# Laffer Curves and Public Goods

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## Abstract

We set out and solve a static neoclassical model with a labor/leisure choice for agents and a government sector producing a Samuelsonian public good. Numerical solutions vary considerably with the elasticity of substitution for commodities in an agent's utility function. We focus on solutions with an income tax rate set by the government (second best solutions). Government revenue varies with the rate of income tax (expressed in a Laffer Curve) and we observe that such curves generally peak "internally" only for case of "high" elasticity values in the utility function of a representative agent. Inelastic substitution possibilities involve the peaking of the Laffer Curve at a corner with the rate of income tax tending to unity. We report on welfare analysis for small changes in the rate of income tax and on first best outcomes (agents charged Samuelson "prices" for the public good).

- JEL classification: H22; H24; H41
- key words: Laffer curves; public goods; income tax incidence

## 1   Introduction

We investigate the peaking behavior of the Laffer Curve embedded in a static, neo-classical model with government product, a Samuelson public good (the total quantity produced enters equally in the utility function of each household in our economy). A Laffer Curve traces total tax revenue

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in the economy as the rate of income tax rises from zero to unity. Our abstract government balances its cost of producing its good with current revenue from an income tax. We focus attention on the central role in defining an equilibrium that the elasticity of substitution in the utility functions of our households plays. We work for the most part with $N$ identical households. Our simulation approach complements that of Saez, Slemrod and Giertz (2012) where the attention moves rapidly from basic theory to empirical investigation. Given CES forms for the utility functions of our households, we observe that the Laffer Curve tends to peak at an interior point as the rate of income tax is changed, when the elasticity of substitution is significantly greater than unity. When the elasticity of substitution is near or less than unity, the Laffer Curve tends to peak at a corner, with the rate of income tax approaching unity. Hours of leisure consumed by a household tend to increase (decrease) with a rise in the rate of tax when the elasticity of substitution is greater than (less than) unity. We also are able to shed light on the welfare cost of financing the government with an income tax rather than with Samuelsonian personal benefit charges. For our elastic case, the income tax (defining our second best case) represents a large "constraint" on the quantity of the government good consumed by a household relative to its chosen consumption in the first best case (when a household pays for the government good with a charge tailored to its marginal benefits).

Our point of departure is the simple textbook case of a single agent working out her labor-leisure choice in the face of an increase in her rate of income tax when her utility function is CES, this latter defined only on current consumption and hours of leisure. For this single agent case, more (less) leisure is selected, with a rise in the income tax rate, when her elasticity of substitution is greater than (less than) unity.

## 2   A Textbook Single Agent Case

In a textbook, single-agent, labor-leisure model, a central result is that hours-worked rise or decline with the rate of income tax as the elasticity of substitution $\sigma$ is less than or greater than unity. Consider then utility defined with the CES function in $u = \{\alpha * q_c^{-\beta} + (1-\alpha) * H_l^{-\beta}\}^{(-1/\beta)}$ for $q_c = [(1-\tau)w(24-H_l)]/p$. Here $q_c$ is quantity of consumption of the consumption good by the agent and $H_l$ is leisure time. $p$ is the price of the consumption good. $24 - H_l$ is laboring time at exogenous wage $w$ per hour. $\tau$ is the rate of income tax. The elasticity of substituion between labor and leisure is $\sigma = 1/(1+\beta)$, for $\beta > -1$ and $\beta \neq 0$. The elasticity of substitution is unity for $\beta = 0$; $\sigma > 1$ for $\beta < 0$ and $\sigma < 1$ for $\beta > 0$. The first order condition corresponding to the agent's choice of $H_l$ is $\left[\frac{pH_l}{(1-\tau)w(24-H_l)}\right]^{\beta+1} = \frac{(1-\alpha)}{\alpha}\left[\frac{p}{(1-\tau)w}\right]$ which reduces to

$$\frac{p}{(1-\tau)w} = \left(\frac{1-\alpha}{\alpha}\right)^{1/\beta}\left(\frac{24-H_l}{H_l}\right)^{((1/(\sigma\beta)))}. \tag{1}$$

It follows directly that for $\sigma > 1$ ($\beta < 0$), $H_l$ increases when $\tau$ increases (the elastic case), and for $\sigma < 1$ ($\beta > 0$), $H_l$ declines when $\tau$ increases (the inelastic case).

## 3   An N Agent Static Model with a Public Good and Durable Capital

This interesting result associated with equation (1) inspires us to work up an $N$ agent model with an explicit production function for consumption goods as well as a government good, this latter treated as a pure public good entering equally as total quantity produced in each agent's utility function. We introduce durable capital as an input in the production of each good, in addition to the hours worked by the N agents. The wage and rental on capital become endogenous along with the price of the government good. The price of the consumption good is our numeraire. There is an explicit production function for the government good, $q_g$

and this quantity and price $p_g$ are endogenous. The government uses an income tax to balance its cost of production with revenue collected $(p_g q_g = \tau[N * w * (H_c + H_g) + r * K]$, for $H_c$ and $H_g$ are hours worked by a household in the consumption and government goods sectors respectively. $K$ is the economy's endowment of durable capital and $r$ is the rental rate for capital.).[1] There is an explicit production function for the private good, $q_c$. Each of the production sectors is constant returns to scale, using inputs efficiently, and is in zero profit equilibrium.

A household's utility function is CES in $\{\alpha * q_c^{-\beta} + z * H_l^{-\beta} + (1 - \alpha - z) * q_g\}^{(-1/\beta)}$ and key equilibrium conditions for household equilibrium are

$$\frac{u_{q_c}}{u_{H_l}} = \frac{p_c}{(1 - \tau)w} \text{ and } q_c = \frac{1}{p_c}[(1 - \tau) * \{w * (H_c + H_g) + r * K/N\}].$$

$u_{q_c}$ and $u_{H_l}$ are marginal utilities (partial derivatives with respect to $q_c$ and $H_l$) for a household. The equilibrium conditions for the model are set out in detail Appendix 1 and the Matlab program for solving the model is in Appendix 2. For this general model, we observe a counterpart to the above "basic" result associated with equation (1): when labor "counts large" in the production of each good, we observe that leisure rises as the rate of income tax rises for our representative agent when the elasticity is greater than unity (but near unity for the agent's CES utility function defined on consumption, leisure and government production), and leisure declines with tax increases when the elasticity is less than unity (but near unity).

In solving for the results in Table 1, each of our N agents has the same tastes and income (identical agents). In the first case, $\beta = 0.05$ ($\sigma = 0.9524$) and $\alpha = 0.1$ and $z = 0.8$. The production functions are Cobb-Douglas (constant returns to scale) in $Nq_c = K_c^{alc}[NH_c]^{1-alc}$, and

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[1] Tax revenue is the only source of government funding in the model and being a single period model, we do not consider the possibility of a deficit.

