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An intertemporal model of growing awareness

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Abstract

This paper presents an intertemporal model of growing awareness. It provides a framework for analyzing problems with long time horizons in the presence of growing awareness and awareness of unawareness. The framework generalizes both the standard event-tree framework and the framework from Karni and Vierø (2017) of awareness of unawareness. Axioms and a representation are provided along with a recursive formulation of intertemporal utility. This allows for tractable and consistent analysis of intertemporal problems with unawareness.

Keywords: Awareness, Unawareness, Intertemporal Utility, Recursive Utility, Reverse Bayesianism
JEL classification: D8, D81, D83, D9

1 Introduction

Under the Bayesian paradigm, the state space is fixed. As new discoveries are made, and new information becomes available, the universe shrinks as some states become null. However, there are many situations in which our universe in fact expands as we become aware of new opportunities. That is, there are, quoting United States former Secretary of Defense Donald Rumsfeld, "unknown unknowns" that we may learn about.[1] In other words, a

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[1] U.S. Department of Defense news briefing February 12, 2002.

decision maker's awareness may grow over time, and the decision maker may be aware of this possibility.

This paper provides a framework for analyzing intertemporal problems with long time horizons in the presence of growing awareness and awareness of unawareness. It thus makes possible the analysis of, for example, many macro and finance problems such as Lucas (1978) tree type asset pricing models, search models, etcetera, when agents are exposed to unawareness.

The analysis builds on the reverse Bayesianism framework of Karni and Vierø (2013, 2015, 2017). However, these papers considered a one-shot increase in a decision maker's awareness. They provided a framework for analyzing such an increase and axiomatized the decision maker's choice behavior in response to the increased awareness. In Karni and Vierø (2013, 2015) the decision maker was myopic with respect to his own unawareness and never anticipated making future discoveries. In Karni and Vierø (2017), the decision maker is aware of her unawareness, so although she cannot know exactly what she is unaware of, she is aware that there may be aspects of the universe that she cannot describe with her current language.

When an agent is looking forward over many future periods, she can envision a plethora of ways that her awareness may grow over time. At each point in time, there is not only the possibilities of making new discoveries or not, but also the possibility of making multiple new discoveries at the same time and different numbers of possible simultaneous discoveries. Thus, the possible paths of resolutions of uncertainty are much more complicated than in a standard event tree. To stay with the tree analogy, under growing awareness branches can sprout in many places in the event tree, and there will be different sprouts, and a different number of sprouts, on different branches.

Because of the complexity of the evolution of the state space over time, one important issue is how to make the problem tractable. Also, given that there is a great number of potential unkowns that the decision maker may discover in the future, the question arises of how much consistency it is reasonable to impose. Furthermore, with a long time horizon the decision maker will form beliefs over the entire future, and connecting these beliefs as awareness grows is a much more challenging task than just considering a one-shot increase in awareness.

Another issue adding complexity is that, in addition to not being measurable with respect to current information, future acts are generally not even fully describable with respect to current awareness. If awareness grows in the future, the decision maker will then know, and derive utility from, a larger set of consequences than she can currently describe. Additional measurability issues thus arise because different such consequences are indistinguishable given the current level of awareness. In order to formulate preferences from the decision maker's current point of view, these issues must be dealt with by the axiomatic structure.

To obtain tractability, one of the new axioms that will be imposed serves the purpose of "preventing the agent's head from exploding". In somewhat more scientific language, the axiom assumes that the decision maker acts as if she simplifies the universe by "collapsing" unknown consequences and parts of the event tree in a particular way. Other new axioms regard the evolution of the decision maker's attitude towards the unknown as awareness grows, and a strengthening of invariance of preferences towards known risks from Karni and Vierø (2017) to also apply across two successive periods.

The main result is an intertemporal representation of preferences. At any point in time, the agent can make complete contingent plans, also for events that involve new discoveries, to the extent that she can describe these plans. The axiomatic structure ensures dynamic consistency in a forward looking way, but not necessarily looking backwards. When awareness does grow, the agent may wish to change her course of action in response to her new awareness. She will, however, still maintain that her original plan was the right one given the awareness she had at the time it was made. Thus, the agent is rational to the extent possible given her limited awareness.

A recursive formulation of the decision maker's utility is also obtained. However, the decision maker can only forecast her future utility function to the extent of her awareness. She is aware that her utility function may change in the future in response to increased awareness, but uses an estimate of her future utility function, based on her current awareness, in the recursive formulation. This recursive formulation makes possible convenient analysis of, and accommodation of awareness and growing awareness in, a large class of problems along the lines of the analysis in, e.g., Sargent (1987).

The intertemporal framework introduced in this paper combines awareness of unawareness with an approach to defining intertemporal acts from Epstein and Schneider (2003). The evolution of awareness and uncertainty is captured by a generalized event tree that has the standard event tree as a special case. As such, the framework is a natural extension of both the standard intertemporal model and the state spaces in Karni and Vierø (2017).

Epstein and Schneider (2003) axiomatize an intertemporal version of multiple-priors utility. As is the case in the present paper, they impose axioms on the entire preference process, i.e. on conditional preferences at each time-event pair. They also connect preferences conditional on different histories, rather than simply applying their axioms to conditional preferences after each history separately. The approach taken in the present paper of specifying acts from the start to the end of the event tree is inspired by Epstein and Schneider's model. The extension of one of the key axioms from Karni and Vierø (2017) to the present intertemporal setting is also inspired by one of Epstein and Schneider's axioms.

In the statistical literature, Walley (1996) and Zabell (1992) have considered related problems. Walley (1996) considers the problem of making inferences from multinomial data in cases where there is no prior information, illustrated by repeated sampling from a bag of marbles whose contents are initially unknown. His approach is not choice theoretic. Rather he proposes using the imprecise Dirichlet model to analyze such problems. Zabell (1992) also considers a problem involving repeated sampling which may result in an observation whose existence was not suspected. Zabell's approach is not choice theoretic either, but limits attention to the probabilities of events.

Halpern, Rong, and Saxena (2010) consider Markov decision problems with unawareness. Their decision maker is initially aware of only a subset of states and actions and their model provides a special explore action by playing which the decision maker may become aware of actions he was previously unaware of. Halpern, Rong, and Saxena provide conditions under which the decision maker can learn to play near-optimally in polynomial time.

Easley and Rustichini (1999) consider a decision maker who must repeatedly choose an action from a finite set. The decision maker knows the set of available actions and that a payoff will occur to each action in each period, but no further structure. The decision maker prefers more payoff to less. He begins with an arbitrary ordering over acts and selects the action with the highest rank. Upon resolution of the period's uncertainty, he observes the payoff to each action and updates his ordering. Easley and Rustichini provide axioms that lead to actions eventually being chosen optimally according to expected utility.

Grant and Quiggin (2013a) consider dynamic games with differential awareness, where players may be unaware of some histories of the game. Unawareness thus materializes as players considering only a restricted version of the game. For such games, Grant and Quiggin provide logical foundations for players using inductive reasoning to conclude that there may be propositions, and hence parts of the game tree, of which they are unaware. Players may also gain inductive support for particular actions leading to unforeseen contingencies. As a result, they may choose strategies subject to heuristic constraints that rule out such actions. Grant and Quiggin (2013b) simplifies the model of Grant and Quiggin (2013a) to a single-person decision problem modelled as a game against nature. This framework is used to formalize and evaluate two versions of the precautionary principle.

There is a number of papers taking a choice theoretic approach to unawareness or related issues. These include Li (2008), Ahn and Ergin (2010), Schipper (2013), Lehrer and Teper (2014), Kochov (2016), Walker and Dietz (2011), Alon (2015), Grant and Quiggin (2015), Dietrich (2017), Piermont (forthcoming), and Dominiak and Tsjerengjimid (2017a). Kochov (2016) uses a three-period model to distinguish between unforeseen and ambiguous events;

thus he considers a different issue than the present. The other papers are either static in nature or consider one-shot increases in awareness.

Walker and Dietz (2011) take a choice theoretic approach to static choice under "conscious unawareness." Schipper (2013) focuses on detecting unawareness. Ahn and Ergin (2010) introduce a model in which the evaluation of acts may depend on the manner in which the underlying events, or contingencies, are described. Lehrer and Teper (2014) model a decision maker who has an increasing ability to distinguish between events, and who has Knightian preferences on the expanded set of acts. Alon (2015) models a decision maker who acts as if she completes the state space with an extra state and assigns the worst consequence to that state. Grant and Quiggin (2015) model unawareness by augmenting a standard Savage (1954) state space with a set of "surprise states".

Dominiak and Tsjerengjimid (2017a) generalizes the preference structure in Karni and Vierø (2013) to allow for the decision maker's ex-post preferences to be ambiguity averse. Dietrich (2017) considers a one-shot increase in awareness in a Savage framework. Piermont (forthcoming) presents a model with a one-shot increase in awareness, where the decision maker may be aware of his unawareness. In his model, the behavioral manifestation of awareness of unawareness is that the decision maker is unwilling to commit to any contingent plan. In other words, when he is aware of his unawareness, the decision maker has a strict preference for delaying choice at a positive cost.

Since the present paper builds on Karni and Vierø (2013, 2015, 2017), it is useful to describe these works in somewhat more detail. Karni and Vierø (2013) considers a one-shot increase in a decision maker's awareness. There are two main contributions. The first is to provide a framework of an expanding universe. What they call the conceivable state space expands as new acts, consequences, or links between them are discovered, that is, when awareness grows. The second contribution is to invoke the revealed preference methodology and axiomatize the decision maker's choice behaviour in the expanding universe. The challenge is that preferences under different levels of awareness are defined over different domains, so they need to be linked. The axioms imply that for a given level of awareness, the decision maker is an expected utility maximizer. The axioms that link behaviour across different state spaces imply that the utility of known risks is invariant to expansions of awareness and also imply reverse Bayesian updating of beliefs: when new discoveries are made, probability mass is shifted proportionally away from events in the prior state space to events created as a result of the expansion of the state space.

Karni and Vierø (2015) has a more general preference structure within the same framework. In both Karni and Vierø (2013) and Karni and Vierø (2015) the decision maker is myopic with respect to his unawareness. Hence, he never anticipates making future discoveries and always acts as if he is fully aware.

The premise of Karni and Vierø (2017) is that if you have become aware of new things in the past, you may anticipate that this can also happen in the future. The paper also considers a one-shot increase in the decision maker's awareness and extends the framework from Karni and Vierø (2013) to allow for decision makers being aware of their unawareness. So, although decision makers cannot know exactly what they are unaware of, they are aware that there may be aspects of the universe that they cannot describe with their current language. The framework has an augmented conceivable state space which is partitioned into fully describable and imperfectly describable states, in the latter of which awareness expands. The axiomatic structure gives that for a given level of awareness, the decision maker is a generalized Expected Utility maximizer: the utility representation consists of a Bernoulli utility function over known consequences, beliefs over the augmented conceivable state space that assign beliefs to expansions in the decision maker's awareness, and an extra parameter that reflects the decision maker's attitude towards the unknown. As in Karni and Vierø (2013), there is reverse Bayesian updating of beliefs and the utility of known risks is invariant to expansions in awareness. However it is now also possible to characterize the decision maker's sense of ignorance and the evolution thereof, which is captured by the probability assigned to expansions in her awareness.

The paper is organized as follows: Section 2 presents the framework for modelling long time horizon problems with awareness of unawareness. Section 3 presents and discusses the axioms. Section 4 contains the representation results, while Section 5 concludes. The proof of the main result is in the appendix.

2 Analytical Framework

Time is discrete, indexed by $t \in \mathscr{T} = \{0, 1, \ldots, T\}$, where T is finite. The decision maker is aware of this. Let the initial state of the world, which is known by the decision maker, be denoted by s_0. Let A be a finite, nonempty, set of basic actions with generic element a. The set of basic actions is available in each period, known by the decision maker, and remains fixed throughout. In contrast, the set of known feasible consequences evolves over time as the decision maker's awareness grows. Let $C(s_0)$ be the initial set of known feasible consequences, which is finite and nonempty.

For any set of consequences C, let c denote a generic element and define $x(C) = \neg C$ to be the abstract "consequence" that has the interpretation "none of the above" and captures consequences of which the decision maker is currently unaware. Also define $\widehat{C} = C \cup \{x(C)\}$ referred to as the set of extended consequences with generic element $\hat{c}$. Label by $\hat{c}^1, \hat{c}^2, \ldots$

the currently unknown consequences in order of discovery.

From a time-0 perspective, the only well-defined consequences are those in $C(s_0)$ and $x(C(s_0))$. From a time-0 perspective any yet undiscovered consequences $\hat{c}^1, \hat{c}^2, \ldots$ are all "none-of-the-above" and thus part of, or indistinguishable from, $x(C(s_0))$ and also indistinguishable from each other. However, the decision maker does know that when she has to make future choices, she may have discovered additional consequences.

As an intermediate construct to developing the state space, we first consider one-step-ahead resolutions of uncertainty and awareness. With this intermediate construct, the state space can then be defined recursively.

