Common Value Auctions with Return Policies

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Abstract

This paper examines the role of return policies in common value auctions. We first characterize the unique symmetric equilibrium in first-price and second-price auctions with continuous signals and discrete common values when certain return policies are provided. We then examine how the return policies affect a seller's revenue. When the lowest common value is zero, a more generous return policy generates a higher seller's revenue; the full refund policy extracts all the surplus and therefore implements the optimal selling mechanism; given any return policy, a second-price auction generates a higher revenue than a first-price auction. In a second-price auction where the lowest common value is not zero but still smaller than the seller's reservation value, then a more generous return policy also generates a higher revenue; otherwise, the optimal return policy could be a full refund, no refund or partial refund policy.

Key words: auctions, return policies, refund

JEL classification: D44, D72, D82

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1 Introduction

Auctions have been used for trading for thousands of years. Due to the rapid growth of internet commerce, online auctions have become extremely popular. These auctions create a problem for both the buyers and the sellers. As a buyer is unable to closely examine the good being auctioned, he may find the good not exactly what he expects when he receives it. Even though this could also happen in store purchases, it is inarguably a more common problem in online auctions. Should sellers allow buyers to return the goods? Should sellers charge a fee for the returns? How would the return policy affect buyers' behavior? Would sellers also benefit from such a return policy? These are the issues we will investigate in this paper.

Each day, there are millions of objects being auctioned on the internet through many online auction sites. We observe that many of the sellers provide very generous return policies in the auctions.[1] The NHL (National Hockey League) online auctions, for example, provide a 7-Day, 100% Money-Back Guarantee. Most sellers in Amazon.com and eBay.com provide at least partial return policies; buyers may need to pay some fees (usually less than 15% of the transaction prices) if they would like to return the merchandises.

Traditionally, standardized consumption goods are categorized as having private values. Non-standardized goods, especially those with uncertain quality, are categorized as having interdependent values.[2] With private values, return policies do not affect a buyer's bidding strategy, since he never bids more than his valuation. As a result, they bid the same as if there were no return policy.[3] In contrast, with interdependent values, return policies induce bidders to bid more aggressively. Return happens with positive probability after the winning bidder receives the good and learns more about its true value. We focus on the common value setting in Wilson [17] since it is the simplest model accounting for interdependent and correlated values.[4] The common value auctions in Wilson [17] are also widely used to model auctions for oil, gas and mineral rights.

The phenomenon, known as the winner's curse, is well recognized in the common value auctions.[5] The fact of winning actually conveys bad information that the winner may have overestimated the object. If buyers do not take this effect into account, they get lower or even negative expected surplus. However, when buyers are fully rational and they fully anticipate the winner's curse, they will adjust their bids downward accordingly, and will not

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[1]Return policies (or money back guarantees) are widely adopted in retailer stores as well. Most large chain stores have some form of full return policy.

[2]Haile [4] examines auctions of goods which have common value components due to resale.

[3]Zhang [20] considers private values which are subject to idiosyncratic shocks after transaction, and illustrates how return policies can be part of the optimal mechanism.

[4]Ideally, the most general model to use is Milgrom and Weber [14]. Unfortunately, with return policies, it is hard to characterize the equilibria. Nevertheless, the insight should be more or less the same.

[5]The winner's curse is first pointed out by Capen et al. [1]. Kagel and Levin [7] documents the winner's curse in experimental designs.

bid more than their estimates of the value conditional on winning. As the number of buyers increases, the winner's curse becomes more severe and buyers bid even less.

However, if a return policy is in place, buyers will bid more aggressively since the winner can get a refund by returning the object if he discovers that he has overestimated its value. As a result, the return policy overcomes some of the winner's curse. In fact, a return policy can do more than mitigating the winner's curse. When the return policy is generous enough, buyers may bid more than the unconditional estimates of the object value. If the seller implements the full refund policy, then it is obvious that every bidder will bid very high in the auction. Of course, returns would negatively impact the seller's revenue as well as the efficiency of trading, as the seller usually has a lower value for keeping the object unsold. By selecting a proper return policy, the seller can achieve higher revenue by balancing the trade-off between higher bids and efficiency loss.

In theory, there exist optimal mechanisms for sellers to maximize revenue.[6] However, those optimal mechanisms are not commonly observed in reality, partly because too much detail regarding the underlining environment is required for the seller to design an optimal mechanism. The discrepancy between theory and common practice prompts the claim that a set of simplicity and robustness criteria should be imposed on the trading mechanisms.[7] Our auction model with return policies satisfies those simplicity criteria, and the return policies do not depend on much of the detail of the environment. As we shall show in this paper, return policies, while being "simple" instruments, are effective in revenue improving.

In our common value auction models with return policies, buyers receive independent signals conditional upon the true common value of the object. While this common value takes discrete values, signals are continuously distributed. A more generous return policy has three effects. First, it induces buyers to bid more aggressively. This is a positive effect. Second, the seller receives a smaller portion of the transaction price when the winning bidder returns the object. This is a negative effect. We call the sum of these two effects as the payment effect. It can be shown that the payment effect is usually positive. Third, returns change the efficiency of the object allocation. This is called the efficiency effect, which could be either positive or negative. How a return policy affects the seller's revenue depends on the magnitude of the payment effect and the efficiency effect.

Our analysis shows that a return policy can often improve a seller's revenue. Under certain conditions, the full refund policy extracts all the surplus and implements the optimal selling mechanism. Although return policies are commonly observed only in online auctions, they should be an effective instrument to improve revenue in any auctions with

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[6]The optimal auction with independent values has been established by Myerson [15]. Matthews [11] and Maskin and Riley [10] characterize the optimal mechanism with risk averse buyers and independent values. With correlated values, (almost) full surplus extraction can be achieved using the mechanism in Cremer and Mclean [2] and McAfee and Reny [12].

[7]Hurwicz [6] illustrates the need for mechanisms that are independent of the parameters of the model. Wilson [19] points out that a desirable property of a trading rule is that it "does not rely on features of the agents". Lopomo [8] [9] restricts to mechanisms with "simplicity" and "robustness".

interdepent values.

There is a huge literature on auctions. The equilibria of common value auctions have already been extensively examined (see Goeree and Offerman [3], Hausch [5], Milgrom and Weber [13], Reece [16], and Wilson [18]). Milgrom and Weber [14] consider a very general model which includes the private values and common values as special cases. However, none of the above papers consider return policies. To the best of our knowledge, Zhang [20] and this paper are the first ones to analyze return policies in auctions.

The rest of this paper is organized as follows. In Section 2, we set up the model. In Section 3, we characterize the bidders' equilibrium strategies in second-price auctions. In Section 4, we characterize the bidders' equilibrium strategies in first-price auctions. In Section 5, we establish the revenue rankings among different auction formats. In Section 6, we conclude. All proofs are relegated to an appendix.

2   The model

Suppose that there are two bidders, bidders 1 and 2.[8] The common value of the object, V, can be either V_H or V_L with $V_H > V_L$. Assume that $V = V_H$ with probability μ_H, and $V = V_L$ with probability $\mu_L = 1 - \mu_H$. Bidders know the distribution of the common value but not its true value before the auction. Bidder i receives a signal x_i, $i = 1, 2$. This signal is correlated to the common value, V, but independently distributed across bidders conditional on V. If $V = V_H$, then x_i follows the distribution with p.d.f. $f_H(\cdot)$ and c.d.f. $F_H(\cdot)$. If $V = V_L$, then x_i follows the distribution with p.d.f. $f_L(\cdot)$ and c.d.f. $F_L(\cdot)$. Assume that $F_H(\cdot)$ and $F_L(\cdot)$ have a common support, $[\underline{x}, \overline{x}]$. To ensure that a higher signal implies a higher probability of $V = V_H$, we will assume that $\rho(x) = \frac{f_H(x)}{f_L(x)}$ is increasing in x, i.e., F_H dominates F_L in likelihood ratio. The lemma below lists a few properties implied by this assumption. The proof is standard and is thus omitted.

Lemma 1 *Suppose that $F_H(x)$ dominates $F_L(x)$ in likelihood ratio, i.e., $\rho(x)$ is increasing in x. Then*

1. *F_H dominates F_L in hazard rate, i.e. $\frac{f_H}{1 - F_H} \leq \frac{f_L}{1 - F_H}$.*

2. *F_H dominates F_L in reversed hazard rate, i.e. $\frac{f_H}{F_H} \geq \frac{f_L}{F_L}$.*

3. *$\frac{F_H}{F_L}$ is increasing.*

Now we consider the return policy. Let p be the transaction price in an auction. Suppose that the seller charges a fee $c = \gamma p$ if the winning bidder (winner) returns the object, where

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$\gamma \in [0, 1]$. Therefore, the winner receives a refund of $(1 - \gamma)p$. Here, we assume that there is no transaction cost for returns on both sides. We focus on the case where $\gamma > 0$. If $\gamma = 0$, there could be multiple equilibria with every bidder bidding higher than or equal to V_H.[9]

The timing of the game is as follows.

1. Nature moves first and selects $V = V_H$ or $V = V_L$. Conditional on V, each bidder draws a signal independently.

2. Either a first-price or second-price auction with return policy γ is held, and the winner is determined.

3. The winner learns the true value of the object and decides whether or not to return the object for a refund.

We assume that the winner can learn about the true value of V after the auction ends. In online auctions, after a buyer receives the object, he would learn more about its value. In auctions for oil, gas and mineral rights, uncertainties resolve over time and the winner will learn the true value eventually. In the analysis, we will focus on the symmetric perfect Bayesian equilibrium with strictly increasing bidding function in the auction. We will start by analyzing the last stage of the game, where the winner makes the return decision. In the following section, we will examine the second-price auctions. We will then examine the first-price auctions in the subsequent section.

3 Second-price auctions

Let us first examine the second-price auctions. The transaction price is equal to the second highest bid in the auction. We assume that both bidders adopt the same strictly increasing bidding function $B^S(\cdot)$ in the auction stage. We can restrict our attention to bidding functions taking values in $[V_L, V_H]$. This is because a buyer with the highest signal should not bid more than V_H; bidding more than V_H would sometimes gives him a negative surplus and is dominated by bidding V_H; if a bidder with the lowest signal bids less than V_L, then by increasing his bid to V_L he may win with a positive probability and thus get a positive surplus.

Now consider the return stage. Assume that buyer 1 is the winner and his signal is x. Suppose that he bids $B^S(\tilde{x})$, wins the auction, and pays $B^S(x_2)$. If the realization of the value of the object is $V = V_H$, he will not return it since his payment is less than V_H. If $V = V_L$, he returns the object when $V_L < (1 - \gamma)B^S(x_2)$, i.e., $x_2 > \left(B^S\right)^{-1}\left(\frac{V_L}{1-\gamma}\right)$; otherwise,

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[9]As we shall show later, when γ converges to zero, the equilibrium converges to the one where every bidder bids V_H. Furthermore, bidding more than V_H is weakly dominated by bidding V_H. Therefore, when $\gamma = 0$, we pick the undominated equilibrium where every bidder bids V_H.

he keeps the object. As a result, there can only be three different situations in the return stage. Case 1: $\left(B^S\right)^{-1}\left(\frac{V_L}{1-\gamma}\right) \geq \overline{x}$, and thus the winner keeps the object all the time. Case 2: $\left(B^S\right)^{-1}\left(\frac{V_L}{1-\gamma}\right) \leq \underline{x}$, and thus the winner returns the object whenever $V = V_L$. Case 3: $\underline{x} < \left(B^S\right)^{-1}\left(\frac{V_L}{1-\gamma}\right) < \overline{x}$, and the winner's return decision is based on a cutoff rule when $V = V_L$.

Note that in a two-bidder second-price common-value auction without return policies, from Milgrom and Weber [14], a bidder with signal x would bid the expected object value conditional on the other bidder having the same signal x:

$$E(V|x,x) = \frac{\mu_H V_H f_H(x)^2 + \mu_L V_L f_L(x)^2}{\mu_H f_H(x)^2 + \mu_L f_L(x)^2} = \frac{\mu_H V_H \rho(x)^2 + \mu_L V_L}{\mu_H \rho(x)^2 + \mu_L}. \tag{1}$$

Define

$$\Gamma^S(x) \equiv 1 - \frac{V_L}{E(V|x,x)} = \frac{(V_H - V_L)\mu_H \rho(x)^2}{V_H \mu_H \rho(x)^2 + \mu_L V_L} \tag{2}$$

as the winning bidder's loss as a percentage to the bidder's bid when the value of the object turns out to be V_L. There are two cutoffs for γ that are important in the characterization of the bidders' equilibrium bidding function. (Recall that γ is the percentage of the transaction price that is retained by the seller if the winning bidder returns the object.)

$$\underline{\gamma}^S = \Gamma^S(\underline{x}), \tag{3}$$

$$\overline{\gamma}^S = \Gamma^S(\overline{x}). \tag{4}$$

The type of equilibrium we will obtain depends crucially on the value of γ. As shown in the proof of Proposition 1, it turns out that if γ is lower than $\underline{\gamma}^S$, the winning bidder would always return the object whenever $V = V_L$. If γ is higher than $\overline{\gamma}^S$, the winning bidder would never return the object when $V = V_L$. If γ is intermediate, the winning bidder would sometimes return the object when $V = V_L$. Furthermore, the intervals for γ in the above three cases do not overlap with each other and they cover the entire interval of $(0,1]$. Thus we can conclude that a unique symmetric perfect Bayesian Nash equilibrium exists for any $\gamma \in (0,1]$. Define x^{S*} as the solution to $\gamma = \Gamma^S(x^{S*})$ for $\gamma \in (\underline{\gamma}^S, \overline{\gamma}^S)$. Since $\Gamma^S(x)$ is strictly increasing, x^{S*} is unique and belongs to $(\underline{x}, \overline{x})$. These results are characterized in the following proposition.

Proposition 1 *In the second-price common value auction with return policy γ, the unique symmetric equilibrium is characterized as follows in three cases.*

Case 1: If $\gamma \geq \overline{\gamma}^S$, each bidder adopts the following strictly increasing bidding function:

$$B^S(x) = B^{S1}(x) \equiv E(V|x, x) = \frac{\mu_H V_H \rho(x)^2 + \mu_L V_L}{\mu_H \rho(x)^2 + \mu_L}. \tag{5}$$

The winning bidder never returns the object.

Case 2: If $\gamma \leq \underline{\gamma}^S$, each bidder adopts the following strictly increasing bidding function:

$$B^S(x) = B^{S2}(x) = \frac{\mu_H V_H \rho(x)^2}{\mu_H \rho(x)^2 + \gamma \mu_L}. \tag{6}$$

The winning bidder always returns the object whenever $V = V_L$.

Case 3: If $\underline{\gamma}^S < \gamma < \overline{\gamma}^S$, each bidder adopts the following strictly increasing bidding function:

$$B^S(x) = \begin{cases} B^{S1}(x), & \text{if } x \leq x^{S*}; \\ \\ B^{S2}(x), & \text{if } x \geq x^{S*}. \end{cases} \tag{7}$$

The winning bidder returns the object when $V = V_L$ if he pays more than $B^S(x^{S*})$.