$q_g = [K - K_c]^{a \lg}[N H_g]^{1-a \lg}$, with $alc = 0.02.$ and $a \lg = 0.06.$[2] $N = 50$ and $K = 40$ and these parameters define the size of our static economy. Labor "counting large" refers to coefficients in production, $alc$ and $a \lg$ being close ot zero. For the results in Table 2, $\beta = -0.05$ or $\sigma = 1.0526$. The following two tables report the results for our economy, of raising the rate of income tax for (a) the elasticity of substitution in the representative agent's utility function slightly less than unity (Table 1) and (b) slightly greater than unity (Table 2).

**Table 1: Marginally inelastic**

($\beta$=0.05, exteme labor (alc=.02;alg=.06), $\alpha = .1$, z=.8)

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| 0.95 | 14.3826 | 0.1554 | 137.3406 | 0.1632 | 2.9751 | 20.8616 |
| 0.85 | 16.2255 | 0.4461 | 118.1607 | 0.4676 | 2.5416 | 20.9907 |
| 0.75 | 17.0105 | 0.7288 | 102.6070 | 0.7625 | 2.1940 | 21.0435 |
| 0.65 | 17.4382 | 1.0075 | 88.1493 | 1.0519 | 1.8738 | 21.0743 |
| 0.55 | 17.6593 | 1.2837 | 74.2171 | 1.3375 | 1.5680 | 21.0944 |
| 0.45 | 17.7215* | 1.5583 | 60.5744 | 1.6201 | 1.2714 | 21.1084 |
| 0.35 | 17.6287 | 1.8320 | 47.0914 | 1.9002 | 0.9814 | 21.1184 |
| 0.25 | 17.3423 | 2.1055 | 33.6806 | 2.1781 | 0.6964 | 21.1255 |
| 0.15 | 16.7295 | 2.3794 | 20.2706 | 2.4541 | 0.4154 | 21.1305 |
| 0.05 | 15.1536 | 2.6544 | 6.7910 | 2.7283 | 0.1377 | 21.1340 |

| $\tau$ | $p_g$ | $r$ | $w$ | $K_c$ | Re $v$ |
|---|---|---|---|---|---|
| 0.95 | 1.0748 | 0.2253 | 0.9327 | 0.6897 | 147.6086 |
| 0.85 | 1.0697 | 0.2008 | 0.9349 | 2.2222 | 126.3982 |
| 0.75 | 1.0655 | 0.1822 | 0.9368 | 4.0000 | 109.3269 |
| 0.65 | 1.0613 | 0.1655 | 0.9386 | 6.0870 | 93.5547 |
| 0.55 | 1.0570 | 0.1498 | 0.9405 | 8.5714 | 78.4473 |
| 0.45 | 1.0524 | 0.1346 | 0.9426 | 11.5789 | 63.7483 |
| 0.35 | 1.0474 | 0.1198 | 0.9448 | 15.2941 | 49.3238 |
| 0.25 | 1.0419 | 0.1053 | 0.9473 | 20.0000 | 35.0918 |
| 0.15 | 1.0357 | 0.0910 | 0.9502 | 26.1538 | 20.9944 |
| 0.05 | 1.0286 | 0.0768 | 0.9534 | 34.5455 | 6.9852 |

Rev in the table is total government revenue, equal to the cost of the government for producing $q_g$. Observe in Table 1 that the utility of an

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[2]When we speak of labor counting large in production, we mean that the coefficient in the Cobb-Douglas production function for capital is small. This choice of parameters leads to the rental rate for capital ending up relatively small.

agent is a maximum when $\tau = 0.45$ and the size of the government sector (revenue and expenditure) rises with the tax rate. The Laffer Curve is peaking as $\tau \to 1$. The leisure of an agent, $H_l$ is declining steadily with increases in $\tau$.

**Table 2: Marginally elastic** ($\beta = -.05$, (alc=.02;alg=.06), z=.8, $\alpha = .1$)

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| 0.95 | 15.8041 | 0.0989 | 89.0274 | 0.1029 | 1.8759 | 22.0211 |
| 0.85 | 17.1420 | 0.3128 | 84.0512 | 0.3255 | 1.7691 | 21.9055 |
| 0.75 | 17.6932 | 0.5346 | 76.2192 | 0.5557 | 1.5991 | 21.8451 |
| 0.65 | 17.9626 | 0.7614 | 67.3804 | 0.7904 | 1.4080 | 21.8016 |
| 0.55 | 18.0600* | 0.9920 | 57.9599 | 1.0282 | 1.2054 | 21.7664 |
| 0.45 | 18.0180 | 1.2260 | 48.1248 | 1.2684 | 0.9954 | 21.7362 |
| 0.35 | 17.8339 | 1.4630 | 37.9535 | 1.5105 | 0.7802 | 21.7093 |
| 0.25 | 17.4683 | 1.7032 | 27.4820 | 1.7543 | 0.5609 | 21.6848 |
| 0.15 | 16.7974 | 1.9466 | 16.7204 | 1.9996 | 0.3385 | 21.6620 |
| 0.05 | 15.2637 | 2.1937 | 5.6563 | 2.2461 | 0.1134 | 21.6405 |

| $\tau$ | $p_g$ | $r$ | $w$ | $K_c$ | Re $v$ |
|---|---|---|---|---|---|
| 0.95 | 1.0551 | 0.1434 | 0.9414 | 0.6897 | 93.9344 |
| 0.85 | 1.0543 | 0.1407 | 0.9417 | 2.2222 | 88.6170 |
| 0.75 | 1.0521 | 0.1336 | 0.9427 | 4.0000 | 80.1900 |
| 0.65 | 1.0493 | 0.1251 | 0.9440 | 6.0870 | 70.6993 |
| 0.55 | 1.0459 | 0.1157 | 0.9455 | 8.5714 | 60.6223 |
| 0.45 | 1.0421 | 0.1059 | 0.9472 | 11.5789 | 50.1529 |
| 0.35 | 1.0378 | 0.0957 | 0.9492 | 15.2941 | 39.3894 |
| 0.25 | 1.0329 | 0.0852 | 0.9514 | 20.0000 | 28.3868 |
| 0.15 | 1.0273 | 0.0744 | 0.9541 | 26.1538 | 17.1762 |
| 0.05 | 1.0206 | 0.0635 | 0.9572 | 34.5455 | 5.7730 |

For this "marginally elastic" case, leisure, $H_l$ is rising with $\tau$. These results on the motion of leisure, $H_l$ with respect to the rise in $\tau$ are similar to those with the single agent, partial model we took up first, above. The utility level is a maximum with $\tau = 0.55$. Utility is somewhat higher at its maximum value when the elasticity of substitution is greater than unity. Again Revenue is rising with rate $\tau$ yielding a Laffer Curve with a corner maximum.