2.1 One-step-ahead resolutions of uncertainty

Define

$$S_1(s_0) \equiv (\widehat{C}(s_0))^A = \{s : A \to \widehat{C}(s_0)\},$$

i.e. the set of all functions from set of basic actions to the initial set of extended consequences. It depicts the possible resolutions of uncertainty at $t = 1$. This object is what was referred to as the augmented conceivable state space in Karni and Vierø (2017). It exhausts all the possible ways one can assign extended consequences to the basic actions. Define also the set

$$\tilde{S}_1(s_0) \equiv (C(s_0))^A = \{s : A \to C(s_0)\},$$

i.e. the set of functions from basic actions to the initial set of known consequences. In Karni and Vierø (2017) this was referred to as the subset of fully describable states. The elements of $S_1(s_0) \setminus \tilde{S}_1(s_0)$ are referred to as imperfectly describable, since their descriptions include the unknown consequence $x = x(C(s_0))$. A generic element of these sets is denoted by s_1. Example 1 provides an illustration.

Example 1 Consider the following situation in which there are two basic actions, $A = \{a_1, a_2\}$, and two feasible consequences, $C(s_0) = \{c_1, c_2\}$. The possible one-step-ahead resolutions of uncertainty are captured by the nine 'states' depicted in the matrix (1), where $x = x(C(s_0))$:

$$\begin{array}{cccccccccc} & s_1^1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 & s_1^6 & s_1^7 & s_1^8 & s_1^9 \\ a_1 & c_1 & c_2 & c_1 & c_2 & x & x & c_1 & c_2 & x \\ a_2 & c_1 & c_1 & c_2 & c_2 & c_1 & c_2 & x & x & x \end{array} \tag{1}$$

The subset of fully describable elements is $\tilde{S}_1(s_0) = \{s_1^1, \ldots, s_1^4\}$, while $\{s_1^5, \ldots, s_1^9\}$ are imperfectly describable.■

As it appears from Example 1 and matrix (1), the elements of $S_1(s_0)$ differ in how many previously unknown consequences will be discovered. In each of the fully describable elements $s_1^1, \ldots, s_1^4$, no new consequence is discovered. In each of elements $s_1^5, \ldots, s_1^8$, one new consequence is discovered, and in element s_1^9 two potentially different new consequences are discovered. The set of known feasible consequences at time 1 thus depends on what is discovered at time 1, i.e. it is a function of which 'state' is realized in the first period.

Define $n_1(s_1)$ as the number of previously unknown consequences discovered in s_1. Note that $n_1(s_1) \in \{0, \ldots, |A|\}$. Let $\{\hat{c}^i(s_1)\}_{i=1}^{n_1(s_1)}$ be the set of new consequences discovered in s_1, with $\{\hat{c}^i(s_1)\}_{i=1}^{n_1(s_1)} = \emptyset$ if $n_1(s_1) = 0$. Then the set of known feasible consequences at time 1 is given by

$$C(s_1) \equiv C(s_0) \cup \{\hat{c}^i(s_1)\}_{i=1}^{n_1(s_1)}.$$

Similar to the definition for the initial state, define $S_2(s_1) \equiv (\widehat{C}(s_1))^A$, that is, $S_2(s_1)$ depicts the possible one-step-ahead resolutions of uncertainty following s_1. Also define the subset of fully describable elements $\tilde{S}_2(s_1) \equiv (C(s_1))^A$ and let $S_2 \equiv \cup_{s_1 \in S_1} S_2(s_1)$, with generic element s_2.

Example 2 Consider a situation with two basic actions, $A = \{a_1, a_2\}$, and initially just one feasible consequence, $C(s_0) = \{c_1\}$. The possible first-period one-step-ahead resolutions of uncertainty are captured by the four 'states' in matrix (2) below, where again $x = x(C(s_0))$:

$$\begin{array}{ccccc} & s_1^1 & s_1^2 & s_1^3 & s_1^4 \\ a_1 & c_1 & c_1 & x & x \\ a_2 & c_1 & x & c_1 & x \end{array} \tag{2}$$

In s_1^1, no new consequence is discovered. Hence, $C(s_1^1) = C(s_0)$, and $S_2(s_1^1) = S_1(s_0)$, i.e. as depicted in (2). In s_1^2, one new consequence, $\hat{c}^1(s_1^2)$, is discovered. Therefore, $C(s_1^2) = C(s_0) \cup \{\hat{c}^1(s_1^2)\}$, and $S_2(s_1^2) = (\widehat{C}(s_1^2))^A$ consists of 9 elements as depicted in matrix (3) below, where $x = x(C(s_1^2))$ and $\hat{c}^1 = \hat{c}^1(s_1^2)$:

$$\begin{array}{cccccccccc} & s_2^1 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 & s_2^7 & s_2^8 & s_2^9 \\ a_1 & c_1 & c_1 & \hat{c}_1 & \hat{c}_1 & c_1 & \hat{c}_1 & x & x & x \\ a_2 & c_1 & \hat{c}_1 & c_1 & \hat{c}_1 & x & x & c_1 & \hat{c}_1 & x \end{array} \tag{3}$$

The situation if s_1^3 is realized is similar to that if s_1^2 is realized, except that the consequence $\hat{c}^1(s_1^3)$ that is discovered in s_1^3 could be different from that which would be discovered if s_1^2 were realized. Since $\hat{c}^1(s_1^3)$ is potentially different from $\hat{c}^1(s_1^2)$, the sets $C(s_1^2)$ and $C(s_1^3)$ are potentially different, as are the entities derived from these sets. Importantly, from an ex-ante perspective, the decision maker cannot distinguish between different such unknown consequences, since she is unaware of their attributes. However, she can reason, like we just did, that they can potentially be different. So although $S_2(s_1^3)$ is also as depicted in (3), with x and $\hat{c}^1$ appropriately redefined, the decision maker can envision that the situation may be different than that following s_1^2.

In s_1^4, two new consequences $\hat{c}^1(s_1^4)$ and $\hat{c}^2(s_1^4)$ are discovered. It could be that $\hat{c}^1(s_1^4) = \hat{c}^2(s_1^4)$, but from an ex-ante perspective using distinct $\hat{c}^1(s_1^4)$ and $\hat{c}^2(s_1^4)$ allows the decision maker to formulate the maximal increase in awareness that she can anticipate. Then $C(s_1^4) = C(s_0) \cup \{\hat{c}^1(s_1^4), \hat{c}^2(s_1^4)\}$ and $S_2(s_1^4) = (\widehat{C}(s_1^4))^A$ consists of 16 elements as in matrix (4), where $x = x(C(s_1^4))$, and $(\hat{c}^1, \hat{c}^2) = (\hat{c}^1(s_1^4), \hat{c}^2(s_1^4))$:

$$\begin{array}{ccccccccccccccccc} & s_2^1 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 & s_2^7 & s_2^8 & s_2^9 & s_2^{10} & s_2^{11} & s_2^{12} & s_2^{13} & s_2^{14} & s_2^{15} & s_2^{16} \\ a_1 & c_1 & c_1 & c_1 & \hat{c}_1 & \hat{c}_1 & \hat{c}_1 & \hat{c}_2 & \hat{c}_2 & \hat{c}_2 & c_1 & \hat{c}_1 & \hat{c}_2 & x & x & x & x \\ a_2 & c_1 & \hat{c}_1 & \hat{c}_2 & c_1 & \hat{c}_1 & \hat{c}_2 & c_1 & \hat{c}_1 & \hat{c}_2 & x & x & x & c_1 & \hat{c}_1 & \hat{c}_2 & x \end{array} \tag{4}$$

The total number of elements in $S_2 = \cup_{s_1 \in S_1} S_2(s_1)$ is 4+9+9+16=38.$\blacksquare$

In general, for $t > 0$, define $n_t(s_t)$ as the number of previously unknown consequences discovered in s_t. Let $\{\hat{c}^i(s_t)\}_{i=1}^{n_t(s_t)}$ be the set of new consequences discovered in s_t, with $\{\hat{c}^i(s_t)\}_{i=1}^{n_t(s_t)} = \emptyset$ if $n_t(s_t) = 0$. Then the set of known feasible consequences in s_t is given by

$$C(s_t) \equiv C(s_{t-1}) \cup \{\hat{c}^i(s_t)\}_{i=1}^{n_t(s_t)}.$$

Define

$$S_{t+1}(s_t) \equiv (\widehat{C}(s_t))^A,$$

which depicts the possible one-step-ahead resolutions of uncertainty following s_t. Define the subset of fully describable elements $\tilde{S}_{t+1}(s_t) \equiv (C(s_t))^A$, and define $S_{t+1} \equiv \cup_{s_t \in S_t} S_{t+1}(s_t)$, with generic element s_{t+1}.

2.2 The state space

The state space can be depicted by an event tree, albeit non-standard. Define a time-t state ω_t by

$$\omega_t \equiv (s_0, s_1, \ldots, s_t),$$

where $s_\tau \in S_\tau(s_{\tau-1})$ for all $\tau \in \{1, \ldots, t\}$. Thus, a time-$t$ state gives the path through the event tree up to and including time t. Define, for all $t \in \{0, \ldots, T\}$,

$$\Omega_t \equiv \{\omega_t = (s_0, s_1, \ldots, s_t) : \; s_\tau \in S_\tau(s_{\tau-1}) \; \forall \, \tau = 1, \ldots, t\} \tag{5}$$

which is referred to as the time-t state space. The time-t state space is the set of all possible evolutions of the decision maker's awareness and uncertainty up to and including time t as she can describe them given her awareness at time 0.

Define the full state space $\mathbf{\Omega}$ by

$$\mathbf{\Omega} \equiv \bigcup_{t=0}^{T} \Omega_t.$$

Thus, the full state space $\mathbf{\Omega}$ is the set of all states at all times, i.e. the set of all partial and complete paths through the event tree. Define also

$$\Omega \equiv \bigcup_{t=0}^{T-1} \Omega_t. \tag{6}$$

This notation for $\mathbf{\Omega} \setminus \Omega_T$ is convenient because the ultimate period is different from the rest, which will be made precise in subsection 2.3.

Example 2 (continued) The event tree for the situation with $A = \{a_1, a_2\}$, $C(s_0) = \{c_1\}$, and T=3 is depicted in Figure 1. The numbers after each time-2 state is the number of branches originating at that state, and thus give the number of possible one-step-ahead resolutions of uncertainty following that second-period state.■

Note that while n, C, S, etcetera are defined recursively one step ahead as functions of s_t, they can also be described as functions of ω_t: $n_t(\omega_t)$, $C(\omega_t)$, $S_{t+1}(\omega_t)$, $\tilde{S}_{t+1}(\omega_t)$, and $S_{t+1} = \cup_{\omega_t \in \Omega_t} S_{t+1}(\omega_t)$. Define, for $\tau \in \{t, \ldots, T\}$,

$$\Omega_\tau(\omega_t) \equiv \{\omega_\tau = (\omega_t, s_{t+1}, \ldots, s_\tau) : s_{t+1} \in S_{t+1}(\omega_t) \text{ and } s_{\hat{t}} \in S_{\hat{t}}(s_{\hat{t}-1}) \; \forall \; \hat{t} = 2, \ldots, \tau\}.$$

This is the set of time-τ states that can be reached from state ω_t, or, in other words, the possible continuation paths through time τ, starting from state ω_t. Also, define

$$\mathbf{\Omega}(\omega_t) \equiv \bigcup_{\tau=t}^{T} \Omega_\tau(\omega_t),$$

which is the set of all partial and full continuation paths from ω_t, starting at ω_t, and

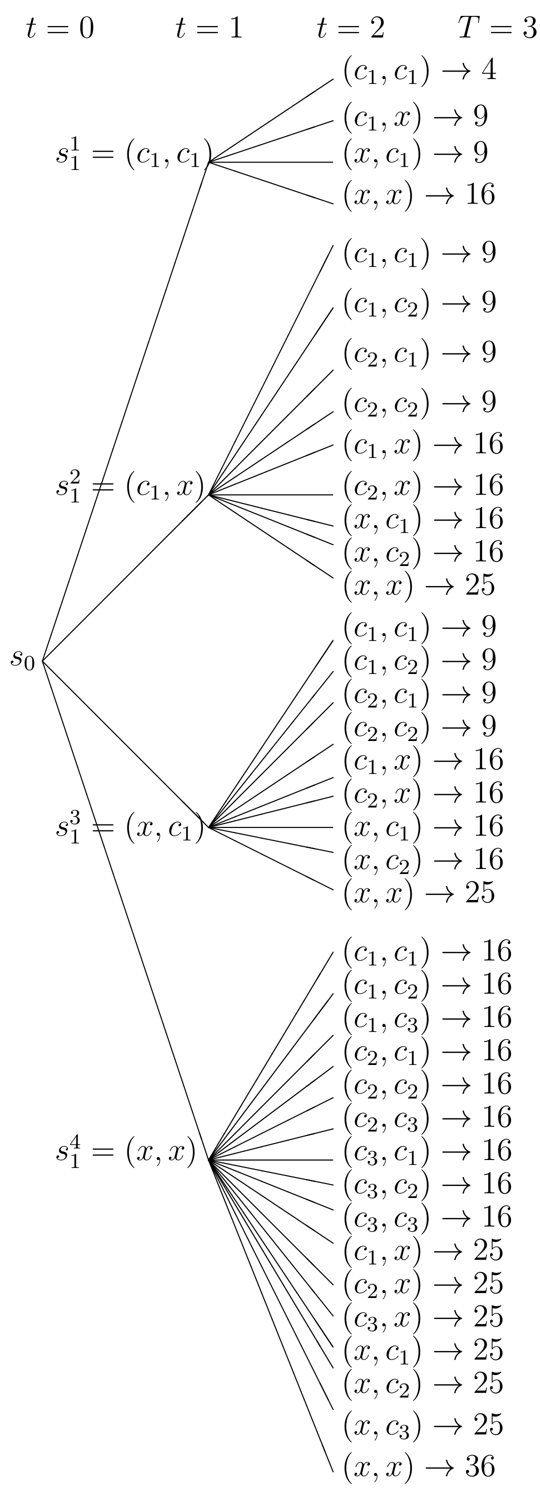
$$\Omega(\omega_t) \equiv \bigcup_{\tau=t}^{T-1} \Omega_\tau(\omega_t), \tag{7}$$

which differs from $\mathbf{\Omega}(\omega_t)$ by excluding the last period.