In Case 1 of the above proposition, since the return policy is never executed, the bidding function coincides with the one with no return policy. Obviously, providing a very strict return policy is equivalent to no returns. The bidding function $B^{S1}(x)$ corresponds to the one in Milgrom and Weber [14].

In Case 2, the function $B^{S2}(x)$ is in fact the equilibrium bidding function for the game when the winner is forced to return the object if the realized value is V_L. It is equal to the price that the bidder will break even if he pays that price (i.e., the other bidder also has signal x) and gets V_H when $V = V_H$, and pays γ percent of that price and gets 0 when $V = V_L$. That is, $B^{S2}(x)$ is the solution to $(V_H - B)\mu_H f_H(x)^2 + (-\gamma B)\mu_L f_L(x)^2 = 0$. When $V_L = 0$, this bid is equivalent to the expected object value of a bidder with signal x conditional on the other bidder having the same signal x and returning the object with probability $1 - \gamma$ when $V = V_L$.

In Case 3, whether the winning bidder returns the object or not depends on how much he pays. It is easy to show that $B^{S1}(x) \geq B^{S2}(x)$ for $x \leq x^{S*}$ and $B^{S1}(x) \leq B^{S2}(x)$ for $x \geq x^{S*}$ with equality at the cutoff x^{S*}. Therefore, the bidding function is the maximum of the two functions in Cases 1 and 2. However, as we shall show in the next section, this pattern is not valid for first-price auctions.

Given any return policy $\gamma \in (0, 1]$, there exists a unique symmetric equilibrium. When $\gamma \to 0$, $B^S(x) \to V_H$. We know that when $\gamma = 0$, there are many equilibria. First, bidding V_H and the winner always returns the object when $V = V_L$ and keeps the object when $V = V_H$ is

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an equilibrium. Second, bidding any amount more than V_H and the winner returns the object all the time is also an equilibrium. However, bidding more than V_H is weakly dominated by bidding V_H. Thus, for simplicity and continuity, we select the equilibrium with $B^S(x) = V_H$ and the winner always returns the object whenever $V = V_L$ and keeps the object whenever $V = V_H$ as the equilibrium for $\gamma = 0$. This is captured in Case 2 with $\gamma = 0$.

4   First-price auctions

Now we examine first-price auction with returns. Here, the transaction price is the winning bid. Again, we focus on an equilibrium where every bidder adopts the same strictly increasing bidding function $B^F(\cdot)$. Similarly to the second-price auctions, we can establish the range of the bidding function to be a subset of $[V_L, V_H]$.

We first examine the winning bidder's return decision. Suppose that a bidder has signal x but bids $B^F(\tilde{x})$. If he wins, he pays $B^F(\tilde{x})$ for the object. When the realization of the object value is $V = V_H$, he will keep the object, since he pays less than V_H. When $V = V_L$, he will return the object if $V_L < (1 - \gamma)B^F(\tilde{x})$, i.e., $\tilde{x} > (B^F)^{-1}(\frac{V_L}{1-\gamma})$. As result, there can only be three different situations in the return stage. Case 1: $(B^F)^{-1}(\frac{V_L}{1-\gamma}) \geq \overline{x}$, and thus the winner keeps the object all the time. Case 2: $(B^F)^{-1}(\frac{V_L}{1-\gamma}) \leq \underline{x}$, and thus the winner returns the object all the time when $V = V_L$. Case 3: $\underline{x} < (B^F)^{-1}(\frac{V_L}{1-\gamma}) < \overline{x}$, and thus the winner's return decision is a cutoff rule when $V = V_L$.

Define two functions:

$$L_1(\alpha|x) = e^{-\int_\alpha^x \frac{\mu_H f_H(s)^2 + \mu_L f_L(s)^2}{\mu_H f_H(s) F_H(s) + \mu_L f_L(s) F_L(s)} ds},\tag{8}$$

$$L_2(\alpha|x; \gamma) = e^{-\int_\alpha^x \frac{\mu_H f_H(s)^2 + \gamma \mu_L f_L(s)^2}{\mu_H f_H(s) F_H(s) + \gamma \mu_L f_L(s) F_L(s)} ds}\tag{9}$$

Lemma 2 $L_1(\alpha|x)$ and $L_2(\alpha|x; \gamma)$ are both proper c.d.f.'s of α with support $[\underline{x}, x]$.

Note that in a two-bidder second-price common-value auction without return policies, according to Milgrom and Weber [14], all bidders would bid according to the same strictly increasing function:

$$\int_{\underline{x}}^x E(V|\alpha, \alpha) dL_1(\alpha|x) = \int_{\underline{x}}^x \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|x).\tag{10}$$

Define

$$\Gamma^F(x) \equiv 1 - \frac{V_L}{\int_{\underline{x}}^x E(V|\alpha, \alpha) dL_1(\alpha|x)} = \frac{\int_{\underline{x}}^x \frac{(V_H - V_L)\mu_H \rho(\alpha)^2}{V_H \mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|x)}{\int_{\underline{x}}^x \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|x)}\tag{11}$$

as the winning bidder's loss as a percentage to the bidder's bid when the value of the object turns out to be V_L. Since the bidding function (10) is strictly increasing, $\Gamma^F(x)$ is strictly increasing. As in the second-price auctions, there are two cutoffs for γ that are important in the characterization of the bidders' equilibrium bidding function.

$$\underline{\gamma}^F = \Gamma^F(\underline{x}), \tag{12}$$

$$\overline{\gamma}^F = \Gamma^F(\overline{x}). \tag{13}$$

Note that $\Gamma^F(\underline{x}) = \Gamma^S(\underline{x})$.

As in the second-price auctions, γ plays an important role in the return decision. Define x^{F*} as the solution to $\gamma = \Gamma^F(x^{F*})$ for $\gamma \in (\underline{\gamma}^F, \overline{\gamma}^F)$. Since $\Gamma^F(x)$ is strictly increasing, x^{F*} is unique and belongs to $(\underline{x}, \overline{x})$. Also define

$$A = \frac{V_L}{1 - \gamma} - \int_{\underline{x}}^{x^{F*}} \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha | x^{F*}; \gamma). \tag{14}$$

Similar to the second-price auctions, we have the following proposition.

Proposition 2 *In a first-price common value auction with return policy γ, the unique symmetric equilibrium can be characterized as follows in three cases.*
Case 1: *When $\gamma \geq \overline{\gamma}^F$, each bidder adopts the following strictly increasing bidding function:*

$$B^F(x) = B^{F1}(x) = \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha | x). \tag{15}$$

The winning bidder never returns the object.

Case 2: *When $\gamma \leq \underline{\gamma}^F$, each bidder adopts the following strictly increasing bidding function:*

$$B^F(x) = B^{F2}(x) = \int_{\underline{x}}^{x} \frac{V_H \mu_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha | x; \gamma). \tag{16}$$

The winning bidder always returns the object when $V = V_L$.

Case 3: *When $\underline{\gamma}^F < \gamma < \overline{\gamma}^F$, each bidder adopts the following strictly increasing bidding function:*

$$B^F(x) = \begin{cases} B^{F1}(x), & \text{if } x \leq x^{F*}, \\ \\ B^{F2}(x) + AL_2(x^{F*} | x; \gamma), & \text{if } x \geq x^{F*}. \end{cases} \tag{17}$$

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The winning bidder returns the object if he pays more than $B^F(x^{F*})$ when $V = V_L$.

In Case 1 of the above proposition, since the return policy is never executed, the bidding function coincides with the one with no return policy. A very strict return policy is equivalent to not allowing returns. The bidding function $B^{F1}(x)$ corresponds to the one in Milgrom and Weber [14]. In Case 2, the function $B^{S2}(x)$ is in fact the equilibrium bidding function for the auction when the winner is forced to return the object if the realized value is V_L. In Case 3, however, the bidding function is no long the maximum of the two individual bidding functions in Cases 1 and 2 as in the second-price auctions.

For the same reason as in the second-price auctions, we choose the equilibrium with $B^F(x) = V_H$ and the winner always returns the object whenever $V = V_L$ and keeps the object whenever $V = V_H$ as the equilibrium for $\gamma = 0$. This is captured in Case 2 with $\gamma = 0$ in the above proposition.

5 Revenue ranking

In this section, we shall make several comparisons in revenue. Suppose that the seller's reservation value of the object is V_0.[10] Consider a return policy with percentage fee of γ for returns. Suppose that the seller implements a more generous return policy (i.e., a lower γ). There are three effects. First, buyers bid more aggressively. This is a positive effect. Second, when the winner returns the object, the seller gets a smaller portion of the transaction price. This is a negative effect. Third, because the cost for returning is lower, the probability of the object being returned to the seller is higher. The effect on the efficiency of the object allocation could be positive or negative, depending on whether the seller's reservation value is higher or lower than the object value. We call the sum of the first two effects the payment effect, and the third effect the efficiency effect. The seller can improve her revenue by balancing the trade-off between the payment effect and the efficiency effect.

In what follows, we shall first examine the special case of $V_L = 0$, and then the general case. We denote the case of $V_L = 0$ the benchmark. In this case, the object is either in perfect condition (high common value), or totally useless (zero common value). The winner always keeps the object when the realized common value is high, and always returns it when the realized common value is zero, regardless of the return policy. Because the return policy does not alter the allocation of the object, there is no efficiency variation among different return policies. This allows us to focus on the role of return policy on the payment effect. Analytically, we can make use of the linkage principle and show how the return policy affects the revenue in this case.[11]

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[10]It is reasonable to assume that $V_0 < V_H$; otherwise, the seller will not have any incentive trying to sell the object.

[11]Unfortunately, when $V_L \neq 0$, the allocation of the object is affected by the return policy and the linkage principle does not apply.

5.1 The linkage principle in the benchmark

Let $M(\tilde{x}, x)$ be the expected payment by a bidder with signal x but reported $\tilde{x}$. We have the following proposition.

Proposition 3 *Let A and B be two auctions with return policies. In both auctions, the bidder with the highest bid wins. Furthermore, the winner always keeps the object if $V = V_H$, and always returns it if $V = V_L$. Suppose that in each auction, there is a symmetric and strictly increasing equilibrium bidding function with the properties that (i) for all x, $M_2^A(x, x) \geq M_2^B(x, x)$; (ii) $M^A(\underline{x}, \underline{x}) = M^B(\underline{x}, \underline{x}) = 0$. Then the seller's expected revenue from A is at least as large as the expected revenue from B.*

In the benchmark, $V_L = 0$. Given any return policy and auction format, the winner with the highest bid wins; and the winner always returns the object when $V = V_L$ and always keeps the object when $V = V_H$. Thus, both first-price and second-price auctions with any return policy can be regarded as a mechanism in the above proposition. Note that the equilibrium strategy for a second-price auction is characterized by Case 2 in Proposition 1, and the equilibrium strategy for a first-price auction is characterized by Case 2 in Proposition 2.

We can rank the expected revenues in the second-price auctions with different γs as follows.

Proposition 4 *Suppose that $V_L = 0$. Then the seller's revenue is decreasing in γ in the second-price auctions. That is, the more generous the return policy is, the more revenue a second-price auction generates.*

The intuition behind this proposition is as follows. Bidders take into consideration of the possible $V_L = 0$ when they calculate their bids. The only situation that γ affects the seller's revenue is when there is some probability that the winning bidder will return the object. Denote a bidder's bid in this case as $B^{S2}(x, \gamma)$, it is the solution to

$$\left[V_H - B^{S2}(x, \gamma)\right] \frac{\mu_H f_H(x)^2}{\mu_H f_H(x) + \mu_L f_L(x)} + \left[-\gamma B^{S2}(x, \gamma)\right] \frac{\mu_L f_L(x)^2}{\mu_H f_H(x) + \mu_L f_L(x)} = 0, \quad (18)$$

or equivalently,

$$f_L(x)\left[B^{S2}(x, \gamma)\mu_H f_H(x)\rho(x) + \gamma B^{S2}(x, \gamma)\mu_L f_L(x)\right] = V_H \mu_H f_H(x)^2, \quad (19)$$

where $\rho(x) = \frac{f_H(x)}{f_L(x)}$.

When γ decreases, $B^{S2}(x, \gamma)$ needs to increase to keep the equation binding. Furthermore, because $B^{S2}(x, \gamma)$ has increased, $\gamma B^{S2}(x, \gamma)$ needs to decrease to keep the equation binding.

Now consider the seller's revenue. The seller receives either $B^{S2}(x,\gamma)$ when $V = V_H$ or $\gamma B^{S2}(x,\gamma)$ when $V = V_L = 0$, if the other bidder's signal (denoted by y) is higher. In this case, the other bidder wins and pays $B^{S2}(x,\gamma)$. The relevant expression entering the seller's expected revenue becomes

$$
B^{S2}(x,\gamma)\frac{\mu_H f_H(y) f_H(x)}{\mu_H f_H(x) + \mu_L f_L(x)} + \gamma B^{S2}(x,\gamma)\frac{\mu_L f_L(y) f_L(x)}{\mu_H f_H(x) + \mu_L f_L(x)}
$$
$$
= \frac{f_L(y)}{\mu_H f_H(x) + \mu_L f_L(x)}\left[B^{S2}(x,\gamma)\mu_H f_H(x)\rho(y) + \gamma B^{S2}(x,\gamma)\mu_L f_L(x)\right], \qquad (20)
$$

where $y > x$. Note our previous assumption that $\rho(x)$ is an increasing function, implying $\rho(y) > \rho(x)$. Therefore, when γ decreases, the increase in B^{S2} together with the decrease in γB^{S2} keeping the left-hand side of (19) constant will increase the value of (20), and thus will increase the seller's revenue. In other words, because the seller receives the bid of a first bidder only when the second bidder has a higher signal, and this higher signal makes $V = V_H$ more likely to occur than the probability used in the first bidder's calculation. Therefore, the total expected revenue for the seller is higher.

Similar ranking in the first-price auctions with different γs can be obtained as well. We have the following proposition.

Proposition 5 *Suppose that $V_L = 0$. Then the seller's revenue is decreasing in γ in the first-price auctions. That is, the more generous the return policy is, the more revenue a first-price auction generates.*

The above two propositions illustrate that the payment effect is positive in both auctions when $V_L = 0$. Since the efficiency effect vanishes when $V_L = 0$, a more generous return policy increases the seller's revenue. This implies that the full refund policy ($\gamma = 0$) is the best return policy. With the full refund policy, in both auctions, all buyers bid up to V_H; and the winner keeps the object when $V = V_H$ and returns it when $V = V_L = 0$. Thus, the seller extracts all possible surplus and the buyers earn zero surplus. As a result, both auctions with full refund policy implement the optimal mechanism which extracts all surplus. This is summarized in the following corollary.

Corollary 1 *Suppose that $V_L = 0$. Then either the first-price or second-price auction with full refund policy implements the optimal mechanism.*

In what follows, we compare the revenues in the first-price and the second-price auctions given the same γ. We have the following proposition.