The large issue is now on the table: does the model yield a Laffer Curve with an interior peak? And what conditions lead to a Laffer Curve with an interior peak? The brief answer is "yes", the model has a solution with an interior Laffer Curve peak and such a solution seemingly requires the elasticity of sustitution in an agent's utility function to be

greater than unity. We turn to solved cases.

# 4    Interior Peaking of the Laffer Curve

The above examples suggest that by allowing a household to move more easily out of hours working, one might get the household to move more rapidly, into leisure-time consumed. This hint in fact works. We move $\beta$ from -0.05 to -0.5 making $\sigma$ now 2. The numerical outputs from repeated solving with different values of $\tau$ are in Table 3. The Laffer Curve now peaks with $\tau = 0.55$ and the utility level of an agent peaks for $\tau = 0.45$ (these values of $\tau$ are approximate for these two peaking locations).

**Table 3: Laffer Curve Peak and Elastic Substitution in Utility**

($\beta$=-0.5, (alc=.02;alg=0.06) and $\alpha = 0.1$ and z=.8)

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| 0.95 | 16.2017 | 0.0010 | 1.0875 | 0.0009 | 0.0173 | 23.9818 |
| 0.85 | 16.7124 | 0.0086 | 2.6894 | 0.0084 | 0.0454 | 23.9462 |
| 0.75 | 16.9961 | 0.0236 | 3.8175 | 0.0230 | 0.0662 | 23.9108 |
| 0.65 | 17.1671 | 0.0457 | 4.5346 | 0.0448 | 0.0798 | 23.8755 |
| 0.55 | 17.2560 | 0.0749 | 4.8649 | 0.0737 | 0.0864 | 23.8399 |
| 0.45 | 17.2730* | 0.1113 | 4.8200 | 0.1097 | 0.0861 | 23.8042 |
| 0.35 | 17.2180 | 0.1549 | 4.4050 | 0.1528 | 0.0789 | 23.7683 |
| 0.25 | 17.0807 | 0.2058 | 3.6203 | 0.2031 | 0.0649 | 23.7320 |

| | $\tau$ | $p_g$ | $r$ | $w$ | $K_c$ | $\mathrm{Re}\,v$ |
|---|---|---|---|---|---|---|
| | 0.95 | 0.8748 | 0.0015 | 1.0339 | 0.6897 | 0.9513 |
| | 0.85 | 0.9107 | 0.0039 | 1.0133 | 2.2222 | 2.4491 |
| | 0.75 | 0.9262 | 0.0059 | 1.0047 | 4.0000 | 3.5359 |
| ... | 0.65 | 0.9354 | 0.0075 | 0.9998 | 6.0870 | 4.2418 |
| | 0.55 | 0.9413 | 0.0087 | 0.9967 | 8.5714 | 4.5792* |
| | 0.45 | 0.9449 | 0.0096 | 0.9948 | 11.5789 | 4.5546 |
| | 0.35 | 0.9470 | 0.0101 | 0.9937 | 15.2941 | 4.1714 |
| | 0.25 | 0.9476 | 0.0103 | 0.9934 | 20.0000 | 3.4305 |

Laffer Curve is peaking at $\tau = 0.55$ and the utility level is a maximum for $\tau = 0.45$. To summerize, we were able to obtain a Laffer Curve with an interior peak quite generally when the elasticity of substitution in utility was greater than unity and relatively "far above" unity.

We turn to a closer analysis of the behavior of our economy in the neighborhood of the peak of the Laffer Curve. We now investigate changing the value of $\tau$ near 0.50. See Table 4. The parameter selection is otherwise the same as that for Table 3 above.

**Table 4: Near the Laffer Peak** ($z=.8$, $\beta=-0.5$, alc$=.02$, alg$=.06$)

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| 0.55 | 17.2560 | 0.0749 | 4.8649 | 0.0737 | 0.0864 | 23.8399 |
| 0.53 | 17.2650 | 0.0816 | 4.8857 | 0.0803 | 0.0869 | 23.8328 |
| 0.51 | 17.2712 | 0.0886 | 4.8916 | 0.0872 | 0.0871 | 23.8257 |
| 0.49 | 17.2746 | 0.0959 | 4.8826 | 0.0944 | 0.0870 | 23.8185 |
| 0.47 | 17.2752* | 0.1035 | 4.8587 | 0.1019 | 0.0867 | 23.8114 |
| 0.45 | 17.2730 | 0.1113 | 4.8200 | 0.1097 | 0.0861 | 23.8042 |

| | $\tau$ | $p_g$ | $r$ | $w$ | $K_c$ | Re $v$ | Re $v/N$ | $\tau Y/q_g$ |
|---|---|---|---|---|---|---|---|---|
| | 0.55 | 0.9413 | 0.0087 | 0.9967 | 8.5714 | 4.5792 | 0.0916 | 0.01883 |
| | 0.53 | 0.9422 | 0.0089 | 0.9962 | 9.1262 | 4.6031 | 0.0921 | 0.01885 |
| ... | 0.51 | 0.9430 | 0.0091 | 0.9958 | 9.7030 | 4.6126* | 0.0923 | 0.01887 |
| | 0.49 | 0.9437 | 0.0093 | 0.9954 | 10.3030 | 4.6076 | 0.0922 | 0.01888 |
| | 0.47 | 0.9443 | 0.0095 | 0.9951 | 10.9278 | 4.5883 | 0.0918 | 0.01889 |
| | 0.45 | 0.9449 | 0.0096 | 0.9948 | 11.5789 | 4.5546 | 0.0911 | 0.01890 |

$\tau Y/q_g$ is payment per unit. The government sets $\tau$ and balances its income and expenditure. A household takes $\tau$ as a parameter as well as the equilibrium value of $q_g$. The "experiment" reported in Table 4 gets to the heart of the behavior of the Laffer Curve at its peak. As $\tau$ is increased from 0.45, payment $\frac{\tau Y}{q_g}$ for a unit of $q_g$ declines and $q_g$ increases , UP TO 4.8916, as $\tau$ reaches 0.51 and tax revenue peaks at 4.6126 (taxable income $Y$ is increasing slightly). $q_g$ is then "hitting" a maximum value at the same time that total revenue is peaking. As $\tau$ is increased to values beyond the one associated with the peaking of tax revenue, payment $\frac{\tau Y}{q_g}$ continues to decline and $q_g$ begins to contract, and continues to contract. Hence the Laffer Curve not only peaks with a maximum of government revenue and expenditure but peaks when a maximum of $q_g$ is being produced. It turns out that willingness-to-pay for $q_g$, namely $\frac{u_{q_g}}{u_{q_c}}$, given above values for $q_g$, is much above the current level of payment. See Figure 1 below. This is a central aspect of the

second best nature of the equilibrium. We report on this in more detail below.