One can also define

$$S_\tau(\omega_t) \equiv \bigcup_{\substack{s_{t+1} \in S_{t+1}(\omega_t) \\ \vdots \\ s_{\tau-1} \in S_{\tau-1}(s_{\tau-2})}} S_\tau(s_{\tau-1})$$

Figure 1: Full state space for Example 2



for $\tau \in \{t+1, \ldots, T\}$. This is the time-$\tau$ part of $\Omega_\tau(\omega_t)$, describing the possible one-step-ahead resolutions of uncertainty from time $\tau - 1$ to time τ when the current state is ω_t. For example, $S_{t+2}(\omega_t) = \bigcup\limits_{s_{t+1} \in S_{t+1}(\omega_t)} S_{t+2}(s_{t+1})$.

If we increase T, A, or $C(s_0)$, the number of states, and hence the possible evolutions of awareness, quickly becomes very large. If the initial set of outcomes is $C_0 = C(s_0)$, there are (abusing notation and also letting the notation for a set denote the number of elements in the set) $(C_0 + 1)^A$ first-period states. Suppressing the variables' dependency on states, there are

$$\sum_{n_1=0}^{A} \binom{A}{n_1} (C_0)^{A-n_1} (C_0 + n_1 + 1)^A$$

time-2 states and

$$\sum_{n_1=0}^{A} \binom{A}{n_1} (C_0)^{A-n_1} \sum_{n_2=0}^{A} \binom{A}{n_2} (C_0 + n_1)^{A-n_2} (C_0 + n_1 + n_2 + 1)^A$$

time-3 states. In comparison, in a standard model without increases in awareness, there would be $(S_1)^t$ time-t states if there were S_1 time-1 states. Thus, due to the possible expansions in the decision maker's awareness, the number of states grows much more rapidly here.

Example 2 (continued) When $A = \{a_1, a_2\}$ and $C(s_0) = \{c_1\}$, there are 4 time-1 states, 38 time-2 states, and 618 time-3 states. In a standard model with 4 time-1 states, there would be 16 time-2 states and 64 time-3 states.■

Let $C(\omega_t)^{+n} = C(\omega_t) \cup \{\hat{c}^1, \ldots, \hat{c}^n\}$ and define the functions $\gamma(n, A, C)$ by $\gamma(n, A, C) = \binom{A}{n}(C)^{A-n}$ and $C(\{n_j\}, \omega_t, \tau) = C(\omega_t)^{+\sum_{j=1}^{\tau} n_j}$. From the point of view of state ω_t, the set of possible resolutions of uncertainty at time τ is given by

$$S_\tau(\omega_t) = \bigcup_{n_{t+1}=0}^{A} \bigcup_{i_1=1}^{\gamma(n_{t+1}, A, C(\omega_t))} \bigcup_{n_{t+2}=0}^{A} \bigcup_{i_2=1}^{\gamma(n_{t+2}, A, C(\omega_t)^{+n_{t+1}})} \cdots \bigcup_{n_\tau=0}^{A} \bigcup_{i_t=1}^{\gamma(n_\tau, A, C(\{n_j\}, \omega_t, \tau))} (\widehat{C}(\{n_j\}, \omega_t, \tau))^A.$$

Many of these states, as well as the consequences the decision maker can obtain in them, are indescribable beyond "there may be a number of currently unknown consequences that I could potentially have discovered by then" from her current point of view. In other words, future acts are not even fully describable with respect to the decision maker's current awareness. An implication is that even constant future acts are not necessarily measurable with respect to the decision maker's current awareness, since they may involve consequences she does not yet know. In contrast, in a standard model without unawareness, the only source of non-measurability is the uncertainty about states, and hence constant acts are always measurable with respect to current information. Some of the axioms that will be imposed on preferences have the role of "collapsing" states and unknown outcomes in a way that keeps the world from exploding and addresses the additional measurability issues that arise from some future consequences being indestinguishable given the current level of awareness.

The framework introduced above captures the important aspects of the problem of awareness of unawareness with long time horizons, namely that there is a plethora of ways that awareness can evolve both in terms of how much, when, and in which order. The framework does so in a systematic way that generalizes the standard approach of using event trees. Furthermore, it is a natural extension of the state spaces in Karni and Vierø (2017).

2.3 Conceivable acts

Since this paper uses the revealed preference methodology, it is a requirement that for a given level of awareness bets can be both meaningfully described using current language and settled once uncertainty has been resolved. Define

$$f(\omega_0) : S_1(\omega_0) \to \Delta(\widehat{C}(\omega_0)) \text{ such that } f(\omega_0)(s) \in \Delta(C(\omega_0)) \text{ for all } s \in \tilde{S}_1(\omega_0), \quad (8)$$

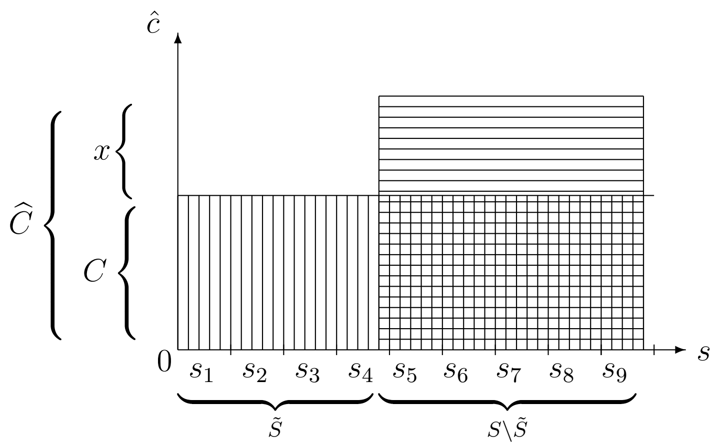
where $\Delta(\cdot)$ denotes the probability simplex.[2] I.e. $f(\omega_0)$ is a function from $S_1(\omega_0)$ into the set of lotteries over the time 0 set of extended consequences for which the range in the fully describable 'states' is restricted to lotteries over the known feasible consequences. See Figure 2 for an illustration. The acts defined in (8) are referred to as restricted Anscombe-Aumann acts.

The reason for the range being restricted in the fully describable 'states' is the requirement that bets should be possible to settle once uncertainty resolves, and that decision makers cannot meaningfully form preferences over acts that assign indescribable consequences to fully describable 'states'. In fully describable 'states', the consequence x remains abstract, and one cannot deliver a consequence that has not yet been discovered. However, there is no problem with promising to deliver a consequence, which is none of the prior consequences, if such a consequence is discovered. Therefore, the acts can assign, to imperfectly describable 'states' only, consequences that will be discovered if these 'states' obtain.[3] As a result, the support of the lotteries in the restricted Anscombe-Aumann acts is L-shaped across 'states',

[2] The usual abuse of notation is adopted, where c is also used to denote the lottery that returns consequence c with probability 1.

[3] For further discussion of this issue, see Karni and Vierø (2017).

Figure 2: Illustration of the support of the lotteries in the restricted Anscombe-Aumann acts



rather than rectangular like the standard Anscombe and Aumann (1963) acts, as Figure 2 shows.

In general, for $\omega_t \in \Omega$, define similar restricted Anscombe-Aumann acts using the set of feasible consequences for state ω_t:

$$f(\omega_t) : S_{t+1}(\omega_t) \to \Delta(\widehat{C}(\omega_t)) \text{ such that } f(\omega_t)(s) \in \Delta(C(\omega_t)) \text{ for all } s \in \tilde{S}_{t+1}(\omega_t). \quad (9)$$

The acts defined in (9) are restricted Anscombe-Aumann acts originating in state ω_t, with the supports of the lotteries restricted to the set of known consequences in the ω_t-fully describable elements of $S_{t+1}(\omega_t)$. They are one-step-ahead acts in the sense that the uncertainty regarding them will be resolved at the end of the current period. Let

$$F(\omega_t) \equiv \{f(\omega_t)\}.$$

This is the set of all restricted Anscombe-Aumann acts originating in state ω_t, defined in (9).

Define

$$f \equiv (f(\omega_t))_{\omega_t \in \Omega}, \quad (10)$$

with Ω defined in (6). The acts defined in (10) are intertemporal acts, consisting of a one-step-ahead act as defined in (9) for each state, that is, for each point in the event tree. Thus, at each point in time (and in each state), two things happen: the uncertainty regarding the previous period's one-step-ahead act $f(\omega_{t-1})$ resolves and a new, current, one-step-ahead act $f(\omega_t)$ may be chosen. The last period differs, since no new one-step-ahead act is chosen. In (9), the notation ω_t is used to denote the originating state and s is used to denote the next period states in which the payoff of the one-step-ahead restricted Anscombe-Aumann act materializes.[4] This convention will be adopted throughout the paper.

The set of all intertemporal acts can now be defined:

$$F \equiv \{f = (f(\omega_t))_{\omega_t \in \Omega}\}. \tag{11}$$

The intertemporal acts reflect that from a time-0 perspective, the only well-defined consequences are those in $C(s_0)$ and $x(C(s_0))$, but that the decision maker knows that when she has to make future choices, she may have discovered additional consequences. The set of all intertemporal acts, defined in (11), is the domain of the decision maker's preferences.

For $E \subseteq S_{t+1}(\omega_t)$, let $\hat{p}_E f$ be the intertemporal act that returns the lottery $\hat{p}$ in all states in the event E and agrees with f elsewhere. Also, let $h_{\omega_t} f$ be the intertemporal act obtained from f by replacing the restricted Anscombe-Aumann act originating at ω_t by $h \in F(\omega_t)$. The act $\hat{p}_E f$ is thus a special case of $h_{\omega_t} f$ for which h agrees with $f(\omega_t)$ for $s \in S_{t+1}(\omega_t) \setminus E$ and is constant at $\hat{p}$ for $s \in E$.

For all $f \in F$, define

$$H_{\omega_t}(f) \equiv \{h_{\omega_t} f | h \in F(\omega_t)\},$$

which is the set of intertemporal acts that agree with f with the exception of the restricted Anscombe-Aumann act originating at ω_t. Also define

$$H_{\Omega_t}(f) \equiv \{h_{\Omega_t} f | h(\omega_t) \in F(\omega_t)\ \forall \omega_t \in \Omega_t\},$$

with Ω_t defined in (5). This is the set of intertemporal acts whose restricted Anscombe-Aumann acts originating at all other times than t agree with f. Finally, define

$$\mathscr{F}_{\omega_t} \equiv \{f \in F | f(\omega_\tau)(s) \in \Delta(C(\omega_t))\ \forall \omega_\tau \in \Omega(\omega_t), \forall s \in S_{\tau+1}(\omega_\tau)\}$$

This is the set of intertemporal acts for which the support of all lotteries in the continuation path from ω_t is restricted to $C(\omega_t)$. Hence, it is the set of intertemporal acts that are measurable with respect to the ω_t-level of awareness.

3 Preferences

The decision maker has a preference ordering on F at any state $\omega_t \in \Omega$, denoted by $\succsim_{\omega_t}$, which expresses the ordering conditional on the awareness level prevailing given the cumulative

[4] Here and henceforth, the term "each state" is used to refer to all states but the ultimate-period states. In order to keep the exposition clean, the distinction of ultimate states will not be mentioned, except when it is directly relevant.

discoveries made in state ω_t. Strict preference $\succ_{\omega_t}$ and indifference $\sim_{\omega_t}$ are defined as usual. Axioms will be imposed on the collection of preference orderings $\{\succsim_{\omega_t}\colon \omega_t \in \Omega\}$. It is henceforth assumed that $C(\omega_0)$ contains at least two elements.

3.1 Axioms

Axiom 1 states that only continuations of acts matter for preferences. Thus, at any point in time and at any state, the decision maker does not care about parts of the event tree that cannot be reached from her current position.

Axiom 1 (Conditional Preference). *For all $\omega_t \in \Omega$, for all $f, f' \in F$, if $f(\omega_\tau) = f'(\omega_\tau)$ for all $\omega_\tau \in \Omega(\omega_t)$ then $f \sim_{\omega_t} f'$.*

Axioms 2 through 5 are, like Axiom 1, imposed on preferences at any time and in any state. They resemble the axioms in Karni and Vierø (2017) that result in their generalized expected utility representation, although the present domains are different than in Karni and Vierø (2017). Axiom 2 contains the standard expected utility axioms.

Axiom 2 (Expected Utility). *For all $\omega_t \in \Omega$,*

(i) (Preorder) the relation $\succ_{\omega_t}$ is asymmetric and negatively transitive on F.

