for firms, since these prices maximize convex combinations of profits.

Assuming, for the moment, that a single pair of such prices exists for all locations a and b, we can now describe the first stage of the game--the location choices. Let $\hat{p}_1(a, b)$ and $\hat{p}_2(a, b)$ be the prices obtained from the selection described above. Firms 1 and 2 locate at a and b; when $a + b > 1$ they locate at $1 - a$ and $1 - b$. The payoff functions of firms 1 and 2 at the location pair (a, b) are given by the profit functions evaluated at $(\hat{p}_1(a, b), \hat{p}_2(a, b))$:

$$\hat{\Pi}_1(a, b) = \frac{\hat{p}_1(a, b)[\hat{p}_2(a, b) - \hat{p}_1(a, b)]}{2t} + \frac{\hat{p}_1(a, b)[1 + a - b]}{2}$$

and

$$\hat{\Pi}_2(a, b) = \frac{\hat{p}_2(a, b)[\hat{p}_1(a, b) - \hat{p}_2(a, b)]}{2t} + \frac{\hat{p}_2(a, b)[1 - a + b]}{2} .$$

An equilibrium for the first stage game is a Nash equilibrium for the noncooperative game whose payoffs are $\hat{\Pi}_1(a, b)$ and $\hat{\Pi}_2(a, b)$, i.e., a pair (a^*, b^*) of locations such that[1]

$$\hat{\Pi}_1(a^*, b^*) \geq \begin{cases} \hat{\Pi}_1(a, b^*) & , \forall a \quad [0, 1] \text{ and } a + b^* \leq 1 \\ \hat{\Pi}_2(1-a, 1-b^*), & \forall a \quad [0, 1] \text{ and } a + b^* > 1 \end{cases}$$

and

$$\hat{\Pi}_2(a^*, b^*) \geq \begin{cases} \hat{\Pi}_2(a^*, b) & , \forall b \quad [0, 1] \text{ and } a^* + b \leq 1 \\ \hat{\Pi}_1(1-a^*, 1-b), & \forall b \quad [0, 1] \text{ and } a^* + b > 1 \end{cases}$$

When both inequalities are strict for all $a \neq a^*$ and $b \neq b^*$, we describe (a^*, b^*) as a strict Nash equilibrium.

III. PRICE DETERMINATION

We now develop the selection of a price pair from the core for any given locations a and b which maximizes any convex combination of profits. For locations sufficiently far apart it is well-known that a unique Nash equilibrium in prices exists (see d'Aspremont et al., [1979]). We first show that this noncooperative Nash equilibrium satisfies the above requirements.

Let Z^* denote the set of core allocations and let $\Phi^* = \{(p_1, p_2); (p_1, p_2)$ correspond to an allocation in $Z^*\}$.

Proposition 3: If (p_1^*, p_2^*) is a pure strategy Nash equilibrium in price, then $(p_1^*, p_2^*) \in \Phi^*$ and (p_1^*, p_2^*) maximizes $\lambda\Pi_1(p_1, p_2) + (1 - \lambda)\Pi_2(p_1, p_2)$ for all $\lambda \in (0, 1)$ and $(p_1, p_2) \in \Phi^*$.

Proof: (i) We show that $(p_1^*, p_2^*) \in \Phi^*$. By definition of a Nash equilibrium, (p_1^*, p_2^*) satisfy

$$\Pi_i(p_i^*, p_j^*) \geq \Pi_i(p_i, p_j^*) \quad i \neq j, i = 1, 2$$

Thus, conditions (1) and (2) are clearly satisfied, and $(p_1^*, p_2^*) \varepsilon \Phi^*$.

(ii) We now establish that (p_1^*, p_2^*) solves

$$\max_{(p_1,p_2)\varepsilon\Phi^*} \lambda\Pi_1(p_1, p_2) + (1 - \lambda)\Pi_2(p_1, p_2)$$

Since the Nash equilibrium exists, (p_1^*, p_2^*) lies at the intersection of the best reply functions:

$$\bar{p}_1(p_2) = \frac{p_2 + t(1 + a - b)}{2} \tag{3}$$

and

$$\bar{p}_2(p_1) = \frac{p_1 + t(1 - a + b)}{2}. \tag{4}$$

No price pair with one price above that given by the corresponding best reply can be associated with an allocation in the core. Indeed, the firm with the lower price can increase profits by reducing price to the best reply level; thus forming a blocking coalition with all customers who gain at the new price. (Figure 1 illustrates the price pairs associated with core allocations.) Since profits in this region are increasing with prices, maximum profits for both firms are obtained at the Nash equilibrium for $(p_1, p_2) \varepsilon \Phi^*$. <u>Q.E.D.</u>

Insert here Figure 1

We now turn to the set of locations for which there does not exist a Nash equilibrium in pure strategies. It is well known that the nonexistence problem results from the fact that, at the price pair given by the intersection of curves given by (3) and (4), at least one firm has an incentive to undercut its rival and serve the whole market. It is precisely this characteristic of preferring to undercut a rival which generates a blocking coalition formed by the undercutter and the whole set of consumers that prevents this candidate equilibrium from being associated with a core allocation. This implies that in order to achieve a core allocation, firms 1 and 2 have to choose prices in the set

$$T = \{(p_1, p_2) \in R_+^2;\ \Pi_1(p_1, p_2) \geq p_2 - t(1 - a - b) \text{ and }$$
$$\Pi_2(p_1, p_2) \geq p_1 - t(1 - a - b)\} .$$