We digress from labor-leisure choice for a moment to report on exogenous shifts in the size of the population in our economy. With public goods, one is on alert for numbers of households sharing the cost of a public good (Flatters, Henderson, Mieszkowski (1974)). Of interest is whether more agents are associated with (a) more of the public good being produced and (b) whether the cost per household declines with more agents. In our model this matter is complicated because there is a fixed amount of capital with returns shared equally across households. As well, more households in our model mean more total labor being supplied to the economy. To probe these issues, we have re-solved out model with population increased from 50 agents to 55 agents, other things remaining unchanged. For reference points, we use two situations: one with utility per agent a maximum at the initial population and the second with tax revenue a maximum at the initial population. The results are reported in Table 5.

**Table 5: Population Expansion** to N=55

(alc=0.02, alg=0.06, z=.8, $\alpha = .1, \beta = -.5$)

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| .475, $N = 50$ | 17.2753 | 0.1016 | 4.8661 | 0.1000 | 0.0868 | 23.8132 |
| .475, $N = 55$ | 17.3575 | 0.1012 | 5.3129 | 0.0998 | 0.0866 | 23.8135 |
| 0.55,N=50 | 17.2560 | 0.0749 | 4.8649 | 0.0737 | 0.0864 | 23.8399 |
| 0.55,N=55 | 17.3382 | 0.0747 | 5.3116 | 0.0735 | 0.0862 | 23.8402 |

| $\tau$ | $p_g$ | $r$ | $w$ | $K_c$ | Re $v$ | $u_{q_c}$ | $u_{q_g}$ | $u_{q_g}/u_{q_c}$ |
|---|---|---|---|---|---|---|---|---|
| .475,$N = 50$ | 0.9442 | 0.0094 | 0.9952 | 10.7692 | 4.5945 | 1.3042 | 0.1884 | 0.1445 |
| .475,$N = 55$ | 0.9477 | 0.0103 | 0.9933 | 10.7692 | 5.0352 | 1.3097 | 0.1807 | 0.1380 |
| 0.55,N=50 | 0.9413 | 0.0087 | 0.9967 | 8.5714 | 4.5792 | 1.5175 | 0.1883 | 0.1241 |
| 0.55,N=55 | 0.9448 | 0.0096 | 0.9948 | 8.5714 | 5.0184 | 1.5240 | 0.1807 | 0.1186 |

$u_{q_c}$ and $u_{q_g}$ are derivatives of the utility function with respect to $q_c$ and $q_g$ respectively. In each case in Table 5, the larger population is associated with utility per agent rising and leisure rising. In each case, the quantity of the public good produced rises with the population

increase, while the quantity of the consumption good per agent declines marginally. Total government revenue increases with population and payment per household, Rev/N declines marginally with the population expansion.

We turn to a case in which leisure per household does not move uniformly up or down with increases in the income tax rate. The capital input "counts large" in this example ($alc = 0.98$ and $a\lg = .96$). We continue to have leisure count large in the household's utility function ($z = 0.8$). For this case, also elastic in utility ($\sigma = 2$), the outputs are reported in Table 6.

**Table 6: Non-Monotonic Motion in Leisure Consumed**

("extreme" weights on capital ($alc = 0.98$; $a\lg = 0.96$), $\alpha = 0.1$, $z = 0.8$, $\beta = -0.5$);

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|------|---------|--------|---------|--------|--------|---------|
| 0.65 | 19.5500 | 0.2609 | 20.4715 | 0.0022 | 0.0122 | 23.9856 |
| 0.55 | 19.2965 | 0.3349 | 17.3837 | 0.0032 | 0.0117 | 23.9851 |
| 0.45 | 19.0044 | 0.4085 | 14.2521 | 0.0043 | 0.0106 | 23.9851 |
| 0.35 | 18.6668 | 0.4816 | 11.0965 | 0.0055 | 0.0089 | 23.9856 |

| | $\tau$ | $p_g$ | $r$ | $w$ | $K_c$ | $\mathrm{Re}\,v$ |
|---|------|--------|--------|--------|---------|---------|
| | 0.65 | 1.1834 | 0.8889 | 2.3838 | 14.3816 | 24.2258 |
| ... | 0.55 | 1.1773 | 0.8912 | 2.1007 | 18.4134 | 20.4659 |
| | 0.45 | 1.1724 | 0.8930 | 1.8982 | 22.4116 | 16.7098 |
| | 0.35 | 1.1684 | 0.8946 | 1.7439 | 26.3768 | 12.9651 |

# 5   Local Analysis of an Increase in the Tax Rate

The textbook measurement of the local incidence of an increase in rate of income tax per household involves perturbing the rate $\tau$ and then measuring $\frac{du}{d\tau}$, for a representative agent, as in:

$$\frac{du}{d\tau} = u_{q_c}\frac{dq_c}{d\tau} + u_{H_l}\frac{dH_l}{d\tau} + u_{q_g}\frac{dq_g}{d\tau}$$

$$\text{or } \Delta Z = \frac{du}{u_{q_c}}\Delta\tau = \left[dq_c + \left[\frac{(1-\tau)w}{p_c}\right]dH_l + \frac{u_{q_g}}{u_{q_c}}dq_g\right]\Delta\tau, \qquad (2)$$

in numeraire units. Recall that $\frac{u_{H_l}}{u_{q_c}} = \frac{(1-\tau)w}{p_c}$. We proceeded to calculate $\Delta Z/\Delta\tau$ for our example in the neighborhood of the utility maximizing

$\tau$ in Table 4, namely $\tau = .475$. The data are in Table 7, entries closely linked to those reported on in Table 4 above. We focused on six shifts: $\tau$ moving from 0.15 to 0.16, 0.25 to 0.26 0.35, to 0.36, 0.45 to 0.46, 0.47 to 0.48 and from 0.49 to 0.50. In addition to our measure in (2), we have available the value for the direct shift in the utility level for each case.