(ii) (Archimedian) for all $h, h', h'' \in F$, if $h \succ_{\omega_t} h'$ and $h' \succ_{\omega_t} h''$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha h + (1-\alpha)h'' \succ_{\omega_t} h'$ and $h' \succ_{\omega_t} \beta h + (1-\beta)h''$.

(iii) (Independence) for all $h, h', h'' \in F$ and for all $\alpha \in (0,1]$, $h \succ_{\omega_t} h'$ if and only if $\alpha h + (1-\alpha)h'' \succ_{\omega_t} \alpha h' + (1-\alpha)h''$.

In Axiom 3 below, the content of parts (i) and (ii) are similar to the standard content of monotonicity, but the statement differs. The difference in statement is necessary because the support of the lotteries in fully describable 'states' is restricted to the set of known consequences, while in the imperfectly describable 'states', the lotteries can involve the unknown consequence that will be discovered. That is, across one-step-ahead 'states' the support of the lotteries is L-shaped rather than rectangular, as Figure 2 illustrates, which necessitates the statement of monotonicity as in Axiom 3. Part (iii) extends monotonicity to also hold for lotteries that occur at different points in time.[5]

Axiom 3 (Monotonicity). *For all $\omega_t \in \Omega$,*

[5] In part (iii), the notation p is abused to denote the restricted Anscombe-Aumann act for which $f(\omega_\tau) = p$ for all $s \in S_{\tau+1}(\omega_\tau)$.

(i) for all $\omega_\tau \in \Omega(\omega_t)$ and $\succ_{\omega_t}$-nonnull $s \in \tilde{S}_{\tau+1}(\omega_\tau)$, for all $p, q \in \Delta(C(\omega_\tau))$, and for all $f \in F$ it holds that $p_s f \succ_{\omega_t} q_s f$ if and only if $p_{S_{\tau+1}(\omega_\tau)} f \succ_{\omega_t} q_{S_{\tau+1}(\omega_\tau)} f$.

(ii) for all $\omega_\tau \in \Omega(\omega_t)$ and $\succ_{\omega_t}$-nonnull $s \in S_{\tau+1}(\omega_\tau) \setminus \tilde{S}_{\tau+1}(\omega_\tau)$, for all $\hat{p}, \hat{q} \in \Delta(\widehat{C}(\omega_\tau))$, and for all $f \in F$ it holds that $\hat{p}_s f \succ_{\omega_t} \hat{q}_s f$ if and only if $\hat{p}_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} f \succ_{\omega_t} \hat{q}_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} f$.

(iii) for all $\omega_\tau \in \Omega(\omega_t)$, for all $p, q \in \Delta(C(\omega_t))$, and for all $f \in F$ it holds that $p_{\omega_\tau} f \succ_{\omega_t} q_{\omega_\tau} f$ if and only if $p_{\Omega(\omega_t)} f \succ_{\omega_t} q_{\Omega(\omega_t)} f$.

Axiom 4 requires non-triviality of each preference relation $\succ_{\omega_t}$ on sets of acts that only differ in one future (or in the current) state. It implies that no state in the continuation path is $\succ_{\omega_t}$-null.

Axiom 4 (Nontriviality)**.** *For all $f \in F$, and for all $\omega_t \in \Omega$, the strict preference relation $\succ_{\omega_t}$ is non-empty on $H_{\omega_\tau}(f)$ for all $\omega_\tau \in \Omega(\omega_t)$.*

Axiom 5 regards intertemporal acts that only differ in the restricted Anscombe-Aumann act originating in a particular future (or in the current) state. Furthermore, those restricted Anscombe-Aumann acts only differ on the imperfectly describable 'states' that follow and are constant on that set of 'states'. The axiom requires that the ranking of such intertemporal acts is independent of the aspects on which the acts agree. This separability is not implied by Independence, since the the payoff $x(C(\omega_\tau))$ is not defined on $\tilde{S}_{\tau+1}(\omega_\tau)$.

Axiom 5 (Separability)**.** *For all $\omega_t \in \Omega$, for all $f, g \in F$, for all $\omega_\tau \in \Omega(\omega_t)$, and for all $\hat{p}, \hat{q} \in \Delta(\widehat{C}(\omega_\tau))$, it holds that $\hat{p}_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} f \succ_{\omega_t} \hat{q}_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} f$ if and only if $\hat{p}_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} g \succ_{\omega_t} \hat{q}_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} g$.*

The following axioms connect preferences across different levels of awareness. Axiom 6 requires that the attitude towards known risks is invariant over time and levels of awareness, both for acts that differ in a single period and in two successive periods. To state the axiom, define

$$L_{\omega_t}(f) = \{h_{\Omega(\omega_t)} f \mid h(\omega_\tau)(s) = l_\tau \in \Delta(C(\omega_t)) \text{ for all } \omega_\tau \in \Omega_\tau(\omega_t),\ s \in S_{\tau+1}(\omega_\tau) \text{ and } \tau \geq t\},$$

and

$$L_{\omega_t}(F) = \bigcup_{f \in F} L_{\omega_t}(f).$$

The objects in $L_{\omega_t}(F)$ return the same lottery, with support being a subset of $\Delta(C(\omega_t))$, in each state at time $\tau + 1$ for all $\tau \geq t$, but can return different lotteries at different times.

Hence, $L_{\omega_t}(F)$ is a subset of F that involves risk but no subjective uncertainty, and only involves currently known consequences. Therefore, $L_{\omega_t}(F) \subset \mathscr{F}_{\omega_t}$.[6]

Axiom 6 (Time- and Awareness-Invariant Risk Preferences)**.** *For all* $l \in L_{\omega_t}(F)$*, for all* $p, p', q, q' \in \Delta(C(\omega_t))$*, if for some* $\omega_{\hat{t}} \in \Omega(\omega_t)$*, and* $\tau \geq \hat{t}$ *it is true that* $p_{\Omega_\tau} p'_{\Omega_{\tau+1}} l \succsim_{\omega_{\hat{t}}} q_{\Omega_\tau} q'_{\Omega_{\tau+1}} l$*, then it is true for every* $\omega_{\hat{t}} \in \Omega(\omega_t)$*, and* $\tau \geq \hat{t}$*.*

Axiom 6 contains elements that concern preferences within an awareness level as well as elements that link preferences across awareness levels. The part that links preferences is stronger than the Invariant Risk Preferences axiom from Karni and Vierø (2017), since it also applies for acts that differ across two successive periods. This was beyond the scope of the framework in Karni and Vierø (2017).[7]

To state the next axiom, the following notation is introduced: For all $\omega_t \in \Omega$ and for all $s_{t+1}, \tilde{s}_{t+1} \in S_{t+1}(\omega_t)$, define the event $\mathscr{E}_{t+2}(\tilde{s}_{t+1}|(\omega_t, s_{t+1}))$ by

$$\begin{aligned}\mathscr{E}_{t+2}(\tilde{s}_{t+1}|(\omega_t, s_{t+1})) \equiv \big\{ & s_{t+2} \in S_{t+2}(\omega_t, s_{t+1}) : \forall a \in A, \text{ if } a(\tilde{s}_{t+1}) \in C(\omega_t) \text{ then } a(s_{t+2}) \\ = a(\tilde{s}_{t+1}) & \text{ and if } a(\tilde{s}_{t+1}) \notin C(\omega_t) \text{ then } a(s_{t+2}) \in \{x(C(\omega_t, s_{t+1}))\} \cup (C(\omega_t, s_{t+1}) \setminus C(\omega_t)).\end{aligned} \tag{12}$$

Definition (12) maps fully describable states into degenerate events and imperfectly describable states into non-degenerate events. The definition can be illustrated using matrices (2) and (3) from Example 2. There, $\mathscr{E}_{t+2}(s_1^1|(\omega_0, s_1^2)) = s_2^1$, $\mathscr{E}_{t+2}(s_1^2|(\omega_0, s_1^2)) = \{s_2^2, s_2^5\}$, $\mathscr{E}_{t+2}(s_1^3|(\omega_0, s_1^2)) = \{s_2^3, s_2^7\}$, and $\mathscr{E}_{t+2}(s_1^4|(\omega_0, s_1^2)) = \{s_2^4 . s_2^6, s_2^8, s_2^9\}$.

Fix two outcomes $c^*, c_* \in C(\omega_0)$ for which $c^*_{\omega_0} f \succ_{\omega_0} c_{*\omega_0} f$ for some[8] $f \in F$. Such two outcomes exist by Axiom 4.

Axiom 7 requires that the ranking of subjective versus objective uncertainty τ periods ahead is independent of the level of detail with which the subjective uncertainty can be described. This is imposed for all future states and the corresponding events following immediately after.

Axiom 7 (Forward Awareness Consistency)**.** *For all* $f \in F$*, for all* $\omega_t \in \Omega$*, for all* $s_{t+1} \in S_{t+1}(\omega_t)$*, for all* $\omega_\tau \in \Omega(\omega_t, s_{t+1}) \cup \{\omega_t\}$*, for all* $s_{\tau+1}, \tilde{s}_{\tau+1} \in S_{\tau+1}(\omega_\tau)$*, for all* $h, g \in H_{\omega_\tau}(f)$*,*

[6] In axiom 6, the notation p is again abused to denote the restricted Anscombe-Aumann act for which $f(\omega_\tau) = p$ for all $s \in S_{\tau+1}(\omega_\tau)$.

[7] Dominiac and Tserenjigmid (2017b) show that in Karni and Vierø (2013), the invariant risk preferences axiom is implied by the other axioms. It is not clear whether this would also be the case with awareness of unawareness. Also, in the present context the axiom is necessary for acts that differ across two successive periods.

[8] and hence, given the axioms, all

and for all $h', g' \in H_{(\omega_\tau, s_{\tau+1})}(f)$, *if*

$$g = (\eta c^* + (1-\eta)c_*)_{S_{\tau+1}(\omega_\tau)} f, \qquad h = c^*_{\tilde{s}_{\tau+1}} c_{*S_{\tau+1}(\omega_\tau)} f,$$
$$g' = (\eta c^* + (1-\eta)c_*)_{S_{\tau+2}(\omega_\tau, s_{\tau+1})} f, \quad h' = c^*_{\mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1}|(\omega_\tau, s_{\tau+1}))} c_{*S_{\tau+2}(\omega_\tau, s_{\tau+1})} f,$$

then $h \succsim_{\omega_t} g$ *if and only if* $h' \succsim_{(\omega_t, s_{t+1})} g'$.

Axiom 7 contains elements of both Awareness Consistency I and II in Karni and Vierø (2017) and also implies a non-increasing sense of unawareness. It ensures consistency of preferences when looking forward. It is not necessarily reasonable to impose such a requirement looking backwards, since the decision maker's awareness may have reached a higher level. Thus, looking backwards, there are things that the decision maker can take into consideration that she was not able to take into consideration previously. However, looking forward, Axiom 7 requires that preferences will be consistent regarding the currently know and well-understood part of the decision maker's universe.[9]

As the name suggests, Axiom 8 requires that the decision maker treats all unknowns as such. She does not a-priori distinguish between, for example, unknowns to be discovered in different states or at different times. Anything that can not be described or imagined at her current state of awareness is treated the same way by the decision maker. This does not preclude that she will have a preference for when to make such discoveries.

When setting $\hat{c} = x(C(\omega_\tau))$, Axiom 8 states that from her current point of view, the decision maker is indifferent between getting, at time $\tau + 1$, a consequence that she cannot describe using her current language, but may be able to describe at time τ, and a consequence that she will still not be able to describe with her time-τ language. Since $\hat{c}$ can also be any consequence discovered between times t and τ, Axiom 8 also states that from her current point of view, the decision maker is indifferent between getting, at time $\tau + 1$, different consequences that she cannot currently describe.

Axiom 8 (Unknowns are Unknowns). *For all* $f \in F$, *for all* $\omega_t \in \Omega$, *for all* $\omega_\tau \in \Omega(\omega_t)$, *and for all* $\hat{c} \in \widehat{C}(\omega_\tau) \setminus C(\omega_t)$, $x(C(\omega_t))_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} f \sim_{\omega_t} \hat{c}_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} f$.

Axiom 9 states that the decision maker's attitude towards the unknown is invariant to her level of awareness. She does not become more fearful or excited towards the unknown as her awareness evolves. Part (i) states that the decision maker's current attitude towards the

[9]There may be situations in which axiom 7 is too strong. For example, the decision maker could become ambiguity averse in response to increases in awareness. Such a possibility is investigated in Dominiak and Tsjerengjimid (2017a) for a one-shot increase in awareness and the decision maker being myopic with respect to his unawareness. It is far from clear how ambiguity aversion would interplay with awareness of unawareness or with the long time horizon.

unknown is independent of which future state she is considering. Part (ii) states that the attitude towards the unknown remains unchanged as the decision maker's awareness grows.

Axiom 9 (Constant Attitude Towards the Unknown). *For all $f \in F$ and for all $\omega_t \in \Omega$,*

(i) if $x(C(\omega_t))_{S_{t+1}(\omega_t)\setminus\tilde{S}_{t+1}(\omega_t)} f \sim_{\omega_t} (\alpha c^* + (1-\alpha)c_*)_{S_{t+1}(\omega_t)\setminus\tilde{S}_{t+1}(\omega_t)} f$ then $x(C(\omega_t))_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} f \sim_{\omega_t} (\alpha c^* + (1-\alpha)c_*)_{S_{\tau+1}(\omega_\tau)\setminus\tilde{S}_{\tau+1}(\omega_\tau)} f$ for all $\omega_\tau \in \Omega(\omega_t)$.