Proposition 6 *Suppose that $V_L = 0$. Given the same return policy γ, a second-price auction generates at least as much revenue as a first-price auction.*

This proposition shows that the result in Milgrom and Weber [14] that second-price auctions generates weakly more revenue than first-price auctions can be generalized to auctions with return policies. In Milgrom and Weber [14], the result can be derived directly from the linkage principle. In contrast, the linkage principle cannot be applied to auctions with return policies. This is because for the linkage principle to work, the difference between the two expected payment functions must be increasing in a bidder's reported type. This property is satisfied among second-price auctions with different γs, as well as among first-price auctions with different γs. But when we compare a first-price auction with a second-price auction with the same γ, the property is no longer valid. (See the proof of this proposition for details.) Nevertheless, revenue ranking is still possible here. This is because in a second-price auction, the seller receives the bid of a bidder only when the other bidder has a higher signal, and this higher signal makes $V = V_H$ more likely to happen than the first bidder originally thought. That is, the object gets returned less often in the seller's revenue calculation than in a bidder's surplus calculation. However, this effect is absent in the first-price auction. Therefore, the total expected revenue for the seller is higher in the second-price auction.

5.2   The general case

Now we consider the general case, where $V_L \neq 0$. In this case, the linkage principle does not apply, since different return policies generate different allocations of the object being auctioned. As the equilibrium bidding function in first-price auctions is very complex, we focus on the second-price auctions here. First-price auctions should have qualitatively similar results. We have the following proposition which is similar to Proposition 4.

Proposition 7 *Suppose that $V_L \leq V_0$. Then the seller's revenue is decreasing in γ in the second-price auctions. That is, the more generous the return policy is, the more revenue the auction will generate. Furthermore, the second-price auction with the full refund policy implements the optimal mechanism.*

As is shown in the proof, the payment effect is always positive in the second-price auctions. When $V_L \neq 0$ but $V_L \leq V_0$, the efficiency effect is positive since the return policy improves efficiency. Therefore, the total effect is positive, which implies that a more generous return policy again increases the seller's revenue and the full refund policy is optimal. The intuition for this proposition is exactly the same as Proposition 4.

When $V_L > V_0$, the efficiency effect is negative. In this case, the net effect of a more generous return policy depends on which of the two effects (the payment effect and the efficiency effect) dominates. The following example shows that the optimal return policy can be a full refund, no refund or partial refund policy.

Example: Suppose that $V_0 = 0$, $V_H = 1$, V_L to be specified, with $\mu_H = \mu_L = 0.5$. For $x \in [0, 1]$, $F_H(x) = x^2$, $F_L(x) = 2x - x^2$, Then $f_H(x) = 2x$, $f_L(x) = 2 - 2x$, and $\rho(x) = \frac{x}{1-x}$.

Note that $\rho(x)$ is indeed strictly increasing as we previously assumed. We will vary the value of V_L and let V_L take the values of 0.02, 0.25, 0.28, and 0.5, respectively. Figure 1 puts all values of V_L in one figure, while the rest of the figures each illustrate one value of V_L. When $V_L = 0.02$, the seller's revenue is decreasing and then increasing in γ with the minimum reached at $\gamma = 0.97$; the optimal return policy is the full refund policy. When $V_L = 0.25$, the seller's revenue first increases, then decreases, and then increases in γ; the optimal return policy is a partial refund policy with $\gamma = 0.82\%$. When $V_L = 0.28$, the seller's revenue first increases, then decreases, and then increases in γ; the optimal return policy is the no refund policy. When $V_L = 0.5$, the seller's revenue is increasing in γ; the no refund policy is optimal again.
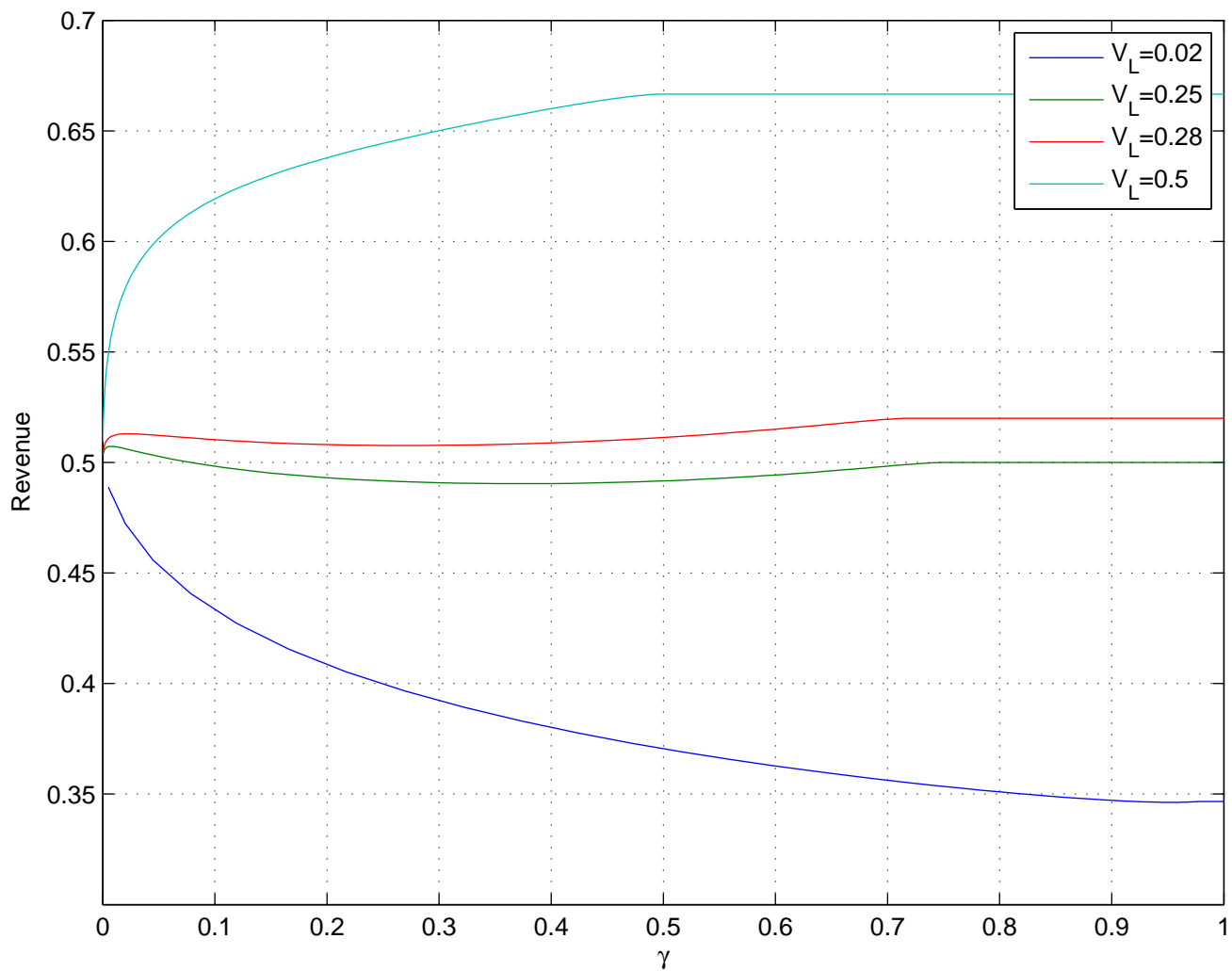
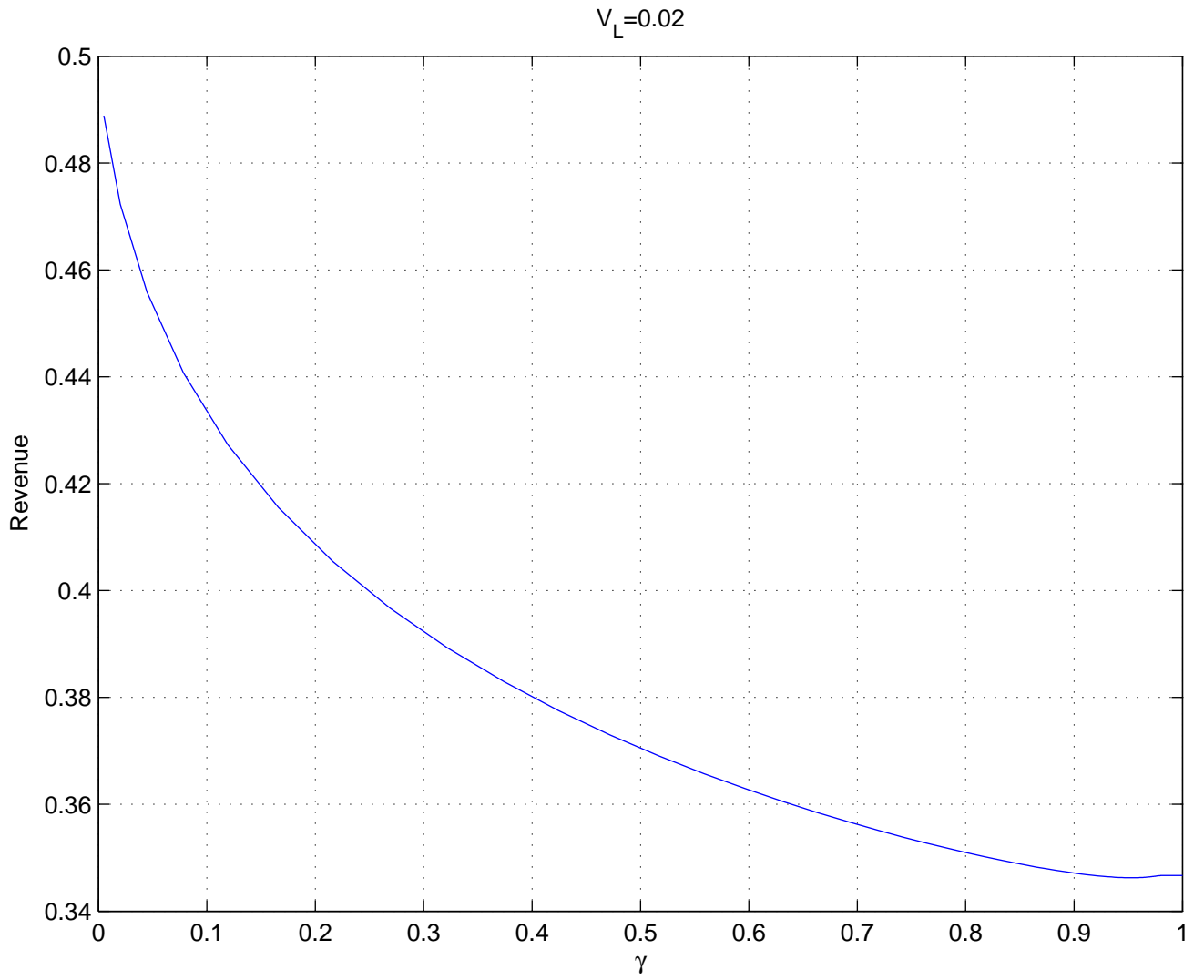Figure 1: The seller's revenue as function of γ for different V_L