In other words, given p_j for which there exists p_i such that $(p_i, p_j) \in T$, firm i quotes a price p_i belonging to

$$T_i(p_j) = \{p_i;\ (p_i, p_j) \in T\} .$$

We now establish that for those locations for which no pure strategy Nash equilibrium exists, we can find a price pair satisfying conditions (1) and (2) and maximizing any convex combination of firms' profits.[2]

Proposition 4: For all location pairs, there exists a price pair $(\hat{p}_1, \hat{p}_2) \in T$ such that

(5) $$\Pi_i(\hat{p}_i, \hat{p}_j) \geq \Pi_i(p_i, \hat{p}_j), \quad \forall\ p_i \in T_i(\hat{p}_j), \quad i = 1, 2, \quad j \neq i .$$

Proof: Given that

(a) $T_i(p_j) \neq \emptyset$ for $p_j \in T_j$ where T_j is the projection of T on the p_j axis;

(b) $T_i(p_j)$ is compact since $\Pi_i(p_i, p_j)$ and $\Pi_j(p_i, p_j)$ are continuous in p_i for $(p_i, p_j) \in T$;

(c) $T_i(p_j)$ is convex since $\Pi_i(p_i, p_j)$ and $\Pi_j(p_i, p_j)$ are concave in p_i for $(p_i, p_j) \in T$;

(d) $\Pi_i(p_i, p_j)$ is continuous and concave in p_i for $(p_i, p_j) \in T$; then $(\hat{p}_1, \hat{p}_2)$ satisfying (5) exists by Theorem 7.3 of Friedman [1977].

Q.E.D.

This result has the following implications. (i) For locations such that a Nash equilibrium exists, the price equilibrium satisfies (5) (see d'Aspremont et al. [1979] for necessary and sufficient conditions of existence). (ii) For locations such that no Nash price equilibrium exists, the solution to (5) involves the constraint $p_i \in T_i(\hat{p}_j)$ for at least one i. In this case, as $\Pi_i(p_i,p_j)$ is strictly concave on $T_i(\hat{p}_j)$, $\hat{p}_i$ must be equal to $F_i(\hat{p}_j) = \max \{p_i;\ p_i \in T_i(\hat{p}_j)\}$. We then say that firm i is constrained in its price choice.

Accordingly, the price domain can be partitioned into four regions (see Figure 2).[3]

Region I: $(\hat{p}_1, \hat{p}_2)$ is such that no firm is constrained in its price choice. Hence $(\hat{p}_1, \hat{p}_2)$ is the solution of the system $p_1 = \bar{p}_1(p_2) = \frac{p_2 + t(1 + a - b)}{2}$ and $p_2 = \bar{p}_2(p_1) = \frac{p_1 + t(1 - a + b)}{2}$, i.e.

$$\hat{p}_1 = t\left(1 + \frac{a - b}{3}\right) \text{ and } \hat{p}_2 = t\left(1 + \frac{b - a}{3}\right) .$$

This is the Nash price equilibrium.

Region II(III): $(\hat{p}_1, \hat{p}_2)$ is such that firm 1 (firm 2) is constrained in its price choice while firm 2 (firm 1) is not. Hence $(\hat{p}_1, \hat{p}_2)$ is the solution of

$$p_1 = F_1(p_2) = \frac{tp_2(1 - a + b) - p_2^2 + 2t^2(1 - a - b)}{2t - p_2}$$

and $$p_2 = \bar{p}_2(p_1) = \frac{p_1 + t(1 - a + b)}{2}$$

whose solution is

$$\hat{p}_1 = t(3 + a - b - 4\sqrt{a}) = F_1(\hat{p}_2) \text{ and } \hat{p}_2 = 2t(1 - \sqrt{a})$$

$$(\hat{p}_1 = 2t(1 - \sqrt{b}) \text{ and } \hat{p}_2 = t(3 - a + b - 4\sqrt{b}) = F_2(\hat{p}_1)) .$$

Region IV: $(\hat{p}_1, \hat{p}_2)$ is such that both firms are constrained in their price choice so that

$$\hat{p}_1 = \frac{t\hat{p}_2(1 - a + b) - \hat{p}_2^2 + 2t^2(1 - a - b)}{2t - \hat{p}_2} = F_1(\hat{p}_2)$$

and

$$\hat{p}_2 = \frac{t\hat{p}_1(1 + a - b) - \hat{p}_1^2 + 2t^2(1 - a - b)}{2t - \hat{p}_1} = F_2(\hat{p}_1)$$

Unfortunately, we have not been able to derive a closed form solution for the above system.