(ii) if $x(C(\omega_t))_{S_{t+1}(\omega_t)\setminus\tilde{S}_{t+1}(\omega_t)} f \sim_{\omega_t} (\alpha c^* + (1-\alpha)c_*)_{S_{t+1}(\omega_t)\setminus\tilde{S}_{t+1}(\omega_t)} f$ then $x(C(\omega_t, s_{t+1}))_{S_{t+2}(\omega_t,s_{t+1})\setminus\tilde{S}_{t+2}(\omega_t,s_{t+1})} f \sim_{(\omega_t,s_{t+1})} (\alpha c^* + (1-\alpha)c_*)_{S_{t+2}(\omega_t,s_{t+1})\setminus\tilde{S}_{t+2}(\omega_t,s_{t+1})} f$ for all $s_{t+1} \in S_{t+1}(\omega_t)$

4 Representation

Theorem 1 provides a representation of preferences over intertemporal acts at each event and awareness level. It also connects preferences, through connecting utilities and beliefs, across events and awareness levels. To facilitate reading the theorem, keep the following notation in mind: In the statement of Theorem 1, ω_t is the current state, ω_τ is used to denote the state in which a restricted Anscombe-Aumann act originates, and s indexes the states in which the uncertainty regarding the restricted Anscombe-Aumann act resolves. The proof of Theorem 1 is in the appendix.

Theorem 1. *The following statements are equivalent:*

(a) $\{\succsim_{\omega_t}\}_{\omega_t\in\Omega}$ satisfy Axioms 1 through 9.

(b) For all $\omega_t \in \Omega$, there exist a real-valued, continuous, non-constant, Bernoulli-utility function u_{ω_t} on $C(\omega_t)$ and a parameter $u^*_{\omega_t}$, unique probability measures π_{ω_t} on $\mathbf{\Omega}$ with $\pi_{\omega_t}(\omega) = 0$ if $\omega \notin \mathbf{\Omega}(\omega_t)$ and $\pi_{\omega_t}(\omega) > 0$ for all $\omega \in \Omega(\omega_t)$, and $\beta > 0$ such that for every ω_t, $\succsim_{\omega_t}$ is represented by $V_{\omega_t}(\cdot)$, where

$$V_{\omega_t}(f) = \sum_{\tau=t}^{T-1} \beta^{\tau-t} \sum_{\omega_\tau\in\Omega_\tau(\omega_t)} \sum_{s\in S_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(\omega_\tau, s) \left(\sum_{c\in C(\omega_t)} f(\omega_\tau)(s)(c) u_{\omega_t}(c) + \Big(1 - \sum_{c\in C(\omega_t)} f(\omega_\tau)(s)(c)\Big) u^*_{\omega_t} \right). \tag{13}$$

The function u_{ω_t} is unique up to positive linear transformations, and the parameter $u^*_{\omega_t} = u^*$ for all ω_t. Also, for all $c \in C(\omega_t)$, $u_{\omega_\tau}(c) = u_{\omega_t}(c)$ for all $\omega_\tau \in \Omega(\omega_t)$. The *probability measures π_{ω_t} satisfy that for all $s_{t+1} \in S_{t+1}(\omega_t)$, for all $\omega_\tau \in \Omega(\omega_t, s_{t+1}) \cup \{\omega_t\}$, and for all $s_{\tau+1}, \tilde{s}_{\tau+1}, \bar{s}_{\tau+1} \in S_{\tau+1}(\omega_\tau)$ we have that*

$$\frac{\pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})}{\pi_{\omega_t}(\omega_\tau, \bar{s}_{\tau+1})} = \frac{\pi_{(\omega_t, s_{t+1})}(\omega_\tau, s_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1}|(\omega_\tau, s_{\tau+1}))}{\pi_{(\omega_t, s_{t+1})}(\omega_\tau, s_{\tau+1}, \mathscr{E}_{\tau+2}(\bar{s}_{\tau+1}|(\omega_\tau, s_{\tau+1}))}. \tag{14}$$

The representation of preferences over intertemporal acts in (13) has the following form: When finding herself in state ω_t, the decision maker acts as if she computes subjective expected utility over states using her ω_t-beliefs and computes the discounted sum of utilities using the time and state invariant discount factor β. The utility functions u_{ω_t} are time and state, and thus awareness, invariant for consequences that are common to the states. The parameter $u^*_{\omega_t}$ reflects the decision maker's attitude towards the unknown, which is also time and state, and thus awareness, invariant. For each state s, the decision maker computes the generalized von Neumann-Morgenstern utility of the lottery that the intertemporal act under evaluation returns in that state. The generalized von Neumann-Morgenstern utility evaluates all outcomes in $C(\omega_t)$ according to u_{ω_t} and collapses all unknown consequences from the ω_t-point of view into one unknown consequence, which is assigned utility value u^*. Furthermore, all possible continuation paths are assigned positive probability.

As awareness (potentially) evolves and we move from one state to the next, beliefs are updated according to reverse Bayesianism, which is described in (14). When awareness grows and new consequences are indeed discovered, the resulting Bernoulli-utility function is an extension of the previous one. The decision maker's attitude towards the unknown remains unchanged in response to the increase in awareness. Hence, she does not become more excited about or fearful towards the unknown. However, the reverse Bayesian updating of beliefs implies a decreasing sense of unawareness: As the decision maker's awareness grows, her posterior assigns a lower probability to making future discoveries than her prior did. Theorem 1 thus succeeds in separating the evolution of the decision maker's attitude towards the unknown from the evolution of her sense of unawareness.

Existence, linearity, and state separability of the representation is a result of Axiom 2. That only the continuation path enters (13) follows from Axiom 1. Axiom 3 aides in identifying the subjective probabilities, and the full support follows from Axiom 4. Axiom 5 ensures that in each state, the attitude towards the unknown, $u^*_{\omega_t}$ is independent of the act under evaluation. Axiom 6 ensures exponential discounting as well as time- and awareness invariance of the discount factor β and that subsequent Bernoulli-utility functions are extensions of preceding ones. The collapsing of all unknown consequences into one, and the time- and awareness invariance of u^*, are results of Axioms 8 and 9. Reverse Bayesian updating of beliefs follows from (13) and Axiom 7.

The next result in Theorem 2 provides a recursive formulation of utility. However, the decision maker can only forecast her future utility function to the extent of her awareness. That is, she can currently only express her future utility with respect to her current set of extended consequences. She does not yet know what will be her Bernoulli-utility of consequences to be discovered between the current and the next period.

To ease notation, define $U_{\omega_t}(\hat{p}) = \sum_{c\in C(\omega_t)} \hat{p}(c)u_{\omega_t}(c) + \big(1-\sum_{c\in C(\omega_t)} \hat{p}(c)\big)u^*$. This is the generalized (with the attitude towards unawareness parameter) von Neumann-Morgenstern utility of the lottery $\hat{p}$.

Theorem 2. *Let $V_{(\omega_t,s)}(f|C(\omega_t))$ be derived from $V_{(\omega_t,s)}(f)$ by setting $u_{(\omega_t,s)}(c) = u^*$ for all $c \in C(\omega_t, s) \setminus C(\omega_t)$. Then the representation in part (b) of Theorem 1 implies that*

$$V_{\omega_t}(f) = \sum_{s\in S_{t+1}(\omega_t)} \pi_{\omega_t}(\omega_t, s) \left[U_{\omega_t}(f(\omega_t)(s)) + \beta V_{(\omega_t,s)}(f|C(\omega_t))\right]. \tag{15}$$

Proof: The result is a direct consequence of Theorem 1.

The function $V_{(\omega_t,s)}(f|C(\omega_t))$ can be thought of as the decision maker's current estimate of her future utility function, given her current awareness. The estimate treats all consequences that the decision maker will potentially discover between now and the next period as currently unknown consequences. As a result, they are all assigned a utility value of u^*.

Once the consequences that the decision maker will potentially discover between the current and the next period are "collapsed" into the current unknown consequence, future lotteries returning different such unknowns with the same probabilities are equivalent from the current point of view. Then the reverse Bayesian updating of beliefs in (14) implies that next period beliefs agree with current beliefs. Hence, the convenient recursive relation in (15) applies. It generalizes the standard recursive approach to include unawareness. For a comprehensive textbook discussion of the standard recursive approach and some of the models it can be used to analyze, see e.g. Sargent (1987).

5 Conclusion

This paper has presented an intertemporal model of growing awareness, which generalizes both the standard event-tree framework and the framework from Karni and Vierø (2017) of awareness of unawareness. At first glance, the problem is seemingly intractable: With a long time horizon, there is a great number of ways in which awareness may grow, both in terms of when increases in awareness occur, what and how much is discovered at any given time, and in which order discoveries are made. The framework provided incorporates all these elements of the problem in a tractable manner.

An axiomatic structure is provided that allows for a representation of preferences over intertemporal acts under awareness of unawareness. The approach to define intertemporal acts is inspired by Epstein and Schneider (2003). The resulting utility function is separable across time and states and has the standard subjective expected utility form as a special case in the absence of awareness of unawareness. With awareness of unawareness present, the decision maker uses a generalized expected utility as in Karni and Vierø (2017) for each state and acts as if acts were describable with respect to uncertainties she can express given her current awareness. A recursive formulation of intertemporal utility is also obtained.

The results in Theorem 1 imply that even when facing highly complex problems with awareness of unawareness and long time horizons, the agent can make complete contingent plans, also for events that involve new discoveries, to the extent that she can describe these plans. The axiomatic structure ensures dynamic consistency in a forward looking way, but not necessarily looking backwards. When awareness does grow, the agent may wish to change her course of action in response to her new awareness. She will, however, still maintain that her original plan was the right one given the awareness she had at the time it was made. Thus, the agent is rational to the extent possible given her limited awareness.

A Proof of Theorem 1

A.1 Sufficiency of Axioms

The set of intertemporal acts F is a convex set, and $\succ_{\omega_t}$ satisfies Axiom 2 for all $\omega_t \in \Omega$. Thus, by the mixture space theorem, there exists, for all ω_t, a real-valued function $V_{\omega_t} : F \to \Re$ such that $\succ_{\omega_t}$ on F is represented by V_{ω_t} and

$$V_{\omega_t}(\alpha f + (1-\alpha)f') = \alpha V_{\omega_t}(f) + (1-\alpha)V_{\omega_t}(f') \tag{16}$$

for all $f, f' \in F$. Moreover, V_{ω_t} is unique up to positive linear transformation: V'_{ω_t} also represents $\succ_{\omega_t}$ if and only if $V'_{\omega_t} = \kappa V_{\omega_t} + \zeta$, with $\kappa > 0$.

Lemma 1. *For all $\omega_t \in \Omega$, the function V_{ω_t} satisfies*

$$V_{\omega_t}(f) = \sum_{\omega_\tau \in \Omega} V_{\omega_t}(\omega_\tau)(f(\omega_\tau)),$$

i.e. V_{ω_t} is separable across states.

Remark: Note that in Lemma 1, $f(\omega_\tau)$ is the restricted Anscombe-Aumann act that originates in state ω_τ.

Proof of Lemma 1: Fix $f^* \in F$ and for each $f \in F, \tau \in \mathscr{T} \setminus \{T\}$, and $\omega_\tau \in \Omega_\tau$, let $f^{\omega_\tau} = f_{\omega_\tau} f^* \in F$ be defined by $f^{\omega_\tau}(\omega_\tau) = f(\omega_\tau)$ and $f^{\omega_\tau}(\omega) = f^*(\omega)$ for $\omega \neq \omega_\tau$. Let $m = \sum_{\omega \in \Omega} 1$. For any $f \in F$,

$$\frac{1}{m} f + \frac{m-1}{m} f^* = \sum_{\omega_\tau \in \Omega} \frac{1}{m} f^{\omega_\tau}. \tag{17}$$

By (16) and (17),

$$\frac{1}{m} \sum_{\omega_\tau \in \Omega} V_{\omega_t}(f^{\omega_\tau}) = \frac{1}{m} V_{\omega_t}(f) + \frac{m-1}{m} V_{\omega_t}(f^*). \tag{18}$$

For each $\omega_\tau \in \Omega$, define $V_{\omega_t}(\omega_\tau) : \Delta(C(\omega_\tau))^{\tilde{S}_{\tau+1}(\omega_\tau)} \times \Delta(\widehat{C}(\omega_\tau))^{S_{\tau+1}(\omega_\tau) \setminus \tilde{S}_{\tau+1}(\omega_\tau)} \to \Re$ (this definition embodies the appropriate restriction on the support in the fully describable 'states') by

$$V_{\omega_t}(\omega_\tau)(g(\omega_\tau)) = V_{\omega_t}(g(\omega_\tau)_{\omega_\tau} f^*) - \frac{m-1}{m} V_{\omega_t}(f^*).$$

For $f \in F$, this definition gives

$$V_{\omega_t}(\omega_\tau)(f(\omega_\tau)) = V_{\omega_t}(f^{\omega_\tau}) - \frac{m-1}{m} V_{\omega_t}(f^*),$$

which implies

$$\frac{1}{m} \sum_{\omega_\tau \in \Omega} V_{\omega_t}(\omega_\tau)(f(\omega_\tau)) = \frac{1}{m} \sum_{\omega_\tau \in \Omega} V_{\omega_t}(f^{\omega_\tau}) - \frac{m-1}{m} V_{\omega_t}(f^*).$$

Combining with (18) and multiplying by m on both sides, we get

$$V_{\omega_t}(f) = \sum_{\omega_\tau \in \Omega} V_{\omega_t}(\omega_\tau)(f(\omega_\tau)).$$

Thus, the representation is additively separable across states.