Figure 2: The seller's revenue as function of γ when $V_L = 0.02$
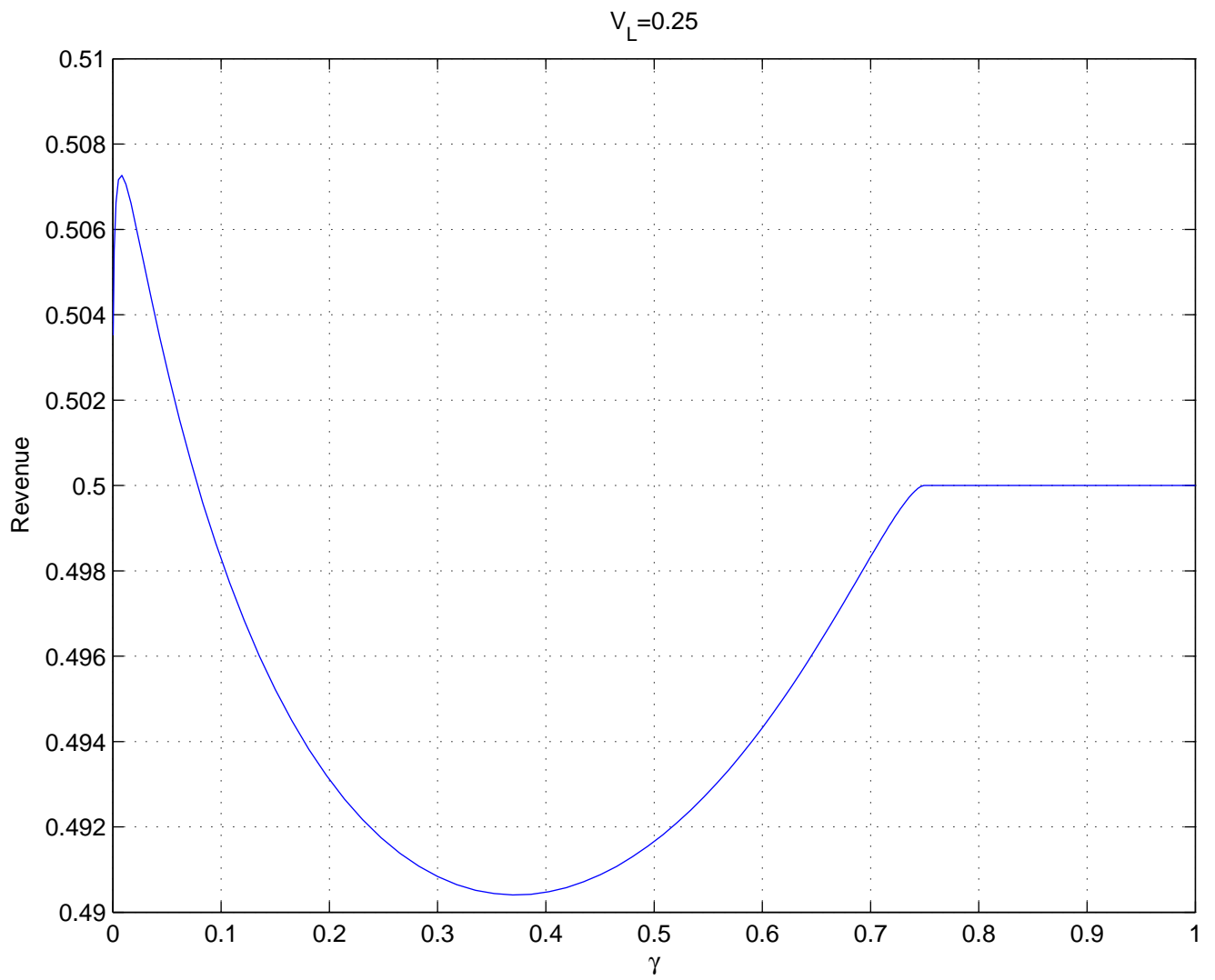
Figure 3: The seller's revenue as function of γ when $V_L = 0.25$
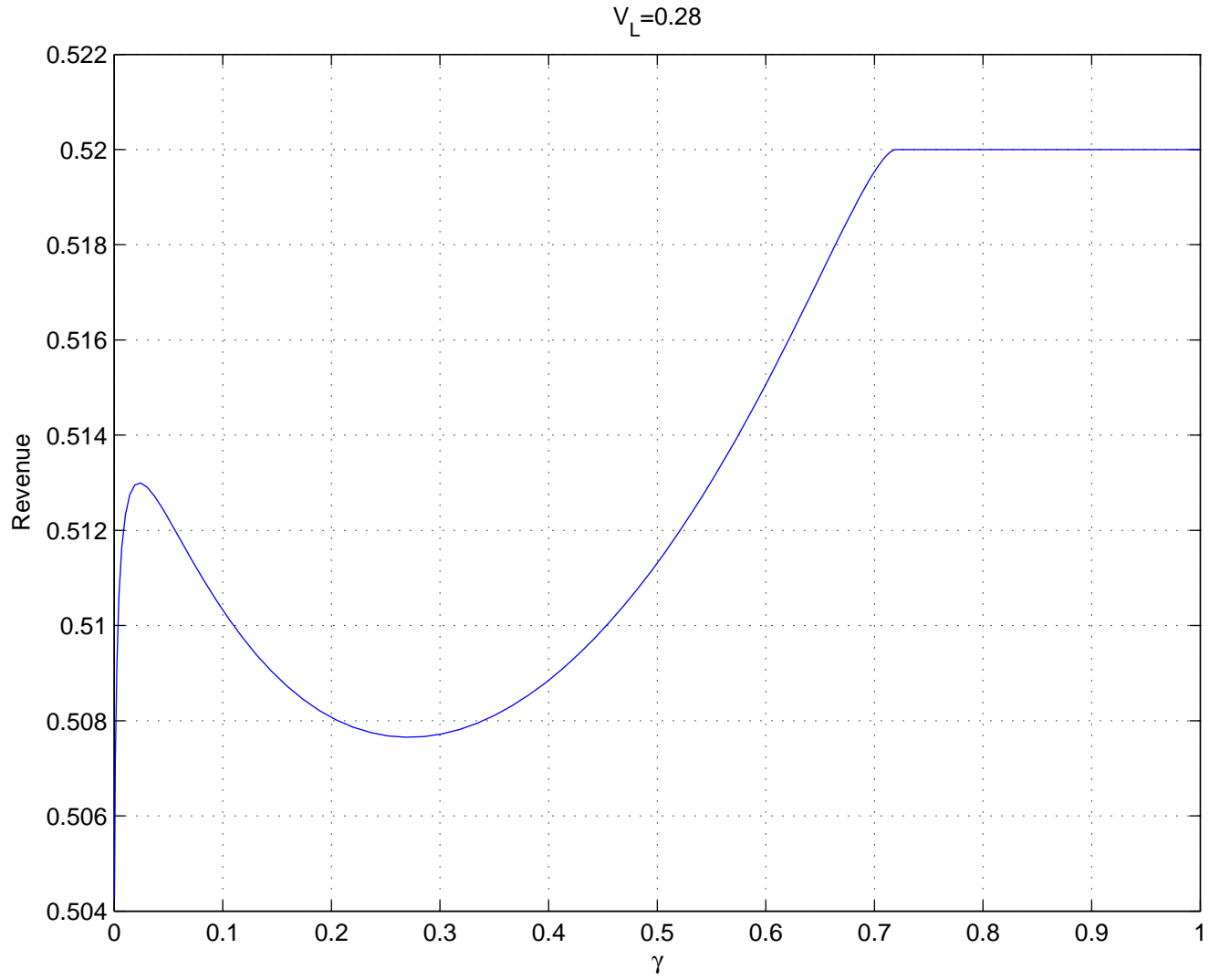
Figure 4: The seller's revenue as function of γ when $V_L = 0.28$
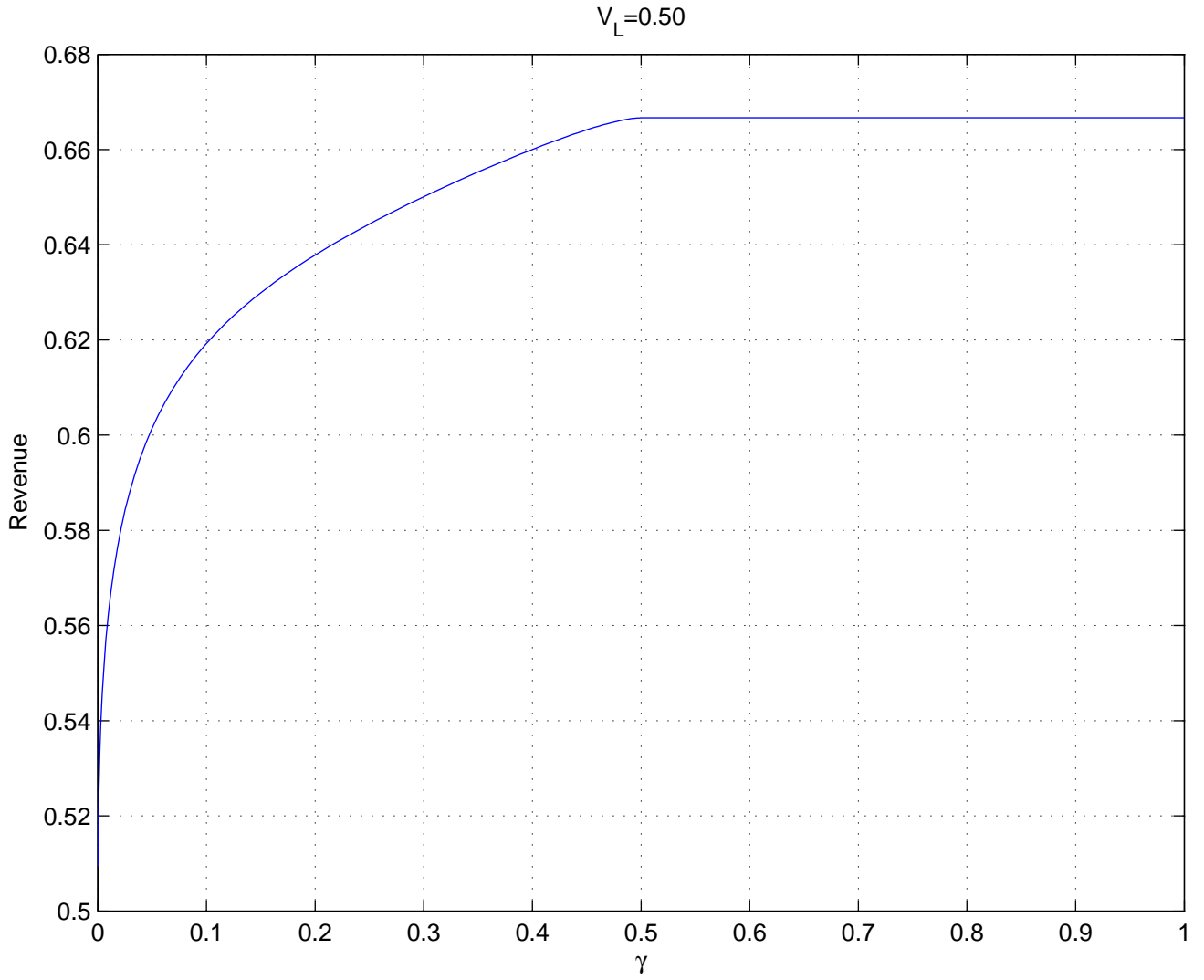
Figure 5: The seller's revenue as function of γ when $V_L = 0.5$

6   Conclusion

This paper investigates how return policies affect buyers' bidding strategies in first-price and second-price auctions and the respective seller's revenue. Providing a return policy undoubtedly induces buyers to bid more aggressively. When the lowest value of the object is zero, the more generous a return policy is, the more expected revenue the auction generates. This is true for both the first-price and the second-price auctions. The standard results in Milgrom and Weber [14] that second-price auction generate more revenue than first-price auctions in common value auctions can be extended to the case of return policies. When the lowest value of the object is non-zero but still lower than the seller's reservation value, the revenue is again higher when the return policy is more generous in the second-price auctions.

Auctions with return policies are more complicated to analyze than standard auctions, as the winning bidder may return the object when he discovers the object's true values. Therefore, a higher bid induced by a more generous return policy may not be beneficial to the sellers. This paper shows that when the efficiency losses from the returns are not significant, a more generous return policy helps the sellers. Since a seller can also use return policies to signal the quality of the object, we should expect to see return policies in many auctions as we have witnessed in online auctions, where buyers have less confidence in the quality of the objects.

7   Appendix

Proof for Proposition 1
Case 1: Never return

We first characterize the symmetric equilibrium bidding function in the case where the winning bidder never returns the object after winning. Let $B^{S1}(\cdot)$ denote the bidding function in this case. Consider buyer 1. Suppose that buyer 1's signal is x and he pretends to have signal $\tilde{x}$ and bids $B^{S1}(\tilde{x})$. Given that when the realization of the value is V_L, bidder 1 will keep the object if he wins, his expected surplus in the auction is given by:

$$
\begin{aligned}
\Pi^{S1}(x, \tilde{x}) &= Pr(V = V_H | x_1 = x) E\{[V - B^{S1}(x_2)] I\{x_2 < \tilde{x}\} | x_1 = x, V = V_H\} \quad (21) \\
&+ Pr(V = V_L | x_1 = x) E\{[V - B^{S1}(x_2)] I\{x_2 < \tilde{x}\} | x_1 = x, V = V_L\} \quad (22) \\
&= \mu_H(x) \int_{\underline{x}}^{\tilde{x}} [V_H - B^{S1}(x_2)] dF_H(x_2) + \mu_L(x) \int_{\underline{x}}^{\tilde{x}} [V_L - B^{S1}(x_2)] dF_L(x_2), \quad (23)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_H(x) &\equiv \Pr(V = V_H | x_1 = x) \quad (24) \\
&= \frac{\Pr(x_1 = x | V = V_H) \Pr(V = V_H)}{\Pr(x_1 = x | V = V_H) \Pr(V = V_H) + \Pr(x_1 = x | V = V_L) \Pr(V = V_L)} \quad (25)
\end{aligned}
$$

20

$$= \frac{f_H(x)\mu_H}{f_H(x)\mu_H + f_L(x)\mu_L}, \tag{26}$$

and where $\mu_L(x) = \Pr(V = V_L | x_1 = x) = 1 - \mu_H(x)$. It is important to note that $\mu_H(x)$ is increasing in x and $\mu_L(x)$ is decreasing in x. Therefore,

$$
\begin{aligned}
\frac{\partial \Pi^{S1}(x, \tilde{x})}{\partial \tilde{x}} \\
&= \mu_H(x)[V_H - B^{S1}(\tilde{x})]f_H(\tilde{x}) + \mu_L(x)[V_L - B^{S1}(\tilde{x})]f_L(\tilde{x}) \\
&= [\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})]\left[\frac{\mu_H(x)V_H f_H(\tilde{x}) + \mu_L(x)V_L f_L(\tilde{x})}{\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})} - B^{S1}(\tilde{x})\right] \\
&= [\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})]\left[\frac{\mu_H f_H(x)V_H f_H(\tilde{x}) + \mu_L f(x)V_L f_L(\tilde{x})}{\mu_H f_H(x)f_H(\tilde{x}) + \mu_L f_L(x)f_L(\tilde{x})} - B^{S1}(\tilde{x})\right] \\
&= [\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})]\left[\frac{\mu_H V_H \rho(x)\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(x)\rho(\tilde{x}) + \mu_L} - B^{S1}(\tilde{x})\right]. \tag{27}
\end{aligned}
$$

The first order condition (FOC) for this bidder's surplus maximization problem gives:

$$\left.\frac{\partial \Pi^{S1}(x, \tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=x} = 0. \tag{28}$$

Solving for $B^{S1}(x)$, we have

$$B^{S1}(x) = \frac{\mu_H V_H \rho(x)^2 + \mu_L V_L}{\mu_H \rho(x)^2 + \mu_L}. \tag{29}$$

The FOC is usually only a necessary condition. We shall show below that the FOC is also a sufficient condition for the above maximization problem. It is easy to check that $\frac{\mu_H V_H \rho(x)\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(x)\rho(\tilde{x}) + \mu_L}$ is increasing in x. Therefore, given the bidding function defined in equation (29), the surplus function $\Pi^{S1}(x, \tilde{x})$ is a unimodal function with the maximum at $\tilde{x} = x$; i.e., increasing for $\tilde{x} \leq x$ and decreasing for $\tilde{x} \geq x$. To see this, for $\tilde{x} \leq x$,

$$
\begin{aligned}
\frac{\partial \Pi^{S1}(x, \tilde{x})}{\partial \tilde{x}} & \tag{30} \\
&= [\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})]\left[\frac{\mu_H V_H \rho(x)\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(x)\rho(\tilde{x}) + \mu_L} - \frac{\mu_H V_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L}\right] \tag{31} \\
&\geq [\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})]\left[\frac{\mu_H V_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L} - \frac{\mu_H V_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L}\right] \tag{32} \\
&= 0, \tag{33}
\end{aligned}
$$

and for $\tilde{x} \geq x$,

$$\frac{\partial \Pi^{S1}(x, \tilde{x})}{\partial \tilde{x}} \tag{34}$$

$$= [\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})] \left[\frac{\mu_H V_H \rho(x)\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(x)\rho(\tilde{x}) + \mu_L} - \frac{\mu_H V_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L} \right] \tag{35}$$

$$\leq [\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})] \left[\frac{\mu_H V_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L} - \frac{\mu_H V_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L V_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L} \right] \tag{36}$$

$$= 0. \tag{37}$$

Therefore, $\tilde{x} = x$ is indeed optimal and the sufficiency of FOC for the maximization is confirmed. Of course, for the above bidding function to be an equilibrium, we need to guarantee that the winner never wants to return the object. Note that the bidding function is increasing. The condition of no return is equivalent to

$$V_L \geq (1 - \gamma)B^{S1}(\overline{x}) \tag{38}$$

$$\Leftrightarrow \quad \gamma \geq \frac{(V_H - V_L)\mu_H \rho(\overline{x})^2}{V_H \mu_H \rho(\overline{x})^2 + \mu_L V_L} \equiv \overline{\gamma}^S. \tag{39}$$