Insert here Figure 2

The uniqueness of the solution to (5) can be established as follows. In Regions I and III, the slope of firm 1's best reply with respect to p_2 equals $\frac{1}{2}$. In Regions II and IV,

$$\frac{\partial F_1}{\partial p_2} = \frac{(2t - p_2)^2 - 4at^2}{(2t - p_2)^2}$$

which is nonnegative and strictly less than one for a positive.[4] The same holds for firm 2 if b is positive. A standard argument then shows that the solution to (5) is unique for all pairs of locations such that $a > 0$ and $b > 0$ (see, e.g., Rosen [1965]). It remains to deal with the case $a = 0$ and $b \geq 0$ ($a \geq 0$ and $b = 0$ can be similarly treated). For $b \leq 15 - 6\sqrt{6}$ (Region I), the solution to (5) is unequivocally given by $p_1 = t(1 - \frac{b}{3})$ and $\hat{p}_2 = t(1 + \frac{b}{3})$; for $15 - 6\sqrt{6} \leq b \leq 1$ (Region III), $\hat{p}_1 = 2t(1 - \sqrt{b})$ and $\hat{p}_2 = t(3 + b - 4\sqrt{b})$.

We now show that $(\hat{p}_1, \hat{p}_2)$ has the same properties as the pure strategy Nash equilibrium as stated in Proposition 3.

Proposition 5: For all location pairs, the solution to (5) $(\hat{p}_1, \hat{p}_2) \in \Phi^*$ and $(\hat{p}_1, \hat{p}_2)$ maximizes $\lambda\Pi_1(p_1, p_2) + (1 - \lambda)\, \Pi_2(p_1, p_2)$ for all $\lambda \in (0, 1)$ and $(p_1, p_2) \in \Phi^*$.

Proof: First, $(\hat{p}_1, \hat{p}_2) \in \Phi^*$ since any solution to (5) trivially satisfies conditions (1) and (2). Second, repeating the argument developed in Proposition 3, in each region with the appropriate best reply functions (i.e., $\bar{p}_i(p_j)$ or $F_i(p_j)$) yields the desired result since the derivatives of functions are always positive. Q.E.D.

To summarize, the selection from the core allocations given by (5) has the desirable property that it maximizes joint profits for any profit-sharing rule between the firms over the allocations in the core.

We have obtained a complete characterization of $(\hat{p}_1, \hat{p}_2)$ when firms are symmetrically located (a = b). It has been shown by d'Aspremont et al. [1979] that, for $a < \frac{1}{2}$, $\hat{p}_1 = \hat{p}_2 = t$ if and only if $a \leq \frac{1}{4}$ (Region I). Thus, for $\frac{1}{4} < a < \frac{1}{2}$, both firms are constrained and it is easy to verify that there exists a unique solution to (5) given by $\hat{p}_1 = \hat{p}_2 = 2t(1 - 2a)$ (Region IV). Finally, when $a = \frac{1}{2}$, there exists a unique solution which is the Bertrand solution, i.e., $\hat{p}_1 = \hat{p}_2 = 0$. Given Proposition 1, the only core allocation is the competitive one. Interestingly, this result is reminiscent of a theorem derived by Shitovitz [1973] in the context of an exchange economy. An illustration of the price pattern for a = b is provided in Figure 3.

Insert here Figure 3

Some of the properties of $\hat{p}_i$ shown in Figure 3 remain valid in the asymmetric case. In particular, $\hat{p}_i$ is a continuous function of a and b. It is also differentiable in a and b except on the boundaries between regions where the RHS and LHS derivatives exist but differ. Furthermore, when firms 1 and 2 are close together, (a, b) is in Region IV and $\hat{p}_i$ is a decreasing function of a and b. The argument is given for $\hat{p}_1$. Taking the total differential of $\hat{p}_1 - F_1(\hat{p}_2, a, b) = 0$ and $\hat{p}_2 - F_2(\hat{p}_1, a, b) = 0$ yields

$$\frac{d\hat{p}_1}{da} = \frac{\partial F_2/\partial a + (\partial F_2/\partial p_1)(\partial F_1/\partial a)}{1 - (\partial F_1/\partial p_2)(\partial F_2/\partial p_1)}$$

with $\partial F_1/\partial a = -\dfrac{t(2t + \hat{p}_2)}{2t - \hat{p}_2}$, $\partial F_2/\partial a = -t$, $\partial F_1/\partial p_2 = \dfrac{(2t - \hat{p}_2)^2 - 4at^2}{(2t - \hat{p}_2)^2}$ and $\partial F_2/\partial p_1 = \dfrac{(2c - \hat{p}_1)^2 - 4bc^2}{(2t - \hat{p}_1)^2}$. As $\partial F_i/\partial p_j > 0$ and $\hat{p}_i$ is smaller than t in Region IV, we obtain $\frac{d\hat{p}_1}{da} < 0$. A similar calculation covers the case $\frac{d\hat{p}_1}{db} < 0$. Thus the equilibrium prices decrease to the competitive level when the distance between the two firms goes to zero. In other words, as differentiation between the firms decreases, the core shrinks to the competitive allocation corresponding here to the Bertrand solution. This suggests that firms have incentives to differentiate themselves to earn positive profits, as we will show in the location game.

IV. LOCATION EQUILIBRIUM

The location game can be described as follows: firms 1 and 2 are the two players; a and b are the strategies; the unit interval is the common strategy set; and $\hat{\Pi}_1(a, b)$, $\hat{\Pi}_2(a, b)$ are the payoff functions for $a + b \leq 1$, and $\hat{\Pi}_2(1 - a, 1 - b)$, $\hat{\Pi}_1(1 - a, 1 - b)$ for $a + b > 1$.

PROPOSITION 6: In the location game, $a^* = b^* = \frac{1}{4}$ is a strict Nash equilibrium.