**Table 7: Examining $\tau$ increments near the Peak of Utility**

(extreme lab coeffs (alc=.02, alg=.06), $\alpha = 0.1$, z=.8, $\beta$=-0.5 (elastic))

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ | $p_g$ | $r$ |
|---|---|---|---|---|---|---|---|---|
| 0.50 | 17.2733 | 0.0922 | 4.8890 | 0.0908 | 0.0871 | 23.8221 | 0.9433 | 0.0092 |
| 0.49 | 17.2746 | 0.0959 | 4.8826 | 0.0944 | 0.0870 | 23.8185 | 0.9437 | 0.0093 |
| 0.48 | 17.2753 | 0.0997 | 4.8725 | 0.0981 | 0.0869 | 23.8150 | 0.9440 | 0.0094 |
| 0.47 | 17.2752 | 0.1035 | 4.8587 | 0.1019 | 0.0867 | 23.8114 | 0.9443 | 0.0095 |
| 0.46 | 17.2745 | 0.1074 | 4.8412 | 0.1058 | 0.0864 | 23.8078 | 0.9447 | 0.0095 |
| 0.45 | 17.2730 | 0.1113 | 4.8200 | 0.1097 | 0.0861 | 23.8042 | 0.9449 | 0.0096 |
| 0.36 | 17.2270 | 0.1503 | 4.4632 | 0.1482 | 0.0799 | 23.7719 | 0.9468 | 0.0101 |
| 0.35 | 17.2180 | 0.1549 | 4.4050 | 0.1528 | 0.0789 | 23.7683 | 0.9470 | 0.0101 |
| 0.26 | 17.0988 | 0.2004 | 3.7155 | 0.1977 | 0.0666 | 23.7357 | 0.9476 | 0.0103 |
| 0.25 | 17.0807 | 0.2058 | 3.6203 | 0.2031 | 0.0649 | 23.7320 | 0.9476 | 0.0103 |
| 0.16 | 16.8626 | 0.2580 | 2.5947 | 0.2544 | 0.0465 | 23.6991 | 0.9470 | 0.0101 |
| 0.15 | 16.8308 | 0.2642 | 2.4618 | 0.2605 | 0.0441 | 23.6954 | 0.9468 | 0.0101 |

| | $\tau$ | $w$ | $K_c$ | $\mathrm{Re}\,v$ | $u_{q_c}$ | $u_{q_g}$ | $u_{q_g}/u_{q_c}$ | $\Delta Z/\Delta\tau$ | $\epsilon$ |
|---|---|---|---|---|---|---|---|---|---|
| | 0.50 | 0.9956 | 10.0000 | 4.6119 | 1.3685 | 0.1880 | 0.1374 | -.001005 | 0.9504 |
| | 0.49 | 0.9954 | 10.3030 | 4.6076 | 1.3420 | 0.1881 | 0.1402 | | |
| | 0.48 | 0.9952 | 10.6122 | 4.5998 | 1.3166 | 0.1883 | 0.1430 | .0000748 | 0.8651 |
| | 0.47 | 0.9951 | 10.9278 | 4.5883 | 1.2921 | 0.1886 | 0.1459 | | |
| | 0.46 | 0.9949 | 11.2500 | 4.5733 | 1.2684 | 0.1889 | 0.1489 | .00124 | 0.8136 |
| ... | 0.45 | 0.9948 | 11.5789 | 4.5546 | 1.2456 | 0.1893 | 0.1520 | | |
| | 0.36 | 0.9938 | 14.8837 | 4.2258 | 1.0708 | 0.1965 | 0.1835 | .008515 | 0.5426 |
| | 0.35 | 0.9937 | 15.2941 | 4.1714 | 1.0542 | 0.1977 | 0.1875 | | |
| | 0.26 | 0.9934 | 19.4737 | 3.5207 | 0.9237 | 0.2145 | 0.2322 | .01971 | 0.3393 |
| | 0.25 | 0.9934 | 20.0000 | 3.4305 | 0.9110 | 0.2172 | 0.2384 | | |
| | 0.16 | 0.9937 | 25.4545 | 2.4571 | 0.8085 | 0.2549 | 0.3153 | .039634 | 0.1845 |
| | 0.15 | 0.9938 | 26.1538 | 2.3309 | 0.7982 | 0.2615 | 0.3276 | | |

These values for $\Delta Z/\Delta$ are very close to the corresponding values of direct measures, $\Delta u/u_{q_c}$. We can infer that the linearized measure in $\Delta Z/\Delta\tau$ captures well the true welfare change emerging from the corre-

sponding small change in tax rate $\tau$. $u_{q_g}/u_{q_c}$ is an "equilibrium point" on the demand curve of a household for the government good. Hence given a rate $\tau$ and the corresponding equilibrium for the model, the distance $u_{q_g}/u_{q_c} - \frac{\tau Y}{q_g}$ is reflecting the welfare cost of raising revenue by an income tax instead of with first best Samuelson, personal charges. In Figure 1, we have sketched how the price of a unit of $q_g$ declines along the demand schedule, $u_{q_g}/u_{q_c}$ with an increase in $\tau$.
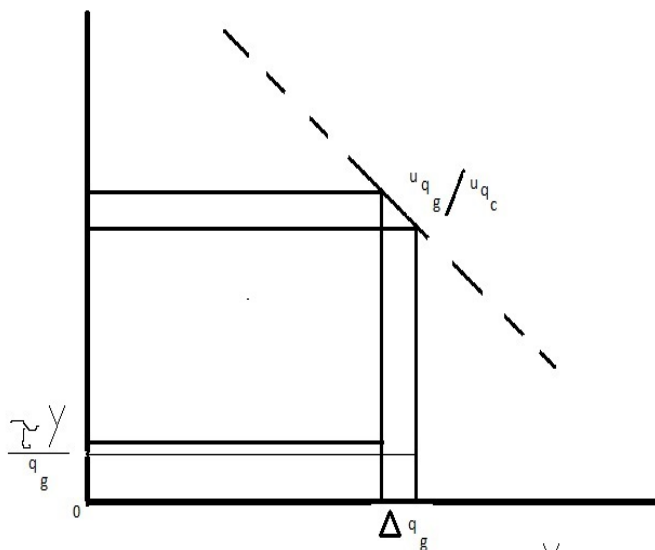


Figure 1: A household's current payment for its government good is $\frac{\tau Y}{q_g}$. Marginal benefit or willingness-to-pay is $u_{q_g}/u_{q_c}$, much higher.

The tax rate is raised a small amount and the quantity of the government good increases a small amount.

Our sketch in Figure 1 is for an equilibrium for a household responding to a small increase in rate $\tau$, for a value of $\tau$ below the value at the peak of the Laffer Curve. $\Delta q_g$ is the quantity response.

The last column of entries in Table 7 contains the elasticity of aggregate taxable income for the small change in $\tau$. This elasticity, $\epsilon$ is approaching unity as $\tau$ approaches the value for aggregate government revenue a maximum. This elasticity incorporates changes in the rental

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rate for $K$, which is somewhat novel for such estimations. Since the rental rate is very small, the shift in this value will not be having much impact on the value of the elasticity. Clearly a more complete model muct have an intertemporal dimension in order to include accumulation of $K$ and a rate of interest. See for example Straub and Werning (2014) or a recent analysis of the taxation of capital income.