Lemma 2. *For all $\omega_\tau \notin \Omega(\omega_t)$, $V_{\omega_t}(\omega_\tau)(f(\omega_\tau)) = k \in \Re$.*

Proof of Lemma 2: This follows from Axiom 1.

Remark: One can set $k = 0$ without affecting anything. For ease of notation, this is adopted.

Lemma 3. *For all $\omega_t \in \Omega$,*

$$V_{\omega_t}(f) = \sum_{\omega_\tau \in \Omega(\omega_t)} \rho_{\omega_t}(\omega_\tau) v_{\omega_t}(\omega_\tau)(f(\omega_\tau)), \tag{19}$$

with $\rho_{\omega_t}(\omega_\tau) > 0$ for all $\omega_\tau \in \Omega(\omega_t)$.

Remark: In Lemma 3, $f(\omega_\tau)$ is, again, the restricted Anscombe-Aumann act that originates in state ω_τ. Therefore, an equivalent way to state (19) is as $V_{\omega_t}(\omega_\tau)(f(\omega_\tau)) = \rho_{\omega_t}(\omega_\tau)v_{\omega_t}(\omega_\tau)(f(\omega_\tau))$.

Proof of Lemma 3: This follows from the Anscombe and Aumann theorem and Axioms 1, 2, 3(iii), and 4, as will now be shown. By Lemmas 1 and 2,

$$V_{\omega_t}(f) = \sum_{\omega_\tau \in \Omega(\omega_t)} V_{\omega_t}(\omega_\tau)(f(\omega_\tau)). \tag{20}$$

Consider the set of acts whose lottery supports are restricted to $C(\omega_t)$ for all $\omega_\tau \in \Omega(\omega_t)$. By Axiom 4 there is a nonnull one-step-ahead resolution of uncertainty for all states. By Axiom 3(iii) and (20),[10]

$$\begin{aligned} V_{\omega_t}(\omega_\tau)(p) > V_{\omega_t}(\omega_\tau)(q) \Leftrightarrow & \sum_{\omega_\tau \in \Omega(\omega_t)} V_{\omega_t}(\omega_\tau)(p) > \sum_{\omega_\tau \in \Omega(\omega_t)} V_{\omega_t}(\omega_\tau)(q) \\ \Leftrightarrow & V_{\omega_t}(\omega'_\tau)(p) > V_{\omega_t}(\omega'_\tau)(q) \end{aligned} \tag{21}$$

for all $\omega_\tau, \omega'_\tau \in \Omega(\omega_t)$. Thus, $V_{\omega_t}(\omega_\tau)$ and $V_{\omega_t}(\omega'_\tau)$ are ordinally equivalent when evaluating restricted Anscombe-Aumann acts whose lottery supports are confined to $C(\omega_t)$ for all $\omega_\tau, \omega'_\tau \in \Omega(\omega_t)$.

Let $v_{\omega_t} \equiv V_{\omega_t}(\omega_t)$. Then, for all $\omega_\tau \in \Omega(\omega_t)$, $V_{\omega_t}(\omega_\tau) = \kappa_{\omega_\tau} v_{\omega_t} + \eta_{\omega_\tau}$, with $\kappa_{\omega_\tau}, \eta_{\omega_\tau} \in \Re$ and $\kappa_{\omega_\tau} > 0$ when restricted to such acts. Hence, by (20), for f, g with $f(\omega_\tau) : S_{\tau+1}(\omega_\tau) \to \Delta(C(\omega_t))$ and $g(\omega_\tau) : S_{\tau+1}(\omega_\tau) \to \Delta(C(\omega_t))$,

$$f \succ_{\omega_t} g \Leftrightarrow \sum_{\omega_\tau \in \Omega(\omega_t)} \kappa_{\omega_\tau} v_{\omega_t}(f(\omega_\tau)) + \eta_{\omega_\tau} > \sum_{\omega_\tau \in \Omega(\omega_t)} \kappa_{\omega_\tau} v_{\omega_t}(g(\omega_\tau)) + \eta_{\omega_\tau}.$$

Cancel out terms, divide both sides by $\sum_{\omega_\tau \in \Omega(\omega_t)} \kappa_{\omega_\tau}$, and define $\rho_{\omega_t}(\omega_\tau) \equiv \frac{\kappa_{\omega_\tau}}{\sum_{\omega_\tau \in \Omega(\omega_t)} \kappa_{\omega_\tau}}$. Then

$$V_{\omega_t}(f) = \sum_{\omega_\tau \in \Omega(\omega_t)} \pi_{\omega_t}(\omega_\tau) v_{\omega_t}(f(\omega_\tau)). \tag{22}$$

By Axiom 4, $\rho_{\omega_t}(\omega_\tau) > 0$ for all $\omega_\tau \in \Omega(\omega_t)$.

For general acts, it follows from (22) that $V_{\omega_t}(f) = \sum_{\omega_\tau \in \Omega(\omega_t)} \rho_{\omega_t}(\omega_\tau) v_{\omega_t}(\omega_\tau)(f(\omega_\tau))$ and that $v_{\omega_t}(\omega_\tau)$ and $v_{\omega_t}(\omega'_\tau)$ agree when evaluating acts whose lottery supports are restricted to $C(\omega_t)$.

[10] Here, the notation p is abused to denote the restricted Anscombe-Aumann act for which $f(\omega_\tau) = p$ for all $s \in S_{\tau+1}(\omega_\tau)$.

Lemma 4. *For all* $\omega_\tau \in \Omega(\omega_t)$,

$$v_{\omega_t}(\omega_\tau)(f(\omega_\tau)) = \sum_{s\in\widetilde{S}_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(s|\omega_\tau) \sum_{c\in C(\omega_\tau)} f(\omega_\tau)(s)(c)u_{\omega_t}(\omega_\tau)(c) + \sum_{s\in S_{\tau+1}(\omega_\tau)\setminus\widetilde{S}_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(s|\omega_\tau) \sum_{\hat{c}\in\widehat{C}(\omega_\tau)} f(\omega_\tau)(s)(\hat{c})u^*_{\omega_t}(\omega_\tau)(\hat{c}) \qquad (23)$$

where u_{ω_t} *and* $u^*_{\omega_t}$ *are unique up to positive linear transformations and agree on* $C(\omega_\tau)$.

Proof of Lemma 4: First note that Axioms 2, 3(i), 3(ii), 4, and 5 all hold on $H_{\omega_\tau}(f)$ for all $\omega_\tau \in \Omega(\omega_t)$ and all $f \in F$.

Consider $h, h' \in H_{\omega_\tau}(f)$. By Lemma 1, the terms in the utilities of h and h' cancel out for all states but ω_τ, since h and h' agree outside of ω_τ. Thus, the choice of conditioning act f is immaterial and, by Lemma 3,

$$h \succ_{\omega_t} h' \Leftrightarrow v_{\omega_t}(\omega_\tau)(h(\omega_\tau)) > v_{\omega_t}(\omega_\tau)(h'(\omega_\tau)).$$

Since $F(\omega_\tau)$ is a convex set, arguments analogous to those preceding Lemma 1 and in the proof of Lemma 1 imply that

$$v_{\omega_t}(\omega_\tau)(h(\omega_\tau)) = \sum_{s\in S_{\tau+1}(\omega_\tau)} v_{\omega_t}(\omega_\tau)(s)(h(\omega_\tau)(s)).$$

The standard induction argument shows that for $p \in \Delta(C(\omega_\tau))$ and $s \in S_{\tau+1}(\omega_\tau)$,

$$v_{\omega_t}(\omega_\tau)(s)(p) = \sum_{c\in C(\omega_\tau)} p(c)u_{\omega_t}(\omega_\tau)(s)(c),$$

with $u_{\omega_t}(\omega_\tau)(s)(c) = v_{\omega_t}(\omega_\tau)(s)(c)$, where the former c denotes the consequence c and the latter c denotes the lottery that returns c with probability 1.

Similar arguments show that for $s \in S_{\tau+1}(\omega_\tau) \setminus \widetilde{S}_{\tau+1}(\omega_\tau)$ and $\hat{p} \in \Delta(\widehat{C}(\omega_\tau))$,

$$v_{\omega_t}(\omega_\tau)(s)(\hat{p}) = \sum_{\hat{c}\in\widehat{C}(\omega_\tau)} \hat{p}(\hat{c})u^*_{\omega_t}(\omega_\tau)(s)(\hat{c}),$$

where $u^*_{\omega_t}(\omega_\tau)(s)(\hat{c}) = v_{\omega_t}(\omega_\tau)(s)(\hat{c})$.

Let $\mathbf{H}_{\omega_\tau}(f) \equiv \{h_{\omega_\tau}f | h : S_{\tau+1}(\omega_\tau) \to \Delta(C(\omega_\tau))\}$, i.e. the subset of $H_{\omega_\tau}(f)$ for which the support of the lotteries in h are restricted to $\Delta(C(\omega_\tau))$. Consider $h, h' \in \mathbf{H}_{\omega_\tau}(f)$. By Lemma 1, the choice of conditioning act f is immaterial. By Axiom 4, there exists at least one $\succ_{\omega_t}$-nonnull state $s' \in S_{\tau+1}(\omega_\tau)$. By Axiom 3(i), for any $p, q \in \Delta(C(\omega_\tau))$,

$$\sum_{c\in C(\omega_\tau)} p(c)u_{\omega_t}(\omega_\tau)(s)(c) > \sum_{c\in C(\omega_\tau)} q(c)u_{\omega_t}(\omega_\tau)(s)(c)$$
$$\Leftrightarrow \sum_{c\in C(\omega_\tau)} p(c)u_{\omega_t}(\omega_\tau)(s')(c) > \sum_{c\in C(\omega_\tau)} q(c)u_{\omega_t}(\omega_\tau)(s')(c)$$

for all $\succ_{\omega_t}$-nonnull $s \in S_{\tau+1}(\omega_\tau)$. Thus, standard arguments following those in the proof of Lemma 3 imply that there exists a unique probability measure $\pi_{\omega_t}(\cdot|\omega_\tau)$ on $S_{\tau+1}(\omega_\tau)$ such that for $h, h' \in \mathbf{H}_{\omega_\tau}(f)$

$$\begin{aligned} h \succ_{\omega_t} h' \Leftrightarrow & \sum_{s \in S_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(s|\omega_\tau) \sum_{c \in C(\omega_\tau)} h(\omega_\tau)(s)(c) u_{\omega_t}(\omega_\tau)(c) \\ > & \sum_{s \in S_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(s|\omega_\tau) \sum_{c \in C(\omega_\tau)} h'(\omega_\tau)(s)(c) u_{\omega_t}(\omega_\tau)(c), \end{aligned}$$

recalling that by Lemma 1 the choice of conditioning act f is immaterial.

Analogous arguments to those above (using Axiom 3(ii) in place of 3(i)) imply that there exists a unique probability measure $\phi_{\omega_t}(\cdot|\omega_\tau)$ on $S_{\tau+1}(\omega_\tau) \setminus \tilde{S}_{\tau+1}(\omega_\tau)$ such that for all $h, h' \in H_{\omega_t}(F)$ that agree in all $s \in \tilde{S}_{\tau+1}(\omega_\tau)$,

$$\begin{aligned} h \succ_{\omega_t} h' \Leftrightarrow & \sum_{s \in S_{\tau+1}(\omega_\tau) \setminus \tilde{S}_{\tau+1}(\omega_\tau)} \phi_{\omega_t}(s|\omega_\tau) \sum_{\hat{c} \in \hat{C}(\omega_\tau)} h(\omega_\tau)(s)(\hat{c}) u^*_{\omega_t}(\omega_\tau)(\hat{c}) \\ > & \sum_{s \in S_{\tau+1}(\omega_\tau) \setminus \tilde{S}_{\tau+1}(\omega_\tau)} \phi_{\omega_t}(s|\omega_\tau) \sum_{\hat{c} \in \hat{C}(\omega_\tau)} h'(\omega_\tau)(s)(\hat{c}) u^*_{\omega_t}(\omega_\tau)(\hat{c}). \end{aligned}$$

Now, arguments analogous to those in the proof of Theorem 1 in Karni and Vierø (2017) complete the proof of Lemma 4.

Lemma 5. *For all $\omega_t \in \Omega$ and all $\omega_\tau \in \Omega(\omega_t)$, $u_{\omega_t}(\omega_\tau)(c) = u_{\omega_t}(\omega_t)(c) \equiv u_{\omega_t}(c)$ for all $c \in C(\omega_t)$.*

Proof of Lemma 5: By the arguments preceeding (22), the functions $v_{\omega_t}(\omega_t)(\cdot)$ and $v_{\omega_t}(\omega_\tau)(\cdot)$ are ordinally equivalent for all $\omega_\tau \in \Omega_\tau(\omega_t)$. Hence, $u_{\omega_t}(\omega_t)(\cdot)$ and $u_{\omega_t}(\omega_\tau)(\cdot)$ in Lemma 4 must be equal on $\Delta(C(\omega_t))$ after suitable linear transformations.