Proof: Let $b = \frac{1}{4}$ and show that

$$\hat{\Pi}_1(\tfrac{1}{4}, \tfrac{1}{4}) = \frac{t}{2} > \hat{\Pi}_1(a, \tfrac{1}{4}) \text{ for all } a \in [0, \tfrac{3}{4}] \text{ and } a \neq \tfrac{1}{4}$$

and

$$\hat{\Pi}_1(\tfrac{1}{4}, \tfrac{1}{4}) = \hat{\Pi}_2(1 - a, \tfrac{3}{4}) \text{ for all } a \in]\tfrac{3}{4}, 1] .$$

First, we know from d'Aspremont et al. [1979] that $\Pi_1(\frac{1}{4}, \frac{1}{4}) > \Pi_1(a, \frac{1}{4})$ for all $a \in [0, \frac{1}{4}[$. Let then $a \in]\frac{1}{4}, \frac{3}{4}[$. First, for $a > \frac{1}{4}$ but close to $\frac{1}{4}$, $(a, \frac{1}{4})$ lies in Region II (see Figure 2). A straightforward calculation leads to $\hat{\Pi}_1(a, \frac{1}{4}) = t\sqrt{a}\,(2.75 + a - 4\sqrt{a})$ which is a decreasing function of a in Region II. For larger values a, $(a, \frac{1}{4})$ is now in Region IV (see also Figure 2). Differentiating $\hat{\Pi}_1(a, \frac{1}{4})$ w.r.t. a in this region, we obtain $\frac{\partial\hat{p}_1}{\partial a}(D_1 + \hat{p}_1\frac{\partial D_1}{\partial p_1}) + \hat{p}_1(\frac{\partial D_1}{\partial p_2}\frac{\partial\hat{p}_2}{\partial a} + \frac{\partial D_1}{\partial a})$. As firm 1 is constrained in its price choice it must be that $D_1 + \hat{p}_1\frac{\partial D_1}{\partial p_1} > 0$. Given that $\frac{\partial\hat{p}_1}{\partial a} < 0$, the first term is therefore negative. The second term is equal to $\hat{p}_1(\frac{1}{2t}\frac{\partial\hat{p}_2}{\partial a} + \frac{1}{2})$. A direct calculation shows that $\frac{\partial\hat{p}_2}{\partial a} \leq -t$. Consequently, $\frac{1}{2t}\frac{\partial\hat{p}_2}{\partial a} + \frac{1}{2} < 0$ and, hence, $\frac{\partial\hat{\Pi}_1}{\partial a} < 0$. This implies that $\hat{\Pi}_1(\frac{1}{4}, \frac{1}{4}) > \hat{\Pi}_1(a, \frac{1}{4})$ for all $a \in]\frac{1}{4}, \frac{3}{4}[$. At $a = \frac{3}{4}$, we have $\hat{\Pi}_1(\frac{3}{4}, \frac{1}{4}) = 0 < \frac{t}{2} = \hat{\Pi}_1(\frac{1}{4}, \frac{1}{4})$. Finally, we have to show that $\hat{\Pi}_1(\frac{1}{4}, \frac{1}{4}) > \hat{\Pi}_1(1 - a, \frac{3}{4})$ for all $a \in]\frac{3}{4}, 1]$. First, for $a > \frac{3}{4}$ but close $\frac{3}{4}$, $(1 - a, \frac{3}{4})$ lies in Region IV so that $\partial\hat{\Pi}_1/\partial a > 0$ as above. Second, for $a > \frac{3}{4}$ such that $(1 - a, \frac{3}{4})$ is in Region III, a direct calculation shows that $\hat{\Pi}_1(1 - a, \frac{3}{4}) = 2t(1 - \frac{\sqrt{3}}{2})^2$. This implies that $\hat{\Pi}_1(1 - a, \frac{3}{4}) \leq 2t(1 - \frac{\sqrt{3}}{2})^2$ for all $a \in]\frac{3}{4}, 1]$. The desired inequality then follows from the fact that $2t(1 - \frac{\sqrt{3}}{2})^2 < \frac{t}{2} = \hat{\Pi}_1(\frac{1}{4}, \frac{1}{4})$. Q.E.D.

The entire set of Nash equilibria can be obtained by analyzing the derivatives of $\hat{\Pi}_1$ and $\hat{\Pi}_2$ w.r.t. a and b in the interior of the four regions I-IV. Essentially, these derivatives behave like those considered in the proof of Proposition 5. Figure 4 gives their signs in each region. Because at least one firm always prefers a unilateral move, only boundary points are possible Nash equilibria. Actually, it is easy to see that all points belonging to the boundary between Regions I and II, and between Regions I and III are Nash equilibria of the location game. However $(\frac{1}{4}, \frac{1}{4})$ is the only strict Nash equilibrium since, at the other equilibria, one firm is indifferent between staying on the boundary and moving inside the corresponding constrained region.[5]

Insert here Figure 4

An alternative location game is sequential entry by the two firms, as analyzed for the constant price case by Presscott and Visscher [1977] and Kats [1986]. Firm 1 will choose a location, looking ahead to the price subgame only. If fixed costs are high enough, it will be the case that both firms know that no further entry will take place. For many locations of firm 1, firm 2's best reply location is not unique, since it receives the same payoff over a range of locations. If firm 2 chooses b to minimize firm 1's profit over the set of locations to which he is indifferent, we have a unique best reply for firm 2. Using the information on $\frac{\partial \Pi_2}{\partial b}$ from Figure 4, firm 2 will choose a location on curve X or Y according to firm's choice of a. The subgame perfect equilibrium of this game is the same as that of the simultaneous location choice game.