Case 2: Always return when $V = V_L$

In this case, the winning bidder always returns the object when $V = V_L$. Given this, buyer 1's surplus when he pretends to have signal $\tilde{x}$ is given by

$$\Pi^{S2}(x, \tilde{x}) = Pr(V = V_H | x_1 = x) E\{[V - B^{S2}(x_2)]I\{x_2 < \tilde{x}\} | x_1 = x, V = V_H\} \tag{40}$$

$$+ Pr(V = V_L | x_1 = x) \left[-E\{\gamma B^{S2}(x_2)I\{x_2 < \tilde{x}\} | x_1 = x, V = V_L\} \right] \tag{41}$$

$$= \mu_H(x) \int_{\underline{x}}^{\tilde{x}} [V_H - B^{S2}(x_2)]dF_H(x_2) - \mu_L(x) \int_{\underline{x}}^{\tilde{x}} \gamma B^{S2}(x_2)dF_L(x_2). \tag{42}$$

Taking the derivative with respect to $\tilde{x}$, we have

$$\frac{\partial \Pi^{S2}(x, \tilde{x})}{\partial \tilde{x}}$$

$$= \mu_H(x)[V_H - B^{S2}(\tilde{x})]f_H(\tilde{x}) - \mu_L(x)\gamma B^{S2}(\tilde{x})f_L(\tilde{x})$$

$$= [\mu_H(x)f_H(\tilde{x}) + \gamma\mu_L(x)f_L(\tilde{x})] \left[\frac{\mu_H(x)V_H f_H(\tilde{x})}{\mu_H(x)f_H(\tilde{x}) + \gamma\mu_L(x)f_L(\tilde{x})} - B^{S2}(\tilde{x}) \right]$$

$$= [\mu_H(x)f_H(\tilde{x}) + \gamma\mu_L(x)f_L(\tilde{x})] \left[\frac{\mu_H f_H(x)V_H f_H(\tilde{x})}{\mu_H f_H(x)f_H(\tilde{x}) + \gamma\mu_L f_L(x)f_L(\tilde{x})} - B^{S2}(\tilde{x}) \right]$$

$$= [\mu_H(x)f_H(\tilde{x}) + \gamma\mu_L(x)f_L(\tilde{x})] \left[\frac{\mu_H V_H \rho(x)\rho(\tilde{x})}{\mu_H \rho(x)\rho(\tilde{x}) + \gamma\mu_L} - B^{S2}(\tilde{x}) \right], \tag{43}$$

The first order condition for bidder 1's surplus maximization problem is

$$\left.\frac{\partial \Pi^{S2}(x, \tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=x} = 0. \tag{44}$$

Solving for $B^{S2}(x)$, we have

$$B^{S2}(x) = \frac{\mu_H V_H \rho(x)^2}{\mu_H \rho(x)^2 + \gamma \mu_L}. \tag{45}$$

The FOC is usually only a necessary condition. It is easy to check that $\frac{\mu_H V_H \rho(x)\rho(\tilde{x})}{\mu_H \rho(x)\rho(\tilde{x})+\gamma \mu_L}$ is increasing in x. Similar to the argument in Case 1, the surplus function $\Pi^{S2}(x, \tilde{x})$ is a unimodal function with maximum at $\tilde{x} = x$ when using the bidding function defined in equation (45). As a result, the sufficiency of FOC for the maximization is confirmed.

Again, for this bidding function to be in equilibrium, the condition for "always returning" has to be satisfied. Given that the bidding function is increasing, this condition is equivalent to

$$V_L \le (1 - \gamma)B^{S2}(\underline{x}) \tag{46}$$

$$\Leftrightarrow \quad \gamma \le \frac{(V_H - V_L)\mu_H \rho(\underline{x})^2}{V_H \mu_H \rho(\underline{x})^2 + \mu_L V_L} \equiv \underline{\gamma}^S. \tag{47}$$

Case 3: Cutoff rule when $V = V_L$

In this case, there is an endogenously determined cutoff in the winning bidder's return decision. We denote this cutoff as x^{S*}. Buyer 1's surplus by pretending to be have signal $\tilde{x}$ is given by

$$\Pi^S(x, \tilde{x}) = \begin{cases} \Pi^{S1}(x, \tilde{x}), & \text{if } \tilde{x} \le x^{S*}; \\ \\ \mu_H(x)\left\{\int_{\underline{x}}^{\tilde{x}}[V_H - B^S(x_2)]dF_H(x_2)\right\} \\ +\mu_L(x)\left\{\int_{\underline{x}}^{x^{S*}}[V_L - B^S(x_2)]dF_L(x_2) - \int_{x^{S*}}^{\tilde{x}}\gamma B^S(x_2)dF_L(x_2)\right\}, & \text{if } \tilde{x} \ge x^{S*}. \end{cases} \tag{48}$$

Note that the above function is continuous. Taking the derivative of the above with respect to $\tilde{x}$, we have

$$\frac{\partial \Pi^S(x, \tilde{x})}{\partial \tilde{x}} = \begin{cases} \frac{\partial \Pi^{S1}(x, \tilde{x})}{\partial \tilde{x}}, & \text{if } \tilde{x} \le x^{S*}; \\ \\ \frac{\partial \Pi^{S2}(x, \tilde{x})}{\partial \tilde{x}}, & \text{if } \tilde{x} \ge x^{S*}. \end{cases} \tag{49}$$

Although $\Pi^S(x, \tilde{x}) \ne \Pi^{S2}(x, \tilde{x})$ when $\tilde{x} \ge x^{S*}$, we have $\frac{\partial \Pi^S(x, \tilde{x})}{\partial \tilde{x}} = \frac{\partial \Pi^{S2}(x, \tilde{x})}{\partial \tilde{x}}$. From the first

order condition, we can derive the bidding function as follows:

$$B^S(x) = \begin{cases} B^{S1}(x) = \frac{\mu_H V_H \rho(x)^2 + \mu_L V_L}{\mu_H \rho(x)^2 + \mu_L}, & \text{if } x \leq x^{S*}; \\[3mm] B^{S2}(x) = \frac{\mu_H V_H \rho(x)^2}{\mu_H \rho(x)^2 + \gamma \mu_L}, & \text{if } x \geq x^{S*}. \end{cases} \tag{50}$$

Note that x^{S*} is determined by

$$(1-\gamma)B^{S1}(x^{S*}) = V_L, \tag{51}$$

i.e.,

$$\gamma = \frac{(V_H - V_L)\mu_H \rho(x^{S*})^2}{V_H \mu_H \rho(x^{S*})^2 + \mu_L V_L}. \tag{52}$$

Note that functions $B^{S2}(x)$ and $B^{S1}(x)$ cross each other at x^{S*}.

Now consider the sufficient condition. Given the bidding function (50), from the proof in Cases 1 and 2, we know that $\Pi^{S1}(x, \tilde{x})$ is a unimodal function with the maximum at $\tilde{x} = x$ when $\tilde{x} \leq x^{S*}$; and $\Pi^{S2}(x, \tilde{x})$ is a unimodal function with the maximum at $\tilde{x} = x$ when $\tilde{x} \geq x^{S*}$. We shall show that $\Pi^S(x, \tilde{x})$ is also a unimodal function with maximum at $\tilde{x} = x$. Consider $x \leq x^{S*}$, for example. For $\underline{x} \leq \tilde{x} \leq x$, the payoff is increasing in $\tilde{x}$ from the first formula of (49). For $x \leq \tilde{x} \leq x^{S*}$, the payoff is decreasing in $\tilde{x}$ from the first formula of (49). For $x^{S*} \leq \tilde{x} \leq \overline{x}$, the payoff is decreasing in $\tilde{x}$ from the second formula of (49). Therefore, $\Pi^S(x, \tilde{x})$ achieves its maximal value at $\tilde{x} = x$. Similar arguments can be applied to the case of $x \geq x^{S*}$. Thus, the sufficient condition for the maximization is thus satisfied.

In this equilibrium, when $V = V_L$, the winning bidder returns the object if he pays too much, and keeps the object otherwise. For this to happen, γ has to satisfy the following condition:

$$(1-\gamma)B^{S1}(\overline{x}) > V_L > (1-\gamma)B^{S1}(\underline{x}) \tag{53}$$
$$\Leftrightarrow \quad \underline{\gamma}^S > \gamma > \overline{\gamma}^S. \tag{54}$$

Q.E.D.

Proof for Lemma 2
We only need to examine $L_2(\alpha|x; \gamma)$ since $L_1(\alpha|x) = L_2(\alpha|x; \gamma = 1)$. Note that the function $\frac{\mu_H f_H(x)f_H(s) + \gamma\mu_L f_L(x)f_L(s)}{\mu_H f_H(x)F_H(s) + \gamma\mu_L f_L(x)F_L(s)}$ is increasing in x. To see this,

$$\frac{\partial \frac{\mu_H f_H(x)f_H(s) + \gamma\mu_L f_L(x)f_L(s)}{\mu_H f_H(x)F_H(s) + \gamma\mu_L f_L(x)F_L(s)}}{\partial x}$$

$$\partial \frac{\mu_H f_H(s) + \gamma \mu_L f_L(s)[\frac{f_L(x)}{f_H(x)}]}{\mu_H F_H(s) + \gamma \mu_L F_L(s)[\frac{f_L(x)}{f_H(x)}]}$$
$$= \frac{}{\partial x}$$

$$= \frac{\gamma \mu_L \mu_H [f_L(s) F_H(s) - f_H(s) F_L(s)] \frac{\partial \frac{f_L(x)}{f_H(x)}}{\partial x}}{\{\mu_H F_H(s) + \gamma \mu_L F_L(s)[\frac{f_L(x)}{f_H(x)}]\}^2}$$

$$= \frac{\gamma \mu_L \mu_H F_L(s) F_H(s)[f_L(s)/F_L(s) - f_H(s)/F_H(s)] \frac{\partial \frac{f_L(x)}{f_H(x)}}{\partial x}}{\{\mu_H F_H(s) + \gamma \mu_L F_L(s)[\frac{f_L(x)}{f_H(x)}]\}^2} \geq 0 \qquad (55)$$

The inequality follows from the second and third parts of Lemma 1. Therefore,

$$\int_{\underline{x}}^{x} \frac{\mu_H f_H(s)^2 + \gamma \mu_L f_L(s)^2}{\mu_H f_H(s) F_H(s) + \gamma \mu_L f_L(s) F_L(s)} ds \qquad (56)$$

$$\geq \int_{\underline{x}}^{x} \frac{\mu_H f_H(\underline{x}) f_H(s) + \gamma \mu_L f_L(\underline{x}) f_L(s)}{\mu_H f_H(\underline{x}) F_H(s) + \gamma \mu_L f_L(\underline{x}) F_L(s)} ds \qquad (57)$$

$$= \int_{\underline{x}}^{x} \frac{d\ln[\mu_H f_H(\underline{x}) F_H(s) + \gamma \mu_L f_L(\underline{x}) F_L(s)]}{ds} ds \qquad (58)$$

$$= \ln[\mu_H f_H(\underline{x}) F_H(x) + \gamma \mu_L f_L(\underline{x}) F_L(x)] - \ln[\mu_H f_H(\underline{x}) F_H(\underline{x}) + \gamma \mu_L f_L(\underline{x}) F_L(\underline{x})] \quad (59)$$

$$= \infty \qquad (60)$$

Thus $L_2(\underline{x}|x;\gamma) = 0$. Moreover $L_2(x|x;\gamma) = 1$ and $L_2(\alpha|x;\gamma)$ is nondecreasing. As a result, $L_2(\alpha|x;\gamma)$ is a distribution function. **Q.E.D.**

Proof for Proposition 2
Case 1: Never return
Consider a bidder who has signal x and pretends to have $\tilde{x}$. Given that he always keeps the object if he wins, his expected surplus is given by

$$\Pi^{F1}(x,\tilde{x}) = Pr(V = V_H|x_1 = x) E\{[V - B^{F1}(\tilde{x})] I\{x_2 < \tilde{x}\}|x_1 = x, V = V_H\} \quad (61)$$
$$+ Pr(V = V_L|x_1 = x) E\{[V - B^{F1}(\tilde{x})] I\{x_2 < \tilde{x}\}|x_1 = x, V = V_L\} \quad (62)$$
$$= \mu_H(x)[V_H - B^{F1}(\tilde{x})] F_H(\tilde{x}) + \mu_L(x)[V_L - B^{F1}(\tilde{x})] F_L(\tilde{x}). \quad (63)$$

Taking the derivative of the above with respect to $\tilde{x}$, we have

$$\frac{\partial \Pi^{F1}(x,\tilde{x})}{\partial \tilde{x}}$$
$$= \mu_H(x)[V_H f_H(\tilde{x}) - (B^{F1})'(\tilde{x}) F_H(\tilde{x}) - B^{F1}(\tilde{x}) f_H(\tilde{x})]$$
$$+ \mu_L(x)[V_L f_L(\tilde{x}) - (B^{F1})'(\tilde{x}) F_L(\tilde{x}) - B^{F1}(\tilde{x}) f_L(\tilde{x})]$$
$$= [\mu_H(x) F_H(\tilde{x}) + \mu_L(x) F_L(\tilde{x})] \times$$

$$\times \left\{ \frac{\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})}{\mu_H(x)F_H(\tilde{x}) + \mu_L(x)F_L(\tilde{x})} \left[\frac{V_H\mu_H(x)f_H(\tilde{x}) + V_L\mu_L(x)f_L(\tilde{x})}{\mu_H(x)f_H(\tilde{x}) + \mu_L(x)f_L(\tilde{x})} - B^{F1}(\tilde{x}) \right] - \left(B^{F1} \right)'(\tilde{x}) \right\}$$
$$= [\mu_H(x)F_H(\tilde{x}) + \mu_L(x)F_L(\tilde{x})] \times$$
$$\times \left\{ \frac{\mu_H\rho(x)f_H(\tilde{x}) + \mu_Lf_L(\tilde{x})}{\mu_H\rho(x)F_H(\tilde{x}) + \mu_LF_L(\tilde{x})} \left[\frac{V_H\mu_H\rho(x)\rho(\tilde{x}) + V_L\mu_L}{\mu_H\rho(x)\rho(\tilde{x}) + \mu_L} - B^{F1}(\tilde{x}) \right] - \left(B^{F1} \right)'(\tilde{x}) \right\}. \qquad (64)$$

The first order condition for this optimization problem is

$$\left. \frac{\partial \Pi^{F1}(x, \tilde{x})}{\partial \tilde{x}} \right|_{\tilde{x}=x} = 0. \qquad (65)$$

Therefore,

$$\left(B^{F1} \right)'(x) = \frac{\mu_H(x)f_H(x) + \mu_L(x)f_L(x)}{\mu_H(x)F_H(x) + \mu_L(x)F_L(x)} \left[\frac{V_H\mu_H\rho(x)^2 + V_L\mu_L}{\mu_H\rho(x)^2 + \mu_L} - B^{F1}(x) \right]. \qquad (66)$$

The above differential equation (66) is just one of the necessary conditions for the equilibrium. It is also necessary that $\left[\frac{V_H\mu_H\rho(x)^2+V_L\mu_L}{\mu_H\rho(x)^2+\mu_L} - B^{F1}(x) \right]$ is nonnegative; otherwise, bidding zero would be better. Furthermore, $\left[\frac{V_H\mu_H\rho(\underline{x})^2+V_L\mu_L}{\mu_H\rho(\underline{x})^2+\mu_L} - B^{F1}(\underline{x}) \right]$ must also be nonpositive; otherwise, the bidder with the lowest signal would be better to bid a little bit more. These last two restrictions determine the boundary condition: $B^{F1}(\underline{x}) = \frac{V_H\mu_H\rho(\underline{x})^2+V_L\mu_L}{\mu_H\rho(\underline{x})^2+\mu_L}$. With this boundary condition and the above differential equation (66), we can obtain the following bidding function:

$$B^{F1}(x) = \int_{\underline{x}}^{x} \frac{\mu_H V_H\rho(\alpha)^2 + \mu_L V_L}{\mu_H\rho(\alpha)^2 + \mu_L} dL_1(\alpha|x), \qquad (67)$$

The bidding function is indeed increasing. Note that this bidding function can also be formulated in the format of Milgrom and Weber [14], i.e., $B^{F1}(x) = \int_{\underline{x}}^{x} E(V|\alpha, \alpha)dL_1(\alpha|x)$.