# 6  The First Best Case

We turn now to solving the first best model for the same parameters that we employed above. The government is now charging each household a payment based on its marginal benefit for the current level of $q_g$. For this first best model, each household consumes the amount $q_g$ such that its $u_{q_g}/u_{q_c} = (p_g/N)/p_c$, and we set $p_c = 1$. It is as if the household is buying the government good at a retailer and paying $p_g/N$ per unit purchased. In addition there is no tax distortion in the selection of hours worked and hours of leisure consumed, that is, $u_{H_l}/u_{q_c} = w/p_c$.[3] The government's budget is balanced somewhat indirectly with the sum of charges per household equal to total costs of production of public good $q_g$. The results for this first best case are in the first line of Table 8 (see also the Matlab program in Appendix 3). The second and third lines are results for the second best model (the income tax model) in the neighborhood of the maximized value of the utility of a household (these two second best cases are reproduced from Table 7 above).

**Table 8: First Best Case in line One**

(extreme lab coeffs (alc=.02, alg=.06), $\alpha = 0.1$, z=.8, $\beta$=-0.5 (elastic))

---

[3]To move from the second best (income tax) model to the first best (Samuelson charging) model we simply substitute $u_{q_g}/u_{q_c} = (p_g/N)/p_c$ for the government revenue-expenditure balance equation and remove the $(1 - \tau)$ from the $u_{H_l}/u_{q_c} = (1 - \tau)w/p_c$ equation.

| | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| $p_g/N$ | 25.9407 | 0.1939 | 384.7498 | 0.2082 | 8.8954 | 14.8964 |
| $\tau = 0.48$ | 17.2753 | 0.0997 | 4.8725 | 0.0981 | 0.0869 | 23.8150 |
| $\tau = 0.47$ | 17.2752 | 0.1035 | 4.8587 | 0.1019 | 0.0867 | 23.8114 |

| | $\tau$ | $p_g$ | $r$ | $w$ | $K_c$ | $\mathrm{Re}\,v$ | $u_{q_c}$ | $u_{q_g}$ | $u_{q_g}/u_{q_c}$ |
|---|---|---|---|---|---|---|---|---|---|
| ... | $p_g/N$ | 1.1224 | 0.6526 | 0.9127 | 0.2971 | 431.8619 | 1.1567 | 0.0260 | 0.0224 |
| | 0.48 | 0.9440 | 0.0094 | 0.9952 | 10.6122 | 4.5998 | 1.3166 | 0.1883 | 0.1430 |
| | 0.47 | 0.9443 | 0.0095 | 0.9951 | 10.9278 | 4.5883 | 1.2921 | 0.1886 | 0.1459 |

The utility level attained rises for the first best case to 25.94 from about 17.3 for the second best case. And as we anticipated (inference from Figure 1), the quantity of $q_g$ consumed is much higher for the first best case, rising from about 4.8 to 384.75. The rental rate on capital is much higher and hours of leisure consumed decline by about 9.0. We have then a direct measure of the welfare cost of an income tax raising revenue in place of Samuelson public good personal charges.

**Population Expansion:** We reported on population expansion above for the second best (income tax) model. Here we report on a similar shift in the size of our economy for the first best model. We recomputed our first best equilibrium with $N$ increase from the baseline value of 50 to the new value of 55. The results are reported in Table 9.

**Table 9: Larger Population for First Best Case**

(extreme lab coeffs (alc=.02, alg=.06), $\alpha = 0.1$, z=.8, $\beta$=-0.5 (elastic))

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| $N = 50$ | 25.9407 | 0.1939 | 384.7498 | 0.2082 | 8.8954 | 14.8964 |
| $N = 55$ | 26.8757 | 0.1865 | 442.5467 | 0.2009 | 9.3846 | 14.4145 |

| | $\tau$ | $p_g$ | $r$ | $w$ | $K_c$ | $\mathrm{Re}\,v$ | $u_{q_c}$ | $u_{q_g}$ | $u_{q_g}/u_{q_c}$ |
|---|---|---|---|---|---|---|---|---|---|
| ... | $N = 50$ | 1.1224 | 0.6526 | 0.9127 | 0.2971 | 431.8619 | 1.1567 | 0.0260 | 0.0224 |
| | $N = 55$ | 1.1291 | 0.7547 | 0.9100 | 0.2719 | 499.6895 | 1.2004 | 0.0246 | 0.0205 |

Observe that utility rises with the population increase, leisure declines, the size of the government sector expands significantly and per capita charge for a unit of the government good declines to .02053 (N=55) from .02245 for the case of the smaller population. However, the expenditure per household on the public good rises with the increase in population.

This is different from what we observed for population expansion for the second best model above.

**Elasticity contrast for the First Best Case:** We illustrate the impact of the value of the elasticity of substitution for the first best formulation. We compare cases with $\sigma = 2$ ($\beta = -.5$) and $\sigma = 2/3$ ($\beta = 0.5$).Our parameter selection is N=50; $\alpha$=0.4; z=0.2; alc=0.2; alg=0.35; and K=40. The inelastic case is reported in the first row in Table 10. The utility level rises to 61.9 from 13.4 when the elasticity rises to 2.

**Table 10: First Best with Elasticity Contrasting Solutions**
($\alpha$=0.4; z=0.2; alc=0.2; alg=0.35; K=40; $\beta$=0.5 and $\beta = -0.5$)

|  | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| $\sigma = 2/3$ | 13.4085 | 6.8293 | 69.9137 | 13.0188 | 3.3060 | 7.6753 |
| $\sigma = 2$ | 61.892 | 0.419 | 344.5 | 0.95 | 22.2 | 0.847 |

| | $p_g$ | $r$ | $w$ | $K_c$ | $p_g q_g$ |
|---|---|---|---|---|---|
| ... | 1.5265 | 2.6412 | .4197 | 25.8574 | 106.723 |
| | 1.74 | 5.359 | 0.352 | 0.78 | 599.43 |

When we change the elasticity to $\sigma = 2$, we get the results in the second row of results in Table 8. Startling is the large rise in the level of utility as the utility function is made more elastic. The household has substituted strongly out of leisure and $q_c$ and into the public good with its elasticity changed to greater than unity.

# 7 Different Distributions of Income between Two Groups

We consider our second-best model with heteogeneous agents, each with a distinct ownership $K^i$. Each household will be consuming the same quantity of the government good, $q_g$ and will be paying the same rate of income tax, $\tau$. A household's utility function will be $u(q_c^i, H_l^i, q_g)$ with

$$p_c q_c^i = (1 - \tau)\{rK^i + w[H_c^i + H_g^i]\}, \qquad (i = 1, ..., N).$$

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Each household is paying tax in the amount, $\tau\{rK^i + w[H_c^i + H_g^i]\}$, $(i = 1, ..., N)$. The government's budget constraint will be

$$p_g q_g = \tau[rK + w\sum_{i=1}^{N}\{H_c^i + H_g^i\}], \text{ with } \sum_{i=1}^{N}K^i = K.$$

Equilibrium for a household will satisfy

$$\frac{u_{q_c^i}}{u_{H_l^i}} = \frac{p_c}{(1 - \tau)w}; \quad 24 = H_c^i + H_g^i + H_l^i, \quad (i = 1, ..., N).$$

We proceeded to calculate some solutions to the above second-best model with two types of individuals. Two individuals in the two distinct groups differ in terms of the endowment of $K$ per capita. The Matlab program is Appendix 4.