<u>Proposition 7</u>: If firm 1 chooses its location first, then firm 2 chooses its location, and then the price subgame determines payoffs, the subgame perfect Nash equilibrium is $a = \frac{1}{4}$, $b = \frac{1}{4}$ if $b = \frac{-1 - 5a + 5\sqrt{a} + a\sqrt{a}}{1 + \sqrt{a}}$ is firm 2's choice for b over $a \in (\frac{1}{4}, \frac{1}{2}]$ where Π_2 is constant with respect to b, for b less than or equal to that value.

<u>Proof</u>: For $a \varepsilon [0, \frac{1}{4})$, $\Pi_1(a, b) = \frac{t}{18}(3 + a - b)^2$. Firm 2's best reply to a is given by $b = 15 + a - 6\sqrt{6+a}$ (curve X in Figures 2 and 4). Thus, $\Pi_1(a, b(a)) = \frac{t}{18}(-12 + 6\sqrt{6+a})^2$ and $\frac{d\Pi_1}{da} = \frac{t}{3}(-12 + 6\sqrt{6+a})(6+a)^{-\frac{1}{2}} > 0 \;\; \forall \; a \varepsilon [0, \frac{1}{4})$. Thus $a = \frac{1}{4}$ is best for firm 1 from $a \varepsilon [0, \frac{1}{4}]$. For $a \varepsilon (\frac{1}{4}, \frac{1}{2}]$, firm 2 minimizes Π_1, at no cost to itself by choosing $b = \frac{-1 - 5a + 5\sqrt{a} + a\sqrt{a}}{1 + \sqrt{a}}$ (along the border of regions II and IV in Figure 4). On this curve, $\Pi_1(a, b) = t\sqrt{a}(3 + a - b - 4\sqrt{a})$. Taking firm 2's best reply into account, $\Pi_1(a, b(a)) = \frac{2t\sqrt{a}}{1+\sqrt{a}}(2 - 3\sqrt{a} + a)$ and $\frac{d\Pi_1}{da} = \frac{t}{\sqrt{a}(1 + \sqrt{a})^2}(2 - 6\sqrt{a} + 2a\sqrt{a}) < 0$ for $a \varepsilon (\frac{1}{4}, \frac{1}{2}]$. Thus $a = \frac{1}{4}$ is firm 1's choice from $a \varepsilon [\frac{1}{4}, \frac{1}{2}]$. For $a > \frac{1}{2}$, firm 2 will prefer to choose $b > \frac{1}{2}$, locating to the left of firm 2. Thus, it is sufficient to consider $a \varepsilon [0, \frac{1}{2}]$. Hence, $a = b = \frac{1}{4}$ is the equilibrium pair of locations.

<u>Q.E.D.</u>

If b were chosen to maximize Π_1 over the set of locations to which firm 2 is indifferent, $a = 15 - 6\sqrt{6}$, $b = 0$ would be the equilibrium. Firm 2 clearly benefits by attempting to minimize firm 1's payoff if it can signal its intention to do so. The equilibrium would remain the quartiles if firm 1 made the most pessimistic assumption about firm 2's response and located to maximize its minimum level of profit.

Thus, when the core is chosen as an equilibrium concept for the price subgame, we can say that <u>the process of spatial competition ends up with the two firms established at the socially optimal locations</u>. This is in contrast to the traditional claim that spatial competition is inefficient in providing variety to consumers.

V. CONCLUSIONS

In this paper, we have mixed two game-theoretic solution concepts to model the process of spatial competition. In the tradition of Hotelling, we have assumed a two-stage model in which firms first choose locations and then prices. Instead of using Nash equilibria to describe the second stage, we have considered the core in which firms and consumers may form coalitions. Although the core is a cooperative solution concept, the outcome is not collusive from the firms' point of view, albeit we have chosen a selection from the core corresponding to joint profit maximization. This occurs because simple coalition involving a firm and the whole set of consumers can often block collusive allocations in which firm would reap greater profits. Although the competitive allocations always lie in the core, the process of product differentiation by firms enlarges the core and allows them to earn positive profits. In particular, all core allocations provide the firms with profits at least as great as in the competitive allocation. Therefore, in this model, there do not exist disadvantageous oligopolists, as discussed in Aumann [1973]. However, as transport costs go to zero, differentiation is reduced and the core shrinks to the competitive allocation in accord with Shitovitz [1973] who modelled trading with identical commodities. Two opposing forces are at work in this game. First, coalitions formed by firms and consumers erode profits. Second, differentiation of products is sought to increases profits of firms. The resulting equilibrium is efficient in minimizing transport costs of serving the entire market. Furthermore, it is interesting to observe that the prices ultimately chosen by firms correspond to the noncooperative Nash equilibrium in pure strategies.

FOOTNOTES

[1]The reversal of the subscripts is necessary because the functional forms of profits depend on firm 1 lying to the left of firm 2.

[2]The pair of prices given by Proposition 4 is similar to the equilibrium prices considered by Eaton and Kierzkowski [1984] in a different, but related, context.