In what follows, we shall show that $\tilde{x} = x$ indeed maximizes the bidder's surplus given the above bidding function. In equation (64), $\frac{\mu_H\rho(x)f_H(\tilde{x})+\mu_Lf_L(\tilde{x})}{\mu_H\rho(x)F_H(\tilde{x})+\mu_LF_L(\tilde{x})}$ is increasing in x. To see this:

$$\frac{\partial \frac{\mu_H\rho(x)f_H(\tilde{x})+\mu_Lf_L(\tilde{x})}{\mu_H\rho(x)F_H(\tilde{x})+\mu_LF_L(\tilde{x})}}{\partial x} = \frac{\mu_L\mu_H[f_H(\tilde{x})F_L(\tilde{x}) - f_L(\tilde{x})F_H(\tilde{x})]\rho'(x)}{[\mu_H\rho(x)F_H(\tilde{x}) + \mu_LF_L(\tilde{x})]^2} \qquad (68)$$

$$= \frac{\mu_L\mu_H F_L(\tilde{x})F_H(\tilde{x})[\frac{f_H(\tilde{x})}{F_H(\tilde{x})} - \frac{f_L(\tilde{x})}{F_L(\tilde{x})}]\rho'(x)}{[\mu_H\rho(x)F_H(\tilde{x}) + \mu_LF_L(\tilde{x})]^2} \qquad (69)$$

$$\geq 0, \qquad (70)$$

where the last inequality is implied from the second part of Lemma 1. Similarly, $\frac{V_H\mu_H\rho(x)\rho(\tilde{x})+V_L\mu_L}{\mu_H\rho(x)\rho(\tilde{x})+\mu_L}$

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is increasing in x. Also note that

$$B^{F1}(x) = \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|x) \qquad (71)$$

$$\leq \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(x)^2 + \mu_L V_L}{\mu_H \rho(x)^2 + \mu_L} dL_1(\alpha|x) \qquad (72)$$

$$= \frac{\mu_H V_H \rho(x)^2 + \mu_L V_L}{\mu_H \rho(x)^2 + \mu_L} \qquad (73)$$

Therefore, for $\tilde{x} < x$, from equation (64),

$$\frac{\partial \Pi^{F1}(x,\tilde{x})}{\partial \tilde{x}} \qquad (74)$$
$$= [\mu_H(x)F_H(\tilde{x}) + \mu_L(x)F_L(\tilde{x})] \times \qquad (75)$$
$$\times \left\{ \frac{\mu_H \rho(x)f_H(\tilde{x}) + \mu_L f_L(\tilde{x})}{\mu_H \rho(x)F_H(\tilde{x}) + \mu_L F_L(\tilde{x})} \left[\frac{V_H \mu_H \rho(x)\rho(\tilde{x}) + V_L \mu_L}{\mu_H \rho(x)\rho(\tilde{x}) + \mu_L} - B^{F1}(\tilde{x}) \right] - \left(B^{F1}\right)'(\tilde{x}) \right\} \qquad (76)$$
$$> [\mu_H(x)F_H(\tilde{x}) + \mu_L(x)F_L(\tilde{x})] \times \qquad (77)$$
$$\times \left\{ \frac{\mu_H \rho(x)f_H(\tilde{x}) + \mu_L f_L(\tilde{x})}{\mu_H \rho(x)F_H(\tilde{x}) + \mu_L F_L(\tilde{x})} \left[\frac{V_H \mu_H \rho(\tilde{x})\rho(\tilde{x}) + V_L \mu_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L} - B^{F1}(\tilde{x}) \right] - \left(B^{F1}\right)'(\tilde{x}) \right\} \qquad (78)$$
$$> [\mu_H(x)F_H(\tilde{x}) + \mu_L(x)F_L(\tilde{x})] \times \qquad (79)$$
$$\times \left\{ \frac{\mu_H \rho(\tilde{x})f_H(\tilde{x}) + \mu_L f_L(\tilde{x})}{\mu_H \rho(\tilde{x})F_H(\tilde{x}) + \mu_L F_L(\tilde{x})} \left[\frac{V_H \mu_H \rho(\tilde{x})\rho(\tilde{x}) + V_L \mu_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L} - B^{F1}(\tilde{x}) \right] - \left(B^{F1}\right)'(\tilde{x}) \right\} \qquad (80)$$
$$= 0 \qquad (81)$$

Similarly, for $\tilde{x} > x$, from equation (64),

$$\frac{\partial \Pi^{F1}(x,\tilde{x})}{\partial \tilde{x}} \qquad (82)$$
$$= [\mu_H(x)F_H(\tilde{x}) + \mu_L(x)F_L(\tilde{x})] \times \qquad (83)$$
$$\times \left\{ \frac{\mu_H \rho(x)f_H(\tilde{x}) + \mu_L f_L(\tilde{x})}{\mu_H \rho(x)F_H(\tilde{x}) + \mu_L F_L(\tilde{x})} \left[\frac{V_H \mu_H \rho(x)\rho(\tilde{x}) + V_L \mu_L}{\mu_H \rho(x)\rho(\tilde{x}) + \mu_L} - B^{F1}(\tilde{x}) \right] - \left(B^{F1}\right)'(\tilde{x}) \right\} \qquad (84)$$
$$< [\mu_H(x)F_H(\tilde{x}) + \mu_L(x)F_L(\tilde{x})] \times \qquad (85)$$
$$\times \left\{ \frac{\mu_H \rho(x)f_H(\tilde{x}) + \mu_L f_L(\tilde{x})}{\mu_H \rho(x)F_H(\tilde{x}) + \mu_L F_L(\tilde{x})} \left[\frac{V_H \mu_H \rho(\tilde{x})\rho(\tilde{x}) + V_L \mu_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L} - B^{F1}(\tilde{x}) \right] - \left(B^{F1}\right)'(\tilde{x}) \right\} \qquad (86)$$
$$< [\mu_H(x)F_H(\tilde{x}) + \mu_L(x)F_L(\tilde{x})] \times \qquad (87)$$
$$\times \left\{ \frac{\mu_H \rho(\tilde{x})f_H(\tilde{x}) + \mu_L f_L(\tilde{x})}{\mu_H \rho(\tilde{x})F_H(\tilde{x}) + \mu_L F_L(\tilde{x})} \left[\frac{V_H \mu_H \rho(\tilde{x})\rho(\tilde{x}) + V_L \mu_L}{\mu_H \rho(\tilde{x})\rho(\tilde{x}) + \mu_L} - B^{F1}(\tilde{x}) \right] - \left(B^{F1}\right)'(\tilde{x}) \right\} \qquad (88)$$
$$= 0 \qquad (89)$$

As a result, the payoff function $\Pi^{F1}(x,\tilde{x})$ is a unimodal function of $\tilde{x}$ with its maximum at $\tilde{x} = x$, i.e., it is increasing when $\tilde{x} \leq x$ and decreasing when $\tilde{x} \geq x$. Thus, $\tilde{x} = x$ indeed maximizes the bidder's surplus.

Note that for the bidding function to form an equilibrium, the condition of "Never return" must be satisfied. Because the bidding function is increasing, "Never return" is equivalent to

$$(1-\gamma)B^{F1}(\overline{x}) \leq V_L \tag{90}$$

$$\Leftrightarrow \quad \gamma \geq \frac{B^{F1}(\overline{x}) - V_L}{B^{F1}(\overline{x})} \tag{91}$$

$$\Leftrightarrow \quad \gamma \geq \frac{\int_{\underline{x}}^{\overline{x}} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|\overline{x}) - V_L}{\int_{\underline{x}}^{\overline{x}} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|\overline{x})} \tag{92}$$

$$\Leftrightarrow \quad \gamma \geq \frac{\int_{\underline{x}}^{\overline{x}} \left\{ \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} - V_L \right\} dL_1(\alpha|\overline{x})}{\int_{\underline{x}}^{\overline{x}} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|\overline{x})} \tag{93}$$

$$\Leftrightarrow \quad \gamma \geq \frac{\int_{\underline{x}}^{\overline{x}} \frac{(V_H - V_L)\mu_H \rho(\alpha)^2}{V_H \mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|\overline{x})}{\int_{\underline{x}}^{\overline{x}} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|\overline{x})} = \overline{\gamma}^F. \tag{94}$$

Case 2: Always return when $V = V_L$

In this case, the winning bidder always returns the object when $V = V_L$. A bidder's surplus when having signal x but pretending to be $\tilde{x}$ is given by

$$\Pi^{F2}(x, \tilde{x}) = Pr(V = V_H | x_1 = x)E\{(V - B^F(\tilde{x}))I\{x_2 < \tilde{x}\}|x_1 = x, V = V_H\} \tag{95}$$
$$-Pr(V = V_L | x_1 = x)E\{\gamma B^F(\tilde{x})I\{x_2 < \tilde{x}\}|x_1 = x, V = V_L\} \tag{96}$$
$$= \mu_H(x)[V_H - B^F(\tilde{x})]F_H(\tilde{x}) - \mu_L(x)\gamma B^F(\tilde{x})F_L(\tilde{x}). \tag{97}$$

Taking the derivative of the above with respect to $\tilde{x}$, we have

$$\frac{\partial \Pi^{F2}(x, \tilde{x})}{\partial \tilde{x}}$$
$$= \mu_H(x)[V_H f_H(\tilde{x}) - B^{F2}{}'(\tilde{x})F_H(\tilde{x}) - B^{F2}(\tilde{x})f_H(\tilde{x})]$$
$$+ \mu_L(x)[-\gamma B^{F2}{}'(\tilde{x})F_L(\tilde{x}) - \gamma B^{F2}(\tilde{x})f_L(\tilde{x})]$$

$$= [\mu_H(x)F_H(\tilde{x}) + \gamma\mu_L(x)F_L(\tilde{x})] \times$$
$$\times \left\{ \frac{\mu_H(x)f_H(\tilde{x}) + \gamma\mu_L(x)f_L(\tilde{x})}{\mu_H(x)F_H(\tilde{x}) + \gamma\mu_L(x)F_L(\tilde{x})} \left[\frac{V_H \mu_H(x)f_H(\tilde{x})}{\mu_H(x)f_H(\tilde{x}) + \gamma\mu_L(x)f_L(\tilde{x})} - B^{F2}(\tilde{x}) \right] - \left(B^{F2}\right)'(\tilde{x}) \right\}$$

$$= [\mu_H(x)F_H(\tilde{x}) + \gamma\mu_L(x)F_L(\tilde{x})] \times$$
$$\times \left\{ \frac{\mu_H \rho(x)f_H(\tilde{x}) + \gamma\mu_L f_L(\tilde{x})}{\mu_H \rho(x)F_H(\tilde{x}) + \gamma\mu_L F_L(\tilde{x})} \left[\frac{V_H \mu_H \rho(x)\rho(\tilde{x})}{\mu_H \rho(x)\rho(\tilde{x}) + \gamma\mu_L} - B^{F2}(\tilde{x}) \right] - \left(B^{F2}\right)'(\tilde{x}) \right\}. \tag{98}$$

The first order condition for this optimization problem is

$$\left.\frac{\partial \Pi^{F2}(x, \tilde{x})}{\partial \tilde{x}}\right|_{\tilde{x}=x} = 0. \tag{99}$$

Therefore,

$$\left(B^{F2}\right)'(x) = \frac{\mu_H(x)f_H(x) + \gamma \mu_L(x)f_L(x)}{\mu_H(x)F_H(x) + \gamma \mu_L(x)F_L(x)} \left[\frac{V_H \mu_H \rho(x)^2}{\mu_H \rho(x)^2 + \gamma \mu_L} - B^{F2}(x)\right]. \tag{100}$$

This differential equation (100) is just one necessary condition for the equilibrium. It must also be that $\left[\frac{V_H \mu_H \rho(x)^2}{\mu_H \rho(x)^2 + \gamma \mu_L} - B^{F2}(x)\right]$ be nonnegative; otherwise, bidding zero would be better. Also, $\left[\frac{V_H \mu_H \rho(\underline{x})^2}{\mu_H \rho(\underline{x})^2 + \gamma \mu_L} - B^{F2}(\underline{x})\right]$ must be nonpositive; otherwise, the bidder with the lowest signal would be better bidding a little bit more. These two conditions determine the boundary condition: $B^{F2}(\underline{x}) = \frac{V_H \mu_H \rho(\underline{x})^2}{\mu_H \rho(\underline{x})^2 + \gamma \mu_L}$. With this boundary condition, we obtain the following bidding function from the differential equation (100):

$$B^{F2}(x) = \int_{\underline{x}}^{x} \frac{V_H \mu_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|x; \gamma), \tag{101}$$

The bidding function is indeed increasing.

In what follows, we shall show that $\tilde{x} = x$ indeed maximizes the bidder's surplus given the above bidding function. In equation (98), it is easy to see that $\frac{\mu_H \rho(x)f_H(\tilde{x})}{\mu_H \rho(x)F_H(\tilde{x}) + \gamma \mu_L F_L(\tilde{x})}$ and $\frac{V_H \mu_H \rho(x)\rho(\tilde{x}) + \gamma V_L \mu_L}{\mu_H \rho(x)\rho(\tilde{x}) + \gamma \mu_L}$ are both increasing in x. Following a similar argument as in Case 1, we can show that the payoff function $\Pi^{F2}(x, \tilde{x})$ is a unimodal function of $\tilde{x}$ with its maximum at $\tilde{x} = x$ given the bidding function, i.e., it is increasing when $\tilde{x} \leq x$ and decreasing when $\tilde{x} \geq x$. Therefore, $\tilde{x} = x$ indeed maximizes the bidder's payoff.