Lemma 6. *For all $\omega_t \in \Omega$ and all $\omega_{\hat{t}} \in \Omega(\omega_t)$, $u_{\omega_{\hat{t}}}(c) = u_{\omega_t}(c)$ for all $c \in C(\omega_t)$.*

Proof of Lemma 6: By Lemma 5, it suffices to consider lottery acts that only differ one step ahead. Consider $l \in L_{\omega_t}(F)$ and $p, q, p' \in \Delta(C(\omega_t))$. By Axiom 6,

$$\begin{aligned} & p_{\Omega_{t+1}(\omega_t)} p'_{\Omega_{t+2}(\omega_t)} l \succsim_{\omega_t} q_{\Omega_{t+1}(\omega_t)} p'_{\Omega_{t+2}(\omega_t)} l \\ \Leftrightarrow & p_{\Omega_{\hat{t}+1}(\omega_{\hat{t}})} p'_{\Omega_{\hat{t}+2}(\omega_t)} l \succsim_{\omega_{\hat{t}}} q_{\Omega_\tau(\omega_{\hat{t}+1})} p'_{\Omega_{\hat{t}+2}(\omega_{\hat{t}})} l. \end{aligned}$$

Thus, v_{ω_t} and $v_{\omega_{\hat{t}}}$ are ordinally equivalent for $p \in \Delta(C(\omega_t))$ for all $\omega_{\hat{t}} \in \Omega_{\hat{t}}(\omega_t)$. Hence, after suitable linear transformation, $u_{\omega_{\hat{t}}}(c)$ and $u_{\omega_t}(c)$ must be equal on $\Delta(C(\omega_t))$.

Lemma 7. *For all $\omega_t \in \Omega$ and all $\omega_\tau \in \Omega(\omega_t)$, $v_{\omega_t}(\omega_\tau) = (\beta_{\omega_t})^{\tau-t} v_{\omega_t}(\omega_t)$ for some $\beta_{\omega_t} > 0$.*

Proof of Lemma 7: By axiom 6, for $p, q, p', q' \in \Delta(C(\omega_t))$,

$$\begin{aligned} &v_{\omega_t(\omega_\tau)}(p) + v_{\omega_t(\omega_{\tau+1})}(p') \geq v_{\omega_t(\omega_\tau)}(q) + v_{\omega_t(\omega_{\tau+1})}(q') \\ \Leftrightarrow &v_{\omega_t(\omega_{\hat{\tau}})}(p) + v_{\omega_t(\omega_{\hat{\tau}+1})}(p') \geq v_{\omega_t(\omega_{\hat{\tau}})}(q) + v_{\omega_t(\omega_{\hat{\tau}+1})}(q'). \end{aligned} \tag{24}$$

Define

$$W_{\omega_t}(\omega_\tau)(p, p') = v_{\omega_t}(\omega_\tau)(p) + v_{\omega_t}(\omega_{\tau+1})(p').$$

Then (24) implies that $W_{\omega_t}(\omega_\tau)$ and $W_{\omega_t}(\omega_{\hat{\tau}})$ are ordinally equivalent for all $\omega_\tau, \omega_{\hat{\tau}} \in \Omega(\omega_t)$. By Lemma 5, $v_{\omega_t}(\omega_\tau)$ and $v_{\omega_t}(\omega_t)$ are ordinally equivalent for all $\omega_\tau \in \Omega(\omega_t)$. Hence, $v_{\omega_t}(\omega_\tau) = \alpha_{\omega_\tau} v_{\omega_t}(\omega_t) + \gamma_{\omega_\tau}$ for all $\omega_\tau \in \Omega(\omega_t)$. Let $v_{\omega_t}(\omega_t) \equiv v_{\omega_t}$. Then

$$v_{\omega_t}(\omega_\tau)(p) + v_{\omega_t}(\omega_{\tau+1})(p') = \alpha_{\omega_\tau} v_{\omega_t}(p) + \gamma_{\omega_\tau} + \alpha_{\omega_{\tau+1}} v_{\omega_t}(p') + \gamma_{\omega_{\tau+1}}$$

and

$$v_{\omega_t}(\omega_{\tau+1})(p) + v_{\omega_t}(\omega_{\tau+2})(p') = \alpha_{\omega_{\tau+1}} v_{\omega_t}(p) + \gamma_{\omega_{\tau+1}} + \alpha_{\omega_{\tau+2}} v_{\omega_t}(p') + \gamma_{\omega_{\tau+2}}.$$

By ordinal equivalence of $W_{\omega_t}(\omega_\tau)$ and $W_{\omega_t}(\omega_{\tau+1})$,

$$\alpha_{\omega_{\tau+1}} v_{\omega_t}(p) + \gamma_{\omega_{\tau+1}} + \alpha_{\omega_{\tau+2}} v_{\omega_t}(p') + \gamma_{\omega_{\tau+2}} = \beta_{\omega_t}[\alpha_{\omega_\tau} v_{\omega_t}(p) + \gamma_{\omega_\tau} + \alpha_{\omega_{\tau+1}} v_{\omega_t}(p') + \gamma_{\omega_{\tau+1}}] + \gamma \tag{25}$$

for some $\alpha > 0$ and $\gamma \in \Re$. It follows from (25) that $\alpha_{\omega_{\tau+1}} = \beta_{\omega_t} \alpha_{\omega_\tau}$ and $\alpha_{\omega_{\tau+2}} = \beta_{\omega_t}^2 \alpha_{\omega_\tau}$, while the constants $\gamma, \gamma_{\omega_{t+1}}, \gamma_{\omega_{t+2}}$ can be set to zero since they will cancel out when comparing acts. Since ω_τ was chosen arbitrarily from $\Omega(\omega_t)$ and $\alpha_{\omega_t} = 1$, it follows that

$$v_{\omega_t}(\omega_\tau) = (\beta_{\omega_t})^{\tau - t} v_{\omega_t}(\omega_t)$$

for all $\omega_\tau \in \Omega(\omega_t)$.

Lemma 8. *For all $\omega_t \in \Omega$, $\beta_{\omega_t} = \beta > 0$.*

Proof of Lemma 8: For all $p, q, p', q' \in \Delta(C(\omega_t))$ and for all $\omega_{\hat{t}}, \omega_{\tilde{t}} \in \Omega(\omega_t)$, $\omega_\tau \in \Omega(\omega_{\hat{t}})$, and $\omega'_\tau \in \Omega(\omega_{\tilde{t}})$, it holds, by Axiom 6, that

$$\begin{aligned} &v_{\omega_{\hat{t}}(\omega_\tau)}(p) + v_{\omega_{\hat{t}}(\omega_{\tau+1})}(p') \geq v_{\omega_{\hat{t}}(\omega_\tau)}(q) + v_{\omega_{\hat{t}}(\omega_{\tau+1})}(q') \\ \Leftrightarrow &v_{\omega_{\tilde{t}}(\omega_{\tau'})}(p) + v_{\omega_{\tilde{t}}(\omega_{\tau'+1})}(p') \geq v_{\omega_{\tilde{t}}(\omega_{\tau'})}(q) + v_{\omega_{\tilde{t}}(\omega_{\tau'+1})}(q'), \end{aligned}$$

which by Lemma 7 implies that

$$(\beta_{\omega_{\hat{t}}})^{\tau - \hat{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p) + (\beta_{\omega_{\hat{t}}})^{\tau + 1 - \hat{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p') = (\beta_{\omega_{\tilde{t}}})^{\tau' - \tilde{t}} v_{\omega_{\tilde{t}}}(\omega_{\tilde{t}})(p) + (\beta_{\omega_{\tilde{t}}})^{\tau' + 1 - \tilde{t}} v_{\omega_{\tilde{t}}}(\omega_{\tilde{t}})(p'). \tag{26}$$

By Lemma 6, we can set $v_{\omega_{\tilde{t}}}(\omega_{\tilde{t}}) = v_{\omega_{\hat{t}}}(\omega_{\hat{t}})$. Hence, (26) gives that

$$(\beta_{\omega_{\hat{t}}})^{\tau - \hat{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p) + (\beta_{\omega_{\hat{t}}})^{\tau + 1 - \hat{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p') = (\beta_{\omega_{\tilde{t}}})^{\tau' - \tilde{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p) + (\beta_{\omega_{\tilde{t}}})^{\tau' + 1 - \tilde{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p'). \tag{27}$$

Consider $\tau, \hat{t}, \tau', \tilde{t}$ such that $\tau - \hat{t} = \tau' - \tilde{t}$. Then (27) implies that

$$(\beta_{\omega_{\hat{t}}})^{\tau-\hat{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p) + (\beta_{\omega_{\hat{t}}})^{\tau+1-\hat{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p') = (\beta_{\omega_{\tilde{t}}})^{\tau-\hat{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p) + (\beta_{\omega_{\tilde{t}}})^{\tau+1-\hat{t}} v_{\omega_{\hat{t}}}(\omega_{\hat{t}})(p'),$$

which implies that $\beta_{\omega_{\hat{t}}} = \beta_{\omega_{\tilde{t}}} \equiv \beta$.

Lemma 9. *For all $\omega_t \in \Omega$, $\omega_\tau \in \Omega(\omega_t)$, and $\hat{c} \in \widehat{C}(\omega_\tau) \setminus C(\omega_t)$, $u^*_{\omega_t}(\omega_\tau)(\hat{c}) = u^*_{\omega_t}(x(C(\omega_t)))$.*

Proof of Lemma 9: By Axiom 8,

$$u^*_{\omega_t}(\omega_\tau)(\hat{c}) = u^*_{\omega_t}(\omega_\tau)(\check{c}) \tag{28}$$

for all $\hat{c}, \check{c} \in \hat{C}(\omega_\tau) \setminus C(\omega_t)$. Also by Axiom 8,

$$u^*_{\omega_t}(\omega_\tau)(x(C(\omega_t))) = u^*_{\omega_t}(\omega_\tau)(x(C(\omega_\tau))). \tag{29}$$

By Axiom 9(ii),

$$u^*_{\omega_t}(\omega_t)(x(C(\omega_t))) = \alpha u^*_{\omega_t}(\omega_t)(c^*) + (1-\alpha) u^*_{\omega_t}(\omega_t)(c_*) \tag{30}$$

$$\Rightarrow u^*_{\omega_t}(\omega_\tau)(x(C(\omega_t))) = \alpha u^*_{\omega_t}(\omega_\tau)(c^*) + (1-\alpha) u^*_{\omega_t}(\omega_\tau)(c_*) \tag{31}$$

By Lemma 4, $u^*_{\omega_t}(\omega_\tau)$ agrees with $u_{\omega_t}(\omega_\tau)$ on $C(\omega_t)$ for all ω_t and $\omega_\tau \in \Omega(\omega_t)$. By Lemma 5, $u_{\omega_t}(\omega_\tau)(c) = u_{\omega_t}(c)$ for all $c \in C(\omega_t)$. Therefore, the right hand sides of (30) and (31) are equal, which implies that $u^*_{\omega_t}(\omega_\tau)(x(C(\omega_t))) = u^*_{\omega_t}(\omega_t)(x(C(\omega_t))) \equiv u^*_{\omega_t}(x(C(\omega_t)))$. Equation (29) now implies that $u^*_{\omega_t}(\omega_\tau)(x(C(\omega_\tau))) = u^*_{\omega_t}(x(C(\omega_t)))$ for all $\omega_\tau \in \Omega(\omega_t)$ and (28) implies that $u^*_{\omega_t}(\omega_\tau)(\hat{c}) = u^*_{\omega_t}(x(C(\omega_t)))$ for all $\omega_\tau \in \Omega(\omega_t)$ and $\hat{c} \in \widehat{C}(\omega_\tau) \setminus C(\omega_t)$.

Lemma 10. *For all $\omega_t \in \Omega$, $u^*_{\omega_t}(x(C(\omega_t))) = u^*(x(C(\omega_t))) \equiv u^*$.*

Proof of Lemma 10: By Lemmas 3, 4, and 9,

$$x(C(\omega_t))_{S_{t+1}(\omega_t)\setminus\tilde{S}_{t+1}(\omega_t)} f \sim_{\omega_t} (\alpha c^* + (1-\alpha)c_*)_{S_{t+1}(\omega_t)\setminus\tilde{S}_{t+1}(\omega_t)} f$$

$$\Leftrightarrow u^*_{\omega_t}(x(C(\omega_t))) = \alpha u_{\omega_t}(c^*) + (1-\alpha) u_{\omega_t}(c_*) \tag{32}$$

and

$$x(C(\omega_t, s_{t+1}))_{S_{t+2}(\omega_t,s_{t+1})\setminus\tilde{S}_{t+2}(\omega_t,s_{t+1})} f \sim_{(\omega_t,s_{t+1})} (\alpha c^* + (1-\alpha)c_*)_{S_{t+2}(\omega_t,s_{t+2})\setminus\tilde{S}_{t+2}(\omega_t,s_{t+2})} f$$

$$\Leftrightarrow u^*_{(\omega_t,s_{t+1})}(x(C(\omega_t, s_{t+1}))) = \alpha u_{(\omega_t,s_{t+1})}(c^*) + (1-\alpha) u_{(\omega_t,s_{t+1})}(c_*) \tag{33}$$

By Lemma 6, $u_{(\omega_t,s_{t+1})}(c) = u_{\omega_t}(c)$ for all $c \in C(\omega_t)$. Thus, the right hand sides of (32) and (33) are equal. By Axiom 9(i), (32) implies (33). Thus, $u^*_{(\omega_t,s_{t+1})}(x(C(\omega_t, s_{t+1}))) =$

$u^*_{\omega_t}(x(C(\omega_t)))$. One can proceed by induction to show that $u^*_{\omega_\tau}(x(C(\omega_\tau))) = u^*_{\omega_t}(x(C(\omega_t)))$ for all $\omega_\tau \in \Omega(\omega_t)$. Setting $t = 0$, it follows that

$$u^*_{\omega_\tau}(x(C(\omega_\tau))) = u^*_{\omega_0}(x(C(\omega_0))) \equiv u^*(x(C(\omega_0))).$$

Since all other $c \in C(\omega_\tau)$ can be evaluated by u_{ω_τ}, $x(C(\omega_\tau))$ is the only 'consequence' that needs to be evaluated by $u^*_{\omega_\tau}$. Thus, one can define $u^* \equiv u^*(x(C(\omega_0)))$ and use u^* in the representation.