[3]The equations for the boundaries of the above regions can be obtained by equating equilibrium prices corresponding to the different regions. Thus the border of I and II is given by $W = \{(a, b);\ p_1 = \bar{p}_1(p_2) = F_1(p_2)$ and $p_2 = \bar{p}_2(p_1) < F_2(p_1)\}$; the border of II and IV is $Y = \{(a, b);\ p_1 = F_1(p_2)$ and $p_2 = \bar{p}_2(p_1) = F_2(p_1)\}$; X and Z are similarly defined.

[4]Evaluating $\frac{\partial F_1}{\partial p_2}$ at $\hat{p}_2 = 2t(1 - \sqrt{a})$, p_2's largest value in Regions II and IV, we see that $\frac{\partial F_1}{\partial p_2}(\hat{p}_2) = 0$. For any smaller value of p_2, $\frac{\partial F_1}{\partial p_2}$ is positive. The reaction function (from below) thus has the property that both derivatives are positive and strictly less than one.

[5]Notice also that the strategies corresponding to the equilibria other than $(\frac{1}{4}, \frac{1}{4})$ are dominated; $a = b = \frac{1}{4}$ is the only Nash equilibrium in undominated strategies. Indeed, inspection of Figure 3 shows that $a > 15 - 6\sqrt{6}$ and $b > 15 - 6\sqrt{6}$ are strategies dominated by $a = 15 - 6\sqrt{6}$ and $b = 15 - 6\sqrt{6}$ respectively. Eliminating those strategies, $a < .25$ and $b < .25$ are dominated by $a = .25$ and $b = .25$, thus giving the desired result.

REFERENCES

Aumann, R., 1973, "Disadvantageous Monopolies", Journal of Economic Theory 6:1-11.

Bester, H., 1986, "Non-Cooperative Bargaining and Spatial Competition", University of Bonn Discussion Paper No. A-73.

d'Aspremont, C., J.J. Gabszewicz and J.-F. Thisse, 1979, "On Hotelling's 'Stability in Competition'", Econometrica 47:1145-1150.

Davidson, C. and R. Deneckere, 1986, "Long-Run Competition in Capacity, Short-Run Competition in Price, and the Cournot Model", Rand Journal of Economics 17:404-415.

Eaton, J. and H. Kierzkowski, 1984, "Oligopolistic Competition, Product Variety, Entry Deterrence and Technology Transfer", Rand Journal of Economics 15:99-107.

Friedman, J.W., 1977, Oligopoly and the Theory of Games. Amsterdam, North-Holland.

Gabszewicz, J.J. and J.-F. Mertens, 1971, "An Equivalence Theorem for the Core of an Economy Whose Atoms Are Not 'Too' Big", Econometrica 39:713-721.

Hotelling, H., 1929, "Stability in Competition", Economic Journal 39:41-57.

Kats, A., 1986, "Sequential Entry in a Model of Spatial Competition", mimeo, VPI.

Kreps, D. and J. Scheinkman, 1983, "Quantity Precommitment and Bertrand Competition Yields Cournot Outcomes", Bell Journal of Economics 14:326-337.

Luce, D. and H. Raiffa, 1958, Games and Decisions, New York: John Wiley.

MacLeod, W.B., 1985, "On the Nonexistence of Equilibria in Differentiated Product Models", Regional Science and Urban Economics 15:245-262.

Milgrom, P., 1987, "An Essay on Price Discrimination", UC Berkeley Discussion Paper 8732.

Okuno, M., A. Postlewaite and J. Roberts, 1980, "Oligopoly and Competition in Large Markets", American Economic Review 70:22-31.

Osborne, M. and C. Pitchik, 1986, "Equilibrium in Hotelling's Model of Spatial Competition", McMaster University Department of Economics Working Paper No. 85-19

Prescott, E.C. and M. Visscher, 1977, "Sequential Location among Firms with Foresight", Bell Journal of Economics 8:378-393.

Rosen, J.B., 1965, "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games", Econometrica 33:520-534.

Shitovitz, B., 1973, "Oligopoly in Markets with a Continuum of Traders", Econometrica 41:467-502.