This equilibrium bidding function is based on the condition that the winning bidder always returns the object if $V = V_L$, which is equivalent to

$$(1 - \gamma)B^{F2}(\underline{x}) \geq V_L, \tag{102}$$

$$\Leftrightarrow (1 - \gamma)\int_{\underline{x}}^{\underline{x}} \frac{V_H \mu_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|\underline{x}; \gamma) \geq V_L \tag{103}$$

$$\Leftrightarrow (1 - \gamma)\frac{V_H \mu_H \rho(\underline{x})^2}{\mu_H \rho(\underline{x})^2 + \gamma \mu_L} \geq V_L \tag{104}$$

$$\Leftrightarrow \gamma \leq \frac{(V_H - V_L)\mu_H \rho(\underline{x})^2}{V_H \mu_H \rho(\underline{x})^2 + \mu_L V_L} = \underline{\gamma}^F. \tag{105}$$

Case 3: Cutoff rule when $V = V_L$

In this case, there is an endogenously determined cutoff in the winning bidder's return

decision. We denote this cutoff as x^{F*}. A bidder 1's surplus by pretending to be have signal $\tilde{x}$ is given by

$$\Pi^F(x, \tilde{x}) = \begin{cases} \Pi^{F1}(x, \tilde{x}), & \text{if } \tilde{x} \leq x^{F*}; \\ \Pi^{F2}(x, \tilde{x}), & \text{if } \tilde{x} \geq x^{F*}. \end{cases} \tag{106}$$

Taking the derivative with respect to $\tilde{x}$, we have

$$\frac{\partial \Pi^F(x, \tilde{x})}{\partial \tilde{x}} = \begin{cases} \frac{\partial \Pi^{F1}(x, \tilde{x})}{\partial \tilde{x}}, & \text{if } \tilde{x} \leq x^{F*}; \\ \\ \frac{\partial \Pi^{F2}(x, \tilde{x})}{\partial \tilde{x}}, & \text{if } \tilde{x} \geq x^{F*}. \end{cases} \tag{107}$$

Note that the above payoff function is continuous.

If $x \leq x^{F*}$, the necessary condition for the optimization implies that $B^F(x) = B^{F1}(x)$. This bidding function also pins down the cutoff x^{F*}, which is determined by $B^{F1}(x^{F*}) = \frac{V_L}{1-\gamma}$. For $x \geq x^{F*}$, the first order condition is again given by equation (98), but the initial condition is different and is replaced by $B^{F2}(x^{F*}) = \frac{V_L}{1-\gamma}$. Solving (98) with this initial condition, we have

$$\begin{aligned} B^F(x) &= \int_{\underline{x}}^x \frac{V_H \mu_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|x; \gamma) + AL_2(x^{F*}|x; \gamma) \tag{108} \\ &= B^{F1}(x) + AL_2(x^{F*}|x; \gamma), \tag{109} \end{aligned}$$

where

$$A = \frac{V_L}{1-\gamma} - \int_{\underline{x}}^{x^{F*}} \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|x^{F*}; \gamma). \tag{110}$$

To summarize, the equilibrium bidding function in this case is characterized by the following:

$$B^F(x) = \begin{cases} B^{F1}(x), & \text{if } x \leq x^{F*}; \\ \\ B^{F2}(x) + AL_2(x^{F*}|x; \gamma), & \text{if } x \geq x^{F*}. \end{cases} \tag{111}$$

In what follows, we shall show that $\tilde{x} = x$ indeed maximizes the bidder's surplus given the above bidding function. Given the bidding function (111), from the proof in Cases 1 and 2, we know that $\Pi^{F1}(x, \tilde{x})$ is a unimodal function with its maximum at $\tilde{x} = x$ when $\tilde{x} \leq x^{F*}$; and $\Pi^{F2}(x, \tilde{x})$ is a unimodal function with its maximum at $\tilde{x} = x$ when $\tilde{x} \geq x^{F*}$. Consider $x \leq x^{F*}$. For $\underline{x} \leq \tilde{x} \leq x$, the payoff function is increasing in $\tilde{x}$ from the first part of (107). For $x \leq \tilde{x} \leq x^{F*}$, the payoff function is decreasing in $\tilde{x}$ from the first part of (107). For $x^{F*} \leq \tilde{x} \leq \overline{x}$, the payoff is decreasing in $\tilde{x}$ from the second part of (107). Similar arguments

can be made for $x \geq x^{F*}$.

In this equilibrium, the winning bidder returns the object if he pays too much, and keeps the object otherwise. For this to happen, γ has to satisfy the following condition:

$$(1-\gamma)B^{F_1}(\overline{x}) > V_L > (1-\gamma)B^{F1}(\underline{x}) \tag{112}$$

$$\Leftrightarrow \ (1-\gamma)\int_{\underline{x}}^{\overline{x}} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|\overline{x}) > V_L > (1-\gamma)\int_{\underline{x}}^{\underline{x}} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|\underline{x})$$

$$\Leftrightarrow \ (1-\gamma)\int_{\underline{x}}^{\overline{x}} \frac{\mu_H V_H \rho(\alpha)^2 + \mu_L V_L}{\mu_H \rho(\alpha)^2 + \mu_L} dL_1(\alpha|\overline{x}) > V_L > (1-\gamma)\frac{\mu_H V_H \rho(\underline{x})^2 + \mu_L V_L}{\mu_H \rho(\underline{x})^2 + \mu_L} \tag{113}$$

$$\Leftrightarrow \ \underline{\gamma}^F < \gamma < \overline{\gamma}^F. \tag{114}$$

The proposition summarizes the equilibrium characterization. **Q.E.D.**

Proof for Proposition 3
The expected payoff of a bidder with signal x but pretending to be $\tilde{x}$ is

$$V_H \mu_H(x) F_H(\tilde{x}) - M(\tilde{x}, x). \tag{115}$$

In equilibrium, it is optimal to choose $\tilde{x} = x$ and the resulting first-order condition implies that

$$M_1(x, x) = V_H \mu_H(x) f_H(x). \tag{116}$$

The seller's expected revenue from this buyer is given by

$$R(x) = M(x, x) + V_0 \mu_L(x) F_L(x). \tag{117}$$

Denote

$$\Delta(x) = R^A(x) - R^B(x) = M^B(x, x) - M^B(x, x), \tag{118}$$

and make use of (116). We obtain

$$\begin{aligned} \Delta'(x) &= M_1^A(x, x) - M_1^B(x, x) + M_2^A(x, x) - M_2^B(x, x) && (119) \\ &= V_H \mu_H(x) f_H(x) - V_H \mu_H(x) f_H(x) + M_2^A(x, x) - M_2^B(x, x) && (120) \\ &= M_2^A(x, x) - M_2^B(x, x) \geq 0. && (121) \end{aligned}$$

The inequality above is by assumption. Since $\Delta(\underline{x}) = 0$, we conclude that for all x, $\Delta(x) \geq 0$.
Q.E.D.

Proof for Proposition 4

When $V_L = 0$, in the second-price auctions,

$$M^{S2}(\tilde{x}, x; \gamma) = \mu_H(x) \int_{\underline{x}}^{\tilde{x}} B^{S2}(x_2) dF_H(x_2) + \mu_L(x) \int_{\underline{x}}^{\tilde{x}} \gamma B^{S2}(x_2) dF_L(x_2) \quad (122)$$

Note that $\mu_H(x) + \mu_L(x) = 0$. Then $\mu'_H(x) = -\mu'_L(x)$ and $\mu'_H(x) > 0$. Therefore,

$$\begin{aligned}
M_2^{S2}(x, x; \gamma) &= \mu'_H(x) \int_{\underline{x}}^{x} B^{S2}(x_2) dF_H(x_2) + \mu'_L(x) \int_{\underline{x}}^{x} \gamma B^{S2}(x_2) dF_L(x_2) \quad (123) \\
&= \mu'_H(x) \int_{\underline{x}}^{x} [f_H(x_2) - \gamma f_L(x_2)] B^{S2}(x_2) dx_2 \\
&= \mu'_H(x) \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(x_2)^2 [f_H(x_2) - \gamma f_L(x_2)]}{\mu_H \rho(x_2)^2 + \gamma \mu_L} dx_2 \quad (124)
\end{aligned}$$

$$\frac{\partial M_2^{S2}(x, x; \gamma)}{\partial \gamma} = \mu'_H(x) \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(x_2)^2 [-f_L(x_2) \mu_H \rho(x_2)^2 - \mu_L f_H(x_2)]}{[\mu_H \rho(x_2)^2 + \gamma \mu_L]^2} dx_2 < 0 \quad (125)$$

Thus, from Proposition 3, the seller's revenue is decreasing in γ. **Q.E.D.**

Proof for Proposition 5
When $V_L = 0$, in the first-price auctions,

$$M^{F2}(\tilde{x}, x; \gamma) = \mu_H(x) B^{F2}(\tilde{x}) F_H(\tilde{x}) + \mu_L(x) \gamma B^{F2}(\tilde{x}) F_L(\tilde{x}). \quad (126)$$

$$\begin{aligned}
M_2^{F2}(x, x; \gamma) &= \mu'_H(x) B^{F2}(x) F_H(x) + \mu'_L(x) \gamma B^{F2}(x) F_L(x) \quad (127) \\
&= \mu'_H(x) \int_{\underline{x}}^{x} \underbrace{\frac{\mu_H V_H \rho(\alpha)^2 [F_H(x) - \gamma F_L(x)]}{\mu_H \rho(\alpha)^2 + \gamma \mu_L}}_{\text{denote as } K(\alpha, x, \gamma)} dL_2(\alpha|x; \gamma) \quad (128)
\end{aligned}$$

It is obvious that $K(\alpha, x, \gamma)$ is decreasing in γ.

$$\begin{aligned}
\frac{\partial L_2(\alpha|x; \gamma)}{\partial \gamma} &= -L_2(\alpha|x; \gamma) \int_{\alpha}^{x} \frac{\mu_L f_L(s)^2 \mu_H f_H(s) F_H(s) - \mu_H f_H(s)^2 \mu_L f_L(s) F_L(s)}{[\mu_H f_H(s) F_H(s) + \gamma \mu_L f_L(s) F_L(s)]^2} ds \\
&= -L_2(\alpha|x; \gamma) \int_{\alpha}^{x} \frac{\mu_L \mu_H f_L(s)^2 f_H(s)^2 [\frac{F_H(s)}{f_H(s)} - \frac{F_L(s)}{f_L(s)}]}{[\mu_H f_H(s) F_H(s) + \gamma \mu_L f_L(s) F_L(s)]^2} ds \geq 0. \quad (129)
\end{aligned}$$

From Lemma 2, we know that $L_2(\alpha|x; \gamma)$ is a distribution. This means that, for $\gamma_1 \leq \gamma_2$, $L_2(\alpha|x; \gamma_1)$ first order stochastic dominates $L_2(\alpha|x; \gamma_2)$. Define $\gamma^* = \frac{F_H(x)}{F_L(x)}$. For $\gamma_1 \leq \gamma_2 \leq \gamma^*$,

$$M_2^{F2}(x, x; \gamma_1) = \mu'_H(x) \int_{\underline{x}}^{x} K(\alpha, x, \gamma_1) dL_2(\alpha|x; \gamma_1)$$

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$$\geq \ \mu'_H(x) \int_{\underline{x}}^{x} K(\alpha, x, \gamma_2) dL_2(\alpha|x; \gamma_1) \tag{130}$$

$$\geq \ \mu'_H(x) \int_{\underline{x}}^{x} K(\alpha, x, \gamma_2) dL_2(\alpha|x; \gamma_2) \tag{131}$$

$$= \ M_2^{F2}(x, x; \gamma_2). \tag{132}$$

The first inequality holds because $K(\alpha, x, \gamma)$ is decreasing in γ. The second inequality holds because $K(\alpha, x, \gamma)$ is increasing in α and $L_2(\alpha|x, \gamma_1)$ first order stochastic dominates $L_2(\alpha|x, \gamma_2)$. Thus $M_2^{F2}(x, x; \gamma))$ is decreasing in γ for $\gamma \leq \gamma^*$.

For $\gamma \geq \gamma^*$

$$M_2^{F2}(x, x; \gamma) \tag{133}$$
$$= \ \mu'_H(x) \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(\alpha)^2 [F_H(x) - \gamma F_L(x)]}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} \frac{\mu_H f_H(\alpha)^2 + \gamma \mu_L f_L(\alpha)^2}{\mu_H f_H(\alpha) F_H(\alpha) + \gamma \mu_L f_L(\alpha) F_L(\alpha)} L_2(\alpha|x; \gamma) d\alpha$$

$$= \ \mu'_H(x) \int_{\underline{x}}^{x} \underbrace{\frac{\mu_H V_H f_H(\alpha)^2 [F_H(x) - \gamma F_L(x)]}{\mu_H f_H(\alpha) F_H(\alpha) + \gamma \mu_L f_L(\alpha) F_L(\alpha)}}_{\text{denote as } Q(\alpha, x, \gamma)} L_2(\alpha|x; \gamma) d\alpha. \tag{134}$$

$$\frac{\partial M_2^{F2}(x, x; \gamma)}{\partial \gamma} = \mu'_H(x) \int_{\underline{x}}^{x} \left[\underbrace{\frac{\partial Q(\alpha, x, \gamma)}{\partial \gamma}}_{\leq 0} \underbrace{L_2(\alpha|x, \gamma)}_{\geq 0} + \underbrace{Q(\alpha, x, \gamma)}_{\leq 0} \underbrace{\frac{\partial L_2(\alpha|x; \gamma)}{\partial \gamma}}_{\geq 0} \right] d\alpha \leq 0. \tag{135}$$

Hence, $M_2^{F2}(x, x; \gamma)$ is decreasing in γ for $\gamma \geq \gamma^*$. As a result, $M_2^{F2}(x, x; \gamma)$ is decreasing in γ all the time. Therefore, from Proposition 3, the seller's revenue is decreasing in γ. **Q.E.D.**

Proof for Proposition 6
From equation (124) and equation (128),

$$M_2^{S2}(x, x; \gamma) - M_2^{F2}(x, x; \gamma) \tag{136}$$

$$= \ \mu'_H(x) \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(x_2)^2 [f_H(x_2) - \gamma f_L(x_2)]}{\mu_H \rho(x_2)^2 + \gamma \mu_L} dx_2 \tag{137}$$

$$- \mu'_H(x) \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(\alpha)^2 [F_H(x) - \gamma F_L(x)]}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|x; \gamma) \tag{138}$$

$$= \ \mu'_H(x) [F_H(x) - \gamma F_L(x)] \Big\{ \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} d \frac{F_H(\alpha) - \gamma F_L(\alpha)}{F_H(x) - \gamma F_L(x)} \tag{139}$$

$$- \int_{\underline{x}}^{x} \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|x; \gamma) \Big\} \tag{140}$$

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$$= \mu'_H(x)[F_H(x) - \gamma F_L(x)]\left\{ \int_{\underline{x}}^x \underbrace{\left[L_2(\alpha|x;\gamma) - \frac{F_H(\alpha) - \gamma F_L(\alpha)}{F_H(x) - \gamma F_L(x)} \right]}_{\text{denote as } P(\alpha,x,\gamma)} d\frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} \right\}$$