The examples reported on in Table 11 have the 25 workers denoted with letter b with less capital owned per person. Group "a" has no subscript. Among other things, a person in the b-group ends up with lower utility in the three runs reported on in Table 11. There are two experiments reported on in Table11. First we solve and then raise the tax rate. See the results in the first two rows of numbers in Table11. Secondly, we re-solve the second case with a different endowment for each person in our a group and our b group.

**Table 11: Distinct Holdings of $K$ for Two Groups**

(Ka=22, Kb=18, Nb=25, ($\sigma = 2/3$; $\beta = 0.5$), $\alpha = 0.4$, $z = 0.2$, alc=0.2, alg=0.35)

| $\tau$ | $u$ | $q_c$ | $q_g$ | $H_c$ | $H_g$ | $H_l$ |
|---|---|---|---|---|---|---|
| 0.22 (Kb=18) | 13.5066 | 6.7410 | 62.0252 | 10.9020 | 4.2524 | 8.8456 |
| 0.27 ($Kb = 18$) | 13.5012 | 6.3507 | 76.0389 | 9.8524 | 5.3830 | 8.7646 |
| 0.27 (Kb=15) | 13.7797 | 6.5087 | 76.0389 | 9.1140 | 5.9033 | 8.9827 |

| | $ub$ | $q_{cb}$ | $H_{cb}$ | $H_{gb}$ | $H_{lb}$ | $p_g$ | $r$ | $w$ | $K_c$ |
|---|---|---|---|---|---|---|---|---|---|
| | 13.1544 | 6.5258 | 13.9857 | 1.4510 | 8.5632 | 1.5082 | 2.4769 | 0.4265 | 26.7811 |
| ... | 13.1271 | 6.1400 | 13.8010 | 1.7252 | 8.4738 | 1.5189 | 2.5719 | 0.4225 | 24.2827 |
| | 12.8445 | 5.9820 | 14.5395 | 1.2048 | 8.2557 | 1.5189 | 2.5719 | 0.4225 | 24.2827 |

The rise in the tax rate from 0.22 to 0.27 results in marginally more total labor being put out in activity in the consumption and government goods sectors combined (see the first two rows of results in Table 11.) When we re-allocate endowments across agents alone, the production side of the economy is unchanged (last two rows in Table11), including total labor supplied by the $N$ agents to activity in the consumption and government goods sectors.[4] Consumption bundles do change for agents with different endowments, including the amounts of leisure by distinct agents. Observe that prices $p_g, r$ and $w$ are unchanged; also $Kc$ remains unchanged.

Total tax revenue is based on total income and the tax rate. Since our total income in the model is independent of the distribution of endowments of total $K$ across individuals, the current tax revenue will be independent also of the distribution of endowments of total $K$ across individuals. Hence our earlier result for the Laffer curve will persist. The Laffer Curve will peak at a corner with $\tau$ tending to unity.

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[4]This result appears to be related to the fact that each individual agent is indifferent as to the number of hours of work she devotes to either of consumption or government goods production. The agent cares only about her total hours worked.

# 8    Concluding Remarks

We set out what seems to be the simplest general model incorporating a government sector producing a public good for $N$ households and balancing its budget with an income tax. We focused attention on the properties of the Laffer Curve within our model. Quite generally the Laffer curve peaked with a tax rate less than unity when the elasticity of substitution in the utility function of a representative household was well above unity and peaked with a tax rate of unity when the elasticity was close to unity or less than unity. We observed that the second best economy, operating with an income tax, tended to under produce the government good relative to a first best economy (one with Samuelsonian marginal benefit charges for the government good). We also observed that the local linearized measure of welfare change for an increase in the rate of tax worked well for the model. Changes in the size of the economy (a larger initial population) were reported on for both the second best and first best models. Population shifts are particularly interesting for economies with public goods (economies with the cost of the government sector shared in a Samuelsonian way). We also reported on a second best economy with two classes of households. Income tax incidence has turned out to be interesting within this framework of analysis.

## References

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APPENDIX 1: EQUILIBRIUM CONDITIONS FOR THE INCOME
TAX MODEL

Each of $N$ agents has utility function $u(q_c, H_l, q_g)$. $H_l$ is defined by
$24 - H_c - H_g$ where $H_c$ is hours worked by the agent in the (private)
consumer goods sector and $H_g$ is hours worked by the agent in the gov-
ernment goods sector. $q_g$ is set by the government so that the budget
is balanced. Every agent consumes quantity $q_g$ and pays indirectly for
$q_g$ via an income tax with rate $\tau$ on taxable income, $Y$. $p_g q_g$ is the total
cost of producing the public good in the economy. Total production of
the private good is $N q_c$. Production in each sector involves a constant
returns to scale production function,

$$N q_c = f(K_c, N H_c)$$

for $K_c$ capital (say buildings and machines) used in the consumption
goods sector. We have

$$q_g = g(K - K_c, N H_g)$$

for $K$ the exogenous endowment of capital to the economy, currently.
$q_g$ is a flow of services that government produces rather than something
like infrastructure that is durable.

For the producers of $N q_c$ and $q_g$, equilibrium requires

$$\frac{f_{K_c}}{f_{[NH_c]}} = \frac{r}{w} \text{ and } \frac{g_{[K-K_c]}}{g_{[NH_g]}} = \frac{r}{w}.$$

$f_{K_c}$ and $f_{[NH_c]}$ are first derivatives of $f(.)$ and $g_{[K-K_c]}$ and $g_{[NH_g]}$ are first
derivatives of $g(.)$. $r$ is the rental rate per unit of $K$ and $w$ is the wage
per hour. The zero profit conditions are

$$p_c N q_c = r K_c + w N H_c \text{ and } p_g q_g = r[K - K_c] + w N H_g.$$

Each household is maximizing her utility, given $q_g$ set by the govern-
ment for $\tau$ yielding government revenue equal to government expendi-
ture. $p_c$, $\frac{rK}{N}$, $w$, $q_g$ and $\tau$ are parameters for a household. Equilibrium

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for a household is defined in part by the first order conditions in

$$\frac{u_{q_c}}{u_{H_l}} = \frac{p_c}{(1-t)w}$$

$$\text{and } q_c = \frac{1}{p_c}\left[(1-\tau) * \left(w * [H_c + H_l] + \frac{rK}{N}\right)\right]$$

$u_{q_c}$, $u_{q_g}$ and $u_{H_l}$ are first derivatives of the utility function. A household's taxable income is $\left(w * [H_c + H_l] + \frac{rK}{N}\right)$ equal to $Y$. Budget balance for the government is

$$p_g q_g = \tau * [N * w * \{H_c + H_g\} + r * K].$$

Leisure for an agent, $H_l$ is defined in $H_l = 24 - H_c - H_g$. We set $p_c = 1$. Other than this last equation, we have 6 equilibrium conditions for our two production sectors, and 2 conditions for a household, plus government budget balance. This gives us a 9 equation system in $r$, $w$, $p_g$, $H_c$, $H_g$, $H_l$, $q_c$, $q_g$ and $u$.