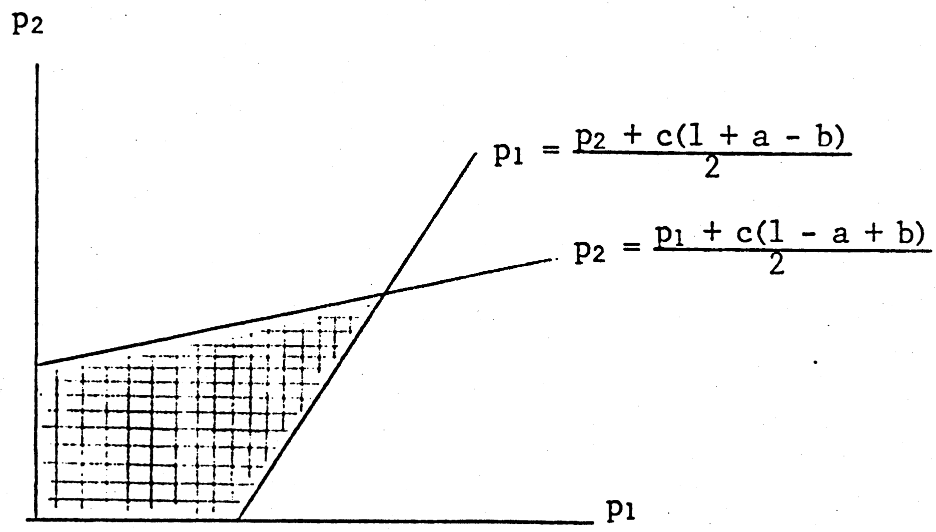
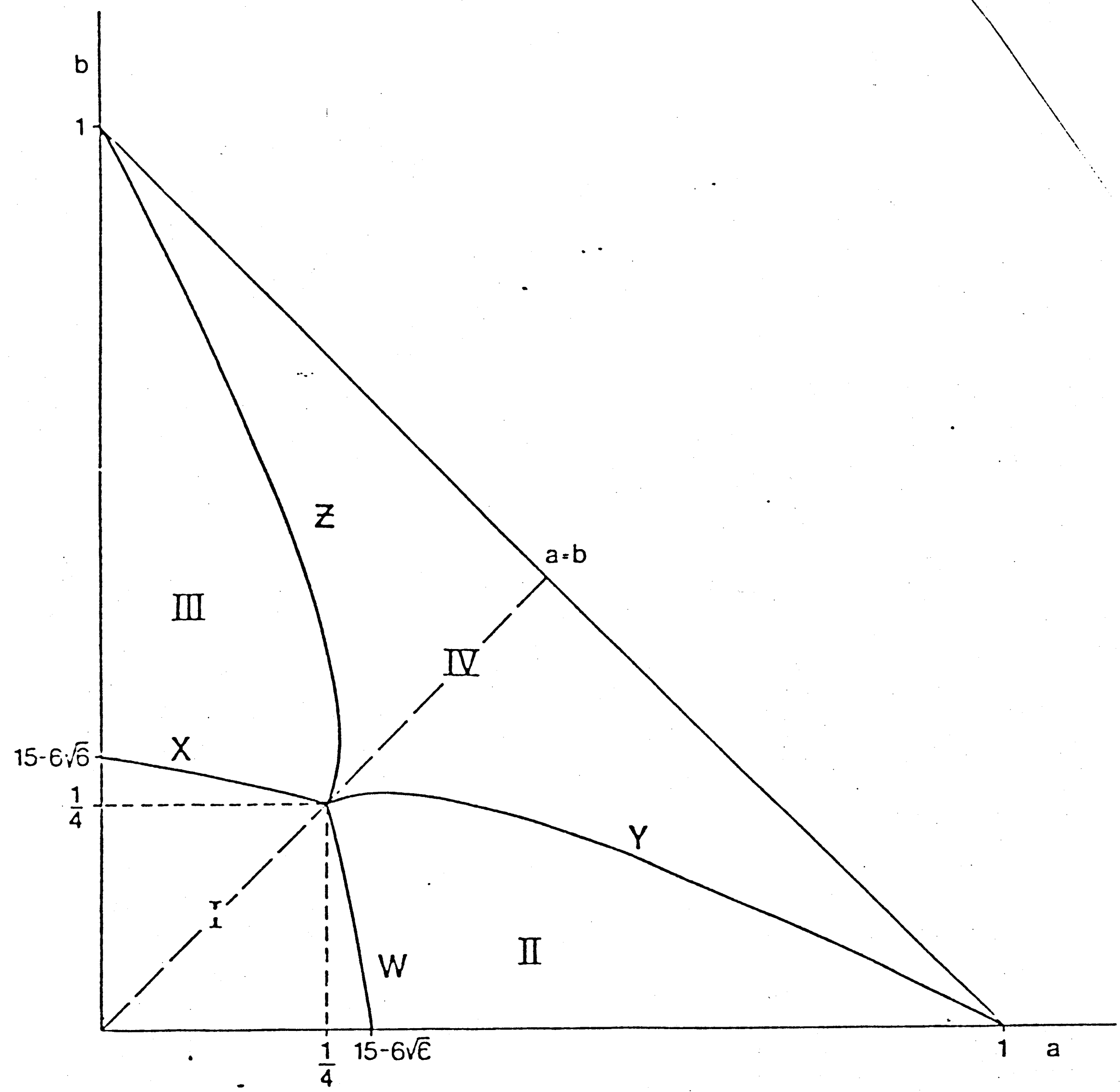


Figure 1: The Shaded Region Consists of Core Price Allocations When (p_1^*, p_2^*) is a Pure Strategy Nash Price Equilibrium