The last equality makes use of integration by parts. Since $L_2(\alpha|x;\gamma)$ is increasing in γ and $\frac{F_H(\alpha) - \gamma F_L(\alpha)}{F_H(x) - \gamma F_L(x)}$ is decreasing in γ, $P(\alpha, x, \gamma)$ is increasing in γ. Since $P(\alpha, x, 0) = 0$, $P(\alpha, x, \gamma)$ is always positive. $\frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L}$ is increasing in γ. Therefore, the integration is always positive. As a result, sign$[M_2^{S2}(x, x; \gamma) - M_2^{F2}(x, x, \gamma)]$ =sign$[F_H(x) - \gamma F_L(x)]$, which could be either positive or negative. Hence, the linkage principle cannot be applied here to rank the revenues. However, we can make a direct comparison. From equations (122) and (126), we obtain

$$M^{S2}(x, x; \gamma) = \int_{\underline{x}}^x \frac{\mu_H f_H(x) f_H(x_2) + \gamma \mu_L f_L(x) f_L(x_2)}{\mu_H f_H(x) + \mu_L f_L(x)} \frac{\mu_H V_H \rho(x_2)^2}{\mu_H \rho(x_2)^2 + \gamma \mu_L} dx_2, \qquad (141)$$

$$M^{F2}(x, x; \gamma) = \frac{\mu_H f_H(x) F_H(x) + \gamma \mu_L f_L(x) F_L(x)}{\mu_H f_H(x) + \mu_L f_L(x)} \int_{\underline{x}}^x \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|x; \gamma). \quad (142)$$

Thus,

$$M^{S2}(x, x; \gamma) - M^{F2}(x, x; \gamma) \qquad (143)$$

$$= \int_{\underline{x}}^x \frac{\mu_H f_H(x) f_H(\alpha) + \gamma \mu_L f_L(x) f_L(\alpha)}{\mu_H f_H(x) + \mu_L f_L(x)} \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} d\alpha \qquad (144)$$

$$- \frac{\mu_H f_H(x) F_H(x) + \gamma \mu_L f_L(x) F_L(x)}{\mu_H f_H(x) + \mu_L f_L(x)} \int_{\underline{x}}^x \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|x; \gamma) \qquad (145)$$

$$= \frac{\mu_H f_H(x) F_H(x) + \gamma \mu_L f_L(x) F_L(x)}{\mu_H f_H(x) + \mu_L f_L(x)} \times \qquad (146)$$

$$\times \left\{ \int_{\underline{x}}^x \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} d\underbrace{\left[\frac{\mu_H f_H(x) F_H(\alpha) + \gamma \mu_L f_L(x) F_L(\alpha)}{\mu_H f_H(x) F_H(x) + \gamma \mu_L f_L(x) F_L(x)} \right]}_{\text{denote as } W(\alpha,x,\gamma)} \qquad (147)$$

$$- \int_{\underline{x}}^x \frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma \mu_L} dL_2(\alpha|x; \gamma) \right\}. \qquad (148)$$

It is straightforward to show that $W(\alpha, x, \gamma)$ can be regarded as a distribution for α on $[\underline{x}, x]$. Furthermore, since $\frac{\mu_H f_H(x) f_H(s) + \gamma \mu_L f_L(x) f_L(s)}{\mu_H f_H(x) F_H(s) + \gamma \mu_L f_L(x) F_L(s)}$ is increasing in x from (55), we have

$$L_2(\alpha|x; \gamma) = e^{-\int_\alpha^x \frac{\mu_H f_H(s)^2 + \gamma \mu_L f_L(s)^2}{\mu_H f_H(s) F_H(s) + \gamma \mu_L f_L(s) F_L(s)} ds} \qquad (149)$$

$$\geq e^{-\int_\alpha^x \frac{\mu_H f_H(x) f_H(s) + \gamma \mu_L f_H(x) f_L(s)}{\mu_H f_H(x) F_H(s) + \gamma \mu_L f_L(x) F_L(s)} ds} \qquad (150)$$

$$= e^{-\int_\alpha^x \frac{d\ln[\mu_H f_H(x) F_H(s) + \gamma \mu_L f_L(x) F_L(s)]}{ds} ds} \qquad (151)$$

$$= e^{ln[\mu_H f_H(x)F_H(\alpha)+\gamma\mu_L f_L(x)F_L(\alpha)]-ln[\mu_H f_H(x)F_H(x)+\gamma\mu_L f_L(x)F_L(x)]} \tag{152}$$

$$= \frac{\mu_H f_H(x)F_H(\alpha) + \gamma\mu_L f_L(x)F_L(\alpha)}{\mu_H f_H(x)F_H(x) + \gamma\mu_L f_L(x)F_L(x)} \tag{153}$$

$$= W(\alpha, x, \gamma). \tag{154}$$

Thus, $W(\alpha, x, \gamma)$ first order stochastic dominates $L_2(\alpha|x; \gamma)$. Since $\frac{\mu_H V_H \rho(\alpha)^2}{\mu_H \rho(\alpha)^2 + \gamma\mu_L}$ is an increasing function of α, we can conclude that $M^{S2}(x, x; \gamma) \geq M^{F2}(x, x; \gamma)$. **Q.E.D.**

Proof for Proposition 7

Since the linkage principle cannot be applied for revenue ranking, we will do a direct comparison. Below, we examine the seller's revenue in the second-price auctions in three cases. Let $R^S(\gamma)$ denote the seller's expected revenue as a function of γ.

Case 1: $\gamma \geq \overline{\gamma}^S$.

In this case, the seller's revenue does not depend on γ:

$$\frac{1}{2}R^S(\gamma) \tag{155}$$

$$= \mu_H \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} B^{S1}(x_2)f_H(x_1)f_H(x_2)dx_1dx_2 + \mu_L \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} B^{S1}(x_2)f_L(x_1)f_L(x_2)dx_1dx_2$$

$$= \int_{\underline{x}}^{\overline{x}} \frac{\mu_H V_H \rho(x_2)^2 + \mu_L V_L}{\mu_H \rho(x_2)^2 + \mu_L}[\mu_H(1 - F_H(x_2))f_H(x_2) + \mu_L(1 - F_L(x_2))f_L(x_2)]dx_2. \tag{156}$$

Case 2: $\gamma \leq \underline{\gamma}^S$.

$$\frac{1}{2}R^S(\gamma) \tag{157}$$

$$= \mu_H \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} B^{S2}(x_2)f_H(x_1)f_H(x_2)dx_1dx_2 + \mu_L \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} \gamma B^{S2}(x_2)f_L(x_1)f_L(x_2)dx_1dx_2$$

$$+\mu_L \int_{\underline{x}}^{\overline{x}} \int_{x_2}^{\overline{x}} V_0 f_L(x_1)f_L(x_2)dx_1dx_2 \tag{158}$$

$$= \mu_H \int_{\underline{x}}^{\overline{x}} B^{S2}(x_2)[1 - F_H(x_2)]f_H(x_2)dx_2 + \mu_L \int_{\underline{x}}^{\overline{x}} \gamma B^{S2}(x_2)[1 - F_L(x_2)]f_L(x_2)dx_2 \tag{159}$$

$$+\frac{\mu_L V_0}{2}. \tag{160}$$

Therefore,

$$\frac{1}{2}(R^S)'(\gamma) \tag{161}$$

$$= \mu_H \int_{\underline{x}}^{\overline{x}} \frac{\partial B^{S2}(x_2)}{\partial \gamma}[1 - F_H(x_2)]f_H(x_2)dx_2 + \mu_L \int_{\underline{x}}^{\overline{x}} \frac{\partial[\gamma B^{S2}(x_2)]}{\partial \gamma}[1 - F_L(x_2)]f_L(x_2)dx_2$$

$$
= \int_{\underline{x}}^{\overline{x}} \Big\{ \frac{\mu_H V_H \rho(x_2)^2 \mu_L}{-[\mu_H \rho(x_2)^2 + \gamma \mu_L]^2} \mu_H [1 - F_H(x_2)] f_H(x_2) \tag{162}
$$

$$
+ \frac{\mu_H V_H \rho(x_2)^2 \mu_H \rho(x_2)^2}{[\mu_H \rho(x_2)^2 + \gamma \mu_L]^2} \mu_L [1 - F_L(x_2)] f_L(x_2) \Big\} dx_2 \tag{163}
$$

$$
= \int_{\underline{x}}^{\overline{x}} \frac{\mu_H V_H \rho(x_2)^2 \mu_H \mu_L}{[\mu_H \rho(x_2)^2 + \gamma \mu_L]^2} \Big\{ \rho(x_2)^2 [1 - F_L(x_2)] f_L(x_2) - [1 - F_H(x_2)] f_H(x_2) \Big\} dx_2 \tag{164}
$$

$$
= \int_{\underline{x}}^{\overline{x}} \frac{\mu_H^2 \mu_L V_H \rho(x_2)^2 f_H(x_2)^2}{[\mu_H \rho(x_2)^2 + \gamma \mu_L]^2} \Big\{ \frac{1 - F_L(x_2)}{f_L(x_2)} - \frac{1 - F_H(x_2)}{f_H(x_2)} \Big\} dx_2 \leq 0 \tag{165}
$$

The inequality follows the first part of Lemma 1. Thus, $R'(\gamma) \leq 0$; the seller's revenue is decreasing in γ.

Case 3: $\underline{\gamma}^S < \gamma < \overline{\gamma}^S$.

In this case,

$$
\frac{1}{2} R^S(\gamma) \tag{166}
$$

$$
= \mu_H \Big\{ \int_{\underline{x}}^{x^{S*}} B^{S1}(x_2)[1 - F_H(x_2)] f_H(x_2) dx_2 + \int_{x^{S*}}^{\overline{x}} B^{S2}(x_2)[1 - F_H(x_2)] f_H(x_2) dx_2 \Big\}
$$

$$
+ \mu_L \Big\{ \int_{\underline{x}}^{x^{S*}} B^{S1}(x_2)[1 - F_L(x_2)] f_L(x_2) dx_2 + \int_{x^{S*}}^{\overline{x}} \gamma B^{S2}(x_2)[1 - F_L(x_2)] f_L(x_2) dx_2 \Big\}
$$

$$
+ \mu_L \int_{x^{S*}}^{\overline{x}} V_0 [1 - F_L(x_2)] f_L(x_2) dx_2, \tag{167}
$$

Note that x^{S*} is also a function of γ.

$$
\frac{1}{2} (R^S)'(\gamma)
$$

$$
= \mu_H \underbrace{[B^{S1}(x^{S*}) - B^{S2}(x^{S*})]}_{=0} [1 - F_H(x^{S*})] f_H(x^{S*}) \frac{dx^{S*}}{d\gamma}
$$

$$
+ \mu_H \int_{x^{S*}}^{\overline{x}} \frac{\partial B^{S2}(x_2)}{\partial \gamma} [1 - F_H(x_2)] f_H(x_2) dx_2 + \mu_L \int_{x^{S*}}^{\overline{x}} \frac{\partial [\gamma B^{S2}(x_2)]}{\partial \gamma} [1 - F_L(x_2)] f_L(x_2) dx_2
$$

$$
+ \mu_L \underbrace{[B^{S1}(x^{S*}) - \gamma B^{S2}(x^{S*})]}_{=V_L} [1 - F_L(x^{S*})] f_L(x^{S*}) \frac{dx^{S*}}{d\gamma} - \mu_L V_0 [1 - F_L(x^{S*})] f_L(x^{S*}) \frac{dx^{S*}}{d\gamma}
$$

$$
= \mu_H \int_{x^{S*}}^{\overline{x}} \frac{\partial B^{S2}(x_2)}{\partial \gamma} [1 - F_H(x_2)] f_H(x_2) dx_2 + \mu_L \int_{x^{S*}}^{\overline{x}} \frac{\partial [\gamma B^{S2}(x_2)]}{\partial \gamma} [1 - F_L(x_2)] f_L(x_2) dx_2
$$

$$
+ \mu_L (V_L - V_0)[1 - F_L(x^{S*})] f_L(x^{S*}) \frac{1}{\frac{d\gamma}{dx^{S*}}}
$$

$$
= \int_{x^{S*}}^{\overline{x}} \frac{\mu_H^2 \mu_L V_H \rho(x_2)^2 f_H(x_2)^2}{[\mu_H \rho(x_2)^2 + \gamma \mu_L]^2} \Big\{ \frac{1 - F_L(x_2)}{f_L(x_2)} - \frac{1 - F_H(x_2)}{f_H(x_2)} \Big\} dx_2
$$

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$$+\mu_L(V_L - V_0)[1 - F_L(x^{S*})]f_L(x^{S*})\frac{[V_H\mu_H\rho(x^{S*}) + \mu_L V_L]^2}{(V_H - V_L)\mu_H\mu_L V_L}\frac{1}{2\rho(x^{S*})\rho'^{S*})}$$

$$= \underbrace{\int_{x^{S*}}^{\overline{x}}\frac{\mu_H^2\mu_L V_H\rho(x_2)^2 f_H(x_2)^2}{[\mu_H\rho(x_2)^2 + \gamma\mu_L]^2}\left\{\frac{1 - F_L(x_2)}{f_L(x_2)} - \frac{1 - F_H(x_2)}{f_H(x_2)}\right\}dx_2}_{payment\ effect\leq 0} \tag{168}$$

$$+ \underbrace{\frac{(V_L - V_0)[1 - F_L(x^{S*})]f_L(x^{S*})^2[V_H\mu_H\rho(x^{S*}) + \mu_L V_L]^2}{2V_L\mu_H(V_H - V_L)f_H(x^{S*})\rho'(x^{S*})}}_{effciency\ effect\leq 0\ if\ V_L\leq V_0\ and\ >0\ if\ V_L>V_0} \tag{169}$$

Therefore, the seller's revenue is decreasing in γ if $V_L \leq V_0$. It is easy to verify that the revenue function is continuous for the entire domain of γ and thus is decreasing for $\gamma \in [0, 1]$. **Q.E.D.**

References

[1] CAPEN, E. C., CLAPP, R. V., AND CAMPBELL, W. M. Competitive bidding in high-risk situations. *Journal of Petroleum Technology 23* (1971), 641–653.

[2] CREMER, J., AND MCLEAN, R. P. Full extraction of the surplus in bayesian and dominant strategy auctions. *Econometrica 56*, 6 (November 1988), 1247–57.

[3] GOEREE, J. K., AND OFFERMAN, T. Competitive bidding in auctions with private and common values. *Economic Journal 113*, 489 (07 2003), 598–613.

[4] HAILE, P. A. Auctions with private uncertainty and resale opportunities. *Journal of Economic Theory 108*, 1 (January 2003), 72–110.

[5] HAUSCH, D. B. An asymmetric common-value auction model. *RAND Journal of Economics 18*, 4 (Winter 1987), 611–621.

[6] HURWICZ, L. On informationally decentralized systems. In *Decision and Organization ed. by R.Radner and C.McGuire* (1972), Amsterdam: North-Holland, pp. 297–336.

[7] KAGEL, J. H., AND LEVIN, D. The winner's curse and public information in common value auctions. *American Economic Review 76*, 5 