/////////////////

APPENDIX 2: (MAY 7, 2015) Matlab Program underlying outputs in Tables 1 and 2

```
function f=dnd(x)
%
N=50; al=.1; z=.8; alc=.02 ;alg=.06; K=40; bet=-0.05;
t=0.05;
qc=x(1);
qg=x(2);
Hc=x(3);
Hg=x(4);
Hl=x(5);
pg=x(6);
r=x(7);
w=x(8);
```

```
Kc=x(9);
u=x(10);
Tx=x(11);
%
pc=1;
f(1)=Kc^alc*(N*Hc)^(1-alc)-qc*N;
f(2)=r*(1-alc)*Kc-alc*N*Hc*w;
f(3)=pc*N*qc-r*Kc-w*N*Hc;
f(4)=(K-Kc)^alg*(N*Hg)^(1-alg)-qg;
f(5)=(1-alg)*(K-Kc)*r-alg*N*Hg*w;
f(6)=r*(K-Kc)+w*N*Hg-pg*qg;
f(7)=24-Hc-Hg-Hl;
f(8)=t*(w*N*(Hc+Hg)+r*K)-(pg*qg);
%
f(9)=(al*Hl^(bet+1))/(z*qc^(bet+1))-(pc/(w*(1-t)));
% f(9)=(al*qg^(bet+1))/((1-al-z)*qc^(bet+1))-(pc/(pg/N));
f(10)=u-(al*qc^(-bet)+z*Hl^(-bet)+(1-al-z)*qg^(-bet))^(-1/bet);
f(11)=Tx-(pg*qg);
//////////
```

Appendix 3: Matlab Program (First Best Case)

```
function f=dnd(x)
%
N=50; al=.1; z=.8; alc=.02 ;alg=.06; K=40; bet=-0.5;
% t=0.15;
qc=x(1);
qg=x(2);
Hc=x(3);
Hg=x(4);
Hl=x(5);
pg=x(6);
```

```
    r=x(7);
    w=x(8);
    Kc=x(9);
    u=x(10);
    Tx=x(11);
    uc=x(12);
    ug=x(13);
    rat=x(14);
    %
    pc=1;
    f(1)=Kc^alc*(N*Hc)^(1-alc)-qc*N;
    f(2)=r*(1-alc)*Kc-alc*N*Hc*w;
    f(3)=pc*N*qc-r*Kc-w*N*Hc;
    f(4)=(K-Kc)^alg*(N*Hg)^(1-alg)-qg;
    f(5)=(1-alg)*(K-Kc)*r-alg*N*Hg*w;
    f(6)=r*(K-Kc)+w*N*Hg-pg*qg;
    f(7)=24-Hc-Hg-Hl;
    % f(8)=t*(w*N*(Hc+Hg)+r*K)-(pg*qg);
    %
    f(8)=(al*Hl^(bet+1))/(z*qc^(bet+1))-(pc/(w));
    f(9)=(al*qg^(bet+1))/((1-al-z)*qc^(bet+1))-(pc/(pg/N));
    f(10)=u-(al*qc^(-bet)+z*Hl^(-bet)+(1-al-z)*qg^(-bet))^(-1/bet);
    f(11)=Tx-(pg*qg);
    % f(12)=ytau-t*(w*(Hc+Hg)+r*(K/N));
    f(12)=uc-(((-1/bet)*(al*qc^(-bet)+z*Hl^(-bet)+(1-al-z)*qg^(-bet))^((-
1-bet)/bet))*(-bet)*(al)/(qc^(bet+1)));
    f(13)=ug-(((-1/bet)*(al*qc^(-bet)+z*Hl^(-bet)+(1-al-z)*qg^(-bet))^((-
1-bet)/bet))*(-bet)*(1-al-z)/(qg^(bet+1)));
    f(14)=rat-(ug/uc);
    //////////
```

Appendix 4: MATLAB Program for the Second Best model with two classes of agents

```
function f=dnmm(x)
%
N=50; al=.4; z=.2; alc=.2; alg=.35; K=40; bet=0.5;Kb=18;Nb=25;
t=0.22;
% bet=(1/sig)-1;
qc=x(1);
qg=x(2);
Hc=x(3);
Hg=x(4);
Hl=x(5);
pg=x(6);
r=x(7);
w=x(8);
Kc=x(9);
qcb=x(10);
Hcb=x(11);
Hgb=x(12);
Hlb=x(13);
ua=x(14);
ub=x(15);
%
pc=1;
f(1)=Kc^alc*((N-Nb)*Hc+(Nb*Hcb))^(1-alc)-(qc*(N-Nb)+(qcb*Nb));
f(2)=r*(1-alc)*Kc-alc*((N-Nb)*Hc+(Nb*Hcb))*w;
f(3)=pc*((N-Nb)*qc+(Nb*qcb))-r*Kc-w*((N-Nb)*Hc+(Nb*Hcb));
f(4)=(K-Kc)^alg*((N-Nb)*Hg+Nb*Hgb)^(1-alg)-qg;
f(5)=(1-alg)*(K-Kc)*r-alg*((N-Nb)*Hg+Nb*Hgb)*w;
f(6)=r*(K-Kc)+w*((N-Nb)*Hg+Nb*Hgb)-pg*qg;
```

f(7)=24-Hc-Hg-Hl;

f(8)=24-Hcb-Hgb-Hlb;

f(9)=Nb*t*(w*(Hcb+Hgb))+(N-Nb)*t*(w*(Hc+Hg))+t*r*K-(pg*qg);

%

f(10)=pc*qc-(1-t)*(w*(Hc+Hg)+r*(K-Kb)/(N-Nb));

f(11)=pc*qcb-(1-t)*(w*(Hcb+Hgb)+r*Kb/Nb);

f(12)=(al*Hl^(bet+1))/(z*qc^(bet+1))-(pc/((1-t)*w));

f(13)=(al*Hlb^(bet+1))/(z*qcb^(bet+1))-(pc/((1-t)*w));

f(14)=ua-(al*qc^(-bet)+z*Hl^(-bet)+(1-al-z)*qg^(-bet))^(-1/bet);

f(15)=ub-(al*qcb^(-bet)+z*Hlb^(-bet)+(1-al-z)*qg^(-bet))^(-1/bet);

.......