$$W : a = 15 + b - 6\sqrt{6 + b}$$

$$X : b = 15 + a - 6\sqrt{6 + a}$$

$$Y : b = \frac{-1 - 5a + 5\sqrt{a} + a\sqrt{a}}{1 + \sqrt{a}}$$

$$Z : a = \frac{-1 - 5b + 5\sqrt{b} + b\sqrt{b}}{1 + \sqrt{b}}$$

<u>Figure 2:</u> <u>Equilibrium Price Regions</u>

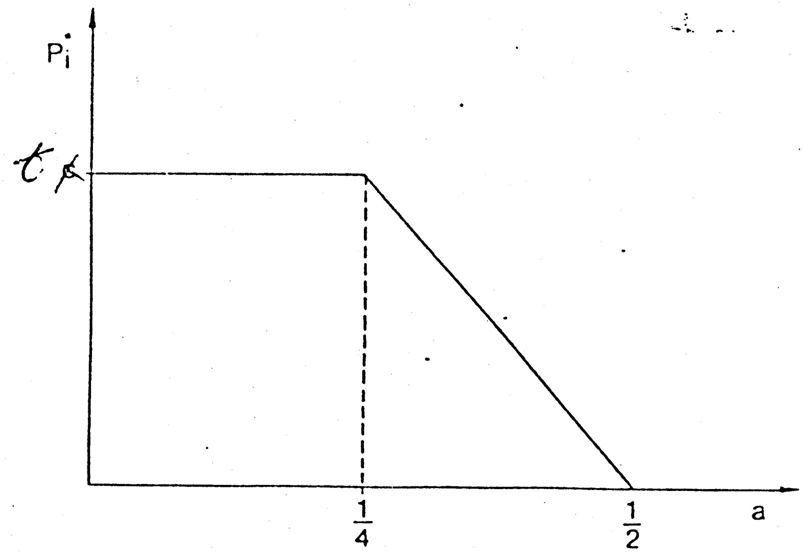


Figure 3: The Solution of Equation (5) for a = b (Symmetric Locations)

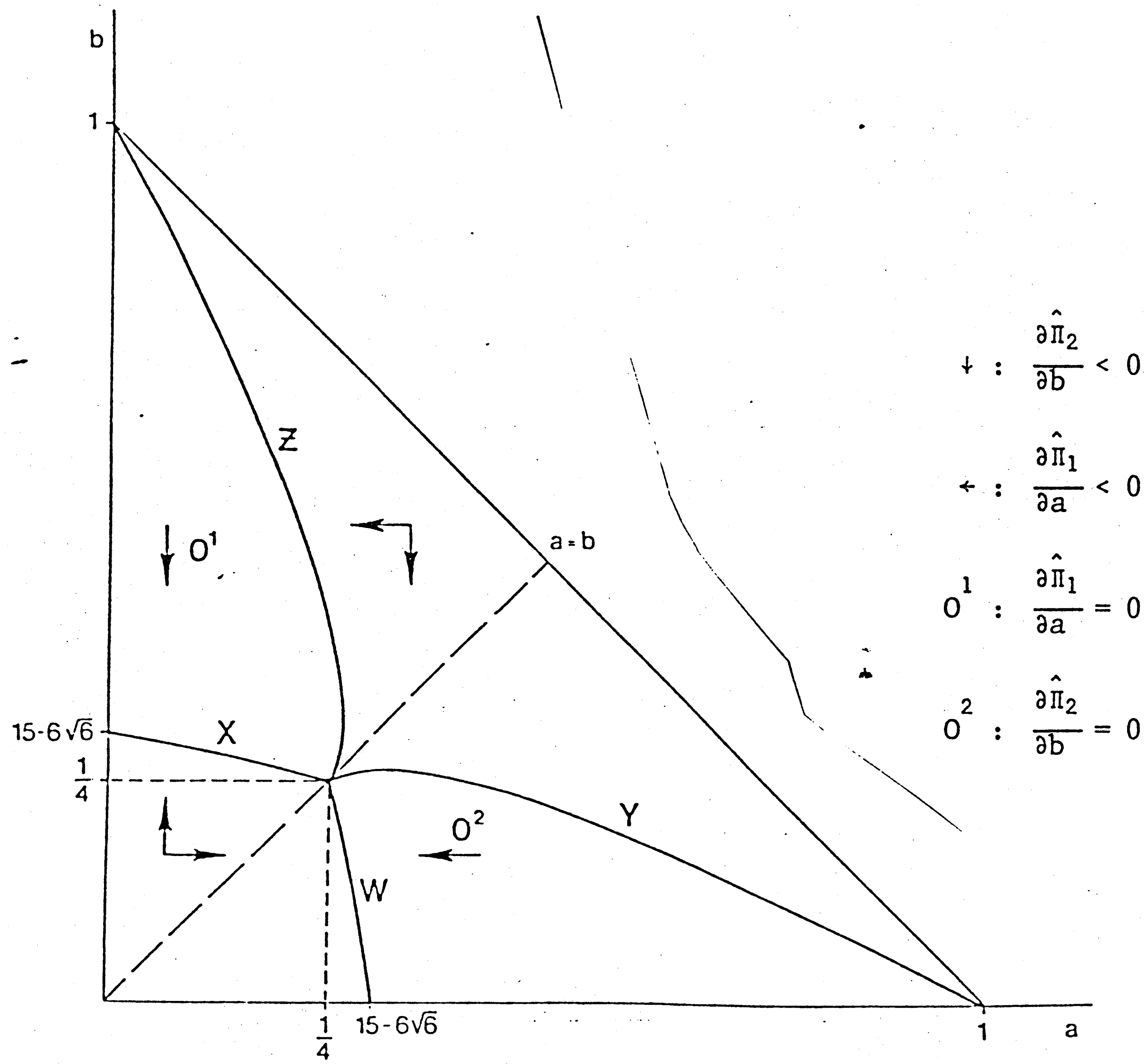


Figure 4: Location Equilibrium Derivatives in Interiors of Price Equilibrium Regions