Lemma 11. *Define $\pi_{\omega_t}(\omega_\tau, s) = p_{\omega_t}(\omega_\tau)\pi_{\omega_t}(s|\omega_\tau)$. Then*

$$V_{\omega_t}(f) = \sum_{\tau=t}^{T-1} \beta^\tau \sum_{\omega_\tau \in \Omega_\tau(\omega_t)} \sum_{s \in S_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(\omega_\tau, s) \left(\sum_{c \in C(\omega_t)} f(\omega_\tau)(s)(c) u_{\omega_t}(c) \right. \\ \left. + \Big(1 - \sum_{c \in C(\omega_t)} f(\omega_\tau)(s)(c)\Big) u^*_{\omega_t} \right)$$

Proof of Lemma 11: This follows from Lemmas 1 through 10.

Lemma 12. *The probability measures π_{ω_t} satisfy that for all $s_{t+1} \in S_{t+1}(\omega_t)$, for all $\omega_\tau \in \Omega(\omega_t)$, and for all $s_{\tau+1}, \tilde{s}_{\tau+1}, \bar{s}_{\tau+1} \in S_{\tau+1}(\omega_\tau)$ we have that*

$$\frac{\pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})}{\pi_{\omega_t}(\omega_\tau, \bar{s}_{\tau+1})} = \frac{\pi_{(\omega_t, s_{t+1})}(\omega_\tau, s_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1}|(\omega_\tau, s_{\tau+1}))}{\pi_{(\omega_t, s_{t+1})}(\omega_\tau, s_{\tau+1}, \mathscr{E}_{\tau+2}(\bar{s}_{\tau+1}|(\omega_\tau, s_{\tau+1}))}.$$

Proof of Lemma 12: Let g, h, g' and h' be as in Axiom 7. Then

$$\begin{aligned} g \succsim_{\omega_t} h \Leftrightarrow & \sum_{s \in S_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(\omega_\tau, s) u_{\omega_t}(\eta c^* + (1-\eta)c_*) \\ & \geq \pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1}) u_{\omega_t}(c^*) + \left(\sum_{s \in S_{\tau+1}(\omega_{t+\tau})} \pi_{\omega_t}(\omega_\tau, s) - \pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1}) \right) u_{\omega_t}(c_*) \\ & \Leftrightarrow \sum_{s \in S_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(\omega_\tau, s)[u_{\omega_t}(\eta c^* + (1-\eta)c_*) - u_{\omega_t}(c_*)] \\ & \geq \pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})[u_{\omega_t}(c^*) - u_{\omega_t}(c_*)], \end{aligned} \tag{34}$$

and

$$
\begin{aligned}
g' \succsim_{(\omega_t,s_{t+1})} h' \Leftrightarrow & \sum_{s\in S_{\tau+2}(\omega_\tau,s_{\tau+1})} \pi_{(\omega_t,s_{t+1})}(\omega_\tau, s_{\tau+1}, s) u_{(\omega_t,s_{t+1})}(\eta c^* + (1-\eta)c_*) \\
\geq & \sum_{s\in \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1}|(\omega_\tau,s_{\tau+1}))} \pi_{(\omega_t,s_{t+1})}(\omega_\tau, s_{\tau+1}, s) u_{(\omega_t,s_{t+1})}(c^*) \\
+ & \left(\sum_{s\in S_{\tau+2}(\omega_\tau,s_{\tau+1})} \pi_{(\omega_t,s_{t+1})}(\omega_\tau, s_{\tau+1}, s) - \sum_{s\in \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1}|(\omega_\tau,s_{\tau+1}))} \pi_{(\omega_t,s_{t+1})}(\omega_\tau, s_{\tau+1}, s) \right) u_{\omega_t}(c_*) \\
\Leftrightarrow & \sum_{s\in S_{\tau+2}(\omega_\tau,s_{\tau+1})} \pi_{(\omega_t,s_{t+1})}(\omega_\tau, s_{\tau+1}, s) [u_{(\omega_t,s_{t+1})}(\eta c^* + (1-\eta)c_*) - u_{(\omega_t,s_{t+1})}(c_*)] \\
\geq & \sum_{s\in \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1}|(\omega_\tau,s_{\tau+1}))} \pi_{(\omega_t,s_{t+1})}(\omega_\tau, s_{\tau+1}, s) [u_{(\omega_t,s_{t+1})}(c^*) - u_{(\omega_t,s_{t+1})}(c_*)], \qquad (35)
\end{aligned}
$$

By Lemma 6, $u_{(\omega_t,s_{t+1})} = u_{\omega_t}$. Thus, when (34) and (35) hold with equality, they imply that

$$
\frac{\pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})}{\sum_{s\in S_{\tau+1}(\omega_\tau)} \pi_{\omega_t}(\omega_\tau, s)} = \frac{\sum_{s\in \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1}|(\omega_\tau,s_{\tau+1}))} \pi_{(\omega_t,s_{t+1})}(\omega_\tau, s_{\tau+1}, s)}{\sum_{s\in S_{\tau+2}(\omega_\tau,s_{\tau+1})} \pi_{(\omega_t,s_{t+1})}(\omega_\tau, s_{\tau+1}, s)}. \qquad (36)
$$

A relationship like the one in (36) holds for all states $\tilde{s}_{\tau+1} \in S_{\tau+1}(\omega_\tau)$. Therefore, we have the result in (14).

Proof of sufficiency of Axioms: The result follows from Lemmas 1 through 12.

A.2 Necessity of Axioms

Necessity of Axiom 1 is obvious. Necessity of Axiom 2 follows from the mixture space theorem. Necessity of Axiom 4 follows from u being non-constant and π_{ω_t} having full support on $\Omega(\omega_t)$.

Axiom 3 is necessary, since the utilities for states where the LHS and RHS acts agree cancel out and one can divide through with the probabilities so that the utilities reduce to the same expressions for the two rankings in the axiom. A similar argument shows necessity of Axiom 5. Necessity of Axiom 8 follows from all $\hat{c} \notin C(\omega_t)$ being assigned the same utility value u^*. Axiom 9 follows from $u^*_{\omega_t}$ being invariant to both the awareness level ω_t and to the state under evaluation ω_τ.

To show necessity of Axiom 6, note that

$$
\begin{aligned}
& p_\tau p'_{\tau+1} l \succsim_{\omega_{\hat{t}}} q_\tau q'_{\tau+1} l && (37)\\
\Leftrightarrow & V_{\omega_{\hat{t}}}(p_\tau p'_{\tau+1} l) \geq V_{\omega_{\hat{t}}}(q_\tau q'_{\tau+1} l) \\
\Leftrightarrow & \beta^{\tau-\hat{t}} \sum_{c\in C(\omega_t)} p(c) u_{\omega_{\hat{t}}}(c) + \beta^{\tau-\hat{t}+1} \sum_{c\in C(\omega_t)} p'(c) u_{\omega_{\hat{t}}}(c) \\
& \quad \geq \beta^{\tau-\hat{t}} \sum_{c\in C(\omega_t)} q(c) u_{\omega_{\hat{t}}}(c) + \beta^{\tau-\hat{t}+1} \sum_{c\in C(\omega_t)} q'(c) u_{\omega_{\hat{t}}}(c) \\
\Leftrightarrow & \sum_{c\in C(\omega_t)} p(c) u_{\omega_{\hat{t}}}(c) + \beta \sum_{c\in C(\omega_t)} p'(c) u_{\omega_{\hat{t}}}(c) \geq \sum_{c\in C(\omega_t)} q(c) u_{\omega_{\hat{t}}}(c) + \beta \sum_{c\in C(\omega_t)} q'(c) u_{\omega_{\hat{t}}}(c) && (38)
\end{aligned}
$$

For different $\omega_{\tilde{t}}, \omega_{\tilde{\tau}}$, it holds that

$$
\begin{aligned}
& p_{\tilde{\tau}} p'_{\tilde{\tau}+1} l \succsim_{\omega_{\tilde{t}}} q_{\tilde{\tau}} q'_{\tilde{\tau}+1} l && (39)\\
\Leftrightarrow & V_{\omega_{\tilde{t}}}(p_{\tilde{\tau}} p'_{\tilde{\tau}+1} l) \geq V_{\omega_{\tilde{t}}}(q_{\tilde{\tau}} q'_{\tilde{\tau}+1} l) \\
\Leftrightarrow & \beta^{\tilde{\tau}-\tilde{t}} \sum_{c\in C(\omega_t)} p(c) u_{\omega_{\tilde{t}}}(c) + \beta^{\tilde{\tau}-\tilde{t}+1} \sum_{c\in C(\omega_t)} p'(c) u_{\omega_{\tilde{t}}}(c) \\
& \quad \geq \beta^{\tilde{\tau}-\tilde{t}} \sum_{c\in C(\omega_t)} q(c) u_{\omega_{\tilde{t}}}(c) + \beta^{\tilde{\tau}-\tilde{t}+1} \sum_{c\in C(\omega_t)} q'(c) u_{\omega_{\tilde{t}}}(c) \\
\Leftrightarrow & \sum_{c\in C(\omega_t)} p(c) u_{\omega_{\tilde{t}}}(c) + \beta \sum_{c\in C(\omega_t)} p'(c) u_{\omega_{\tilde{t}}}(c) \geq \sum_{c\in C(\omega_t)} q(c) u_{\omega_{\tilde{t}}}(c) + \beta \sum_{c\in C(\omega_t)} q'(c) u_{\omega_{\tilde{t}}}(c) && (40)
\end{aligned}
$$

Since $u_{\omega_{\hat{t}}}(c) = u_{\omega_{\tilde{t}}}(c)$ for all $c \in C(\omega_t)$ and for all $\omega_{\hat{t}}, \omega_{\tilde{t}} \in \Omega(\omega_t)$, the expressions in (38) and (40) are equivalent, and the equivalence of (37) and (39) follows.

To show necessity of Axiom 7, note that

$$g \succsim_{\omega_t} h \Leftrightarrow \eta u_{\omega_t}(c^*) + (1-\eta) u_{\omega_t}(c_*) \geq \pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1}) u_{\omega_t}(c^*) + (1 - \pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})) u_{\omega_t}(c_*),$$

which holds if and only if $\eta \geq \pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})$. Also,

$$
\begin{aligned}
g' \succsim_{\omega_t} h' \Leftrightarrow & \eta u_{(\omega_t, s_{t+1})}(c^*) + (1-\eta) u_{(\omega_t, s_{t+1})}(c_*) \\
& \geq \pi_{(\omega_t, s_{t+1})}(\omega_\tau, \tilde{s}_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1} | (\omega_\tau, s_{\tau+1}))) u_{(\omega_t, s_{t+1})}(c^*) \\
& \quad + (1 - \pi_{(\omega_t, s_{t+1})}(\omega_\tau, \tilde{s}_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1} | (\omega_\tau, s_{\tau+1})))) u_{(\omega_t, s_{t+1})}(c_*),
\end{aligned}
$$

which holds if and only if $\eta \geq \pi_{(\omega_t, s_{t+1})}(\omega_\tau, \tilde{s}_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1} | (\omega_\tau, s_{\tau+1})))$.

By (14),

$$\frac{\pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})}{1 - \pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})} = \frac{\pi_{(\omega_t, s_{t+1})}(\omega_\tau, \tilde{s}_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1} | (\omega_\tau, s_{\tau+1})))}{1 - \pi_{(\omega_t, s_{t+1})}(\omega_\tau, \tilde{s}_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1} | (\omega_\tau, s_{\tau+1})))},$$

which is equivalent to $\pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1}) = \pi_{(\omega_t, s_{t+1})}(\omega_\tau, \tilde{s}_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1} | (\omega_\tau, s_{\tau+1})))$. Hence, $\eta \geq \pi_{\omega_t}(\omega_\tau, \tilde{s}_{\tau+1})$ if and only if $\eta \geq \pi_{(\omega_t, s_{t+1})}(\omega_\tau, \tilde{s}_{\tau+1}, \mathscr{E}_{\tau+2}(\tilde{s}_{\tau+1} | (\omega_\tau, s_{\tau+1})))$, which establishes that the axiom holds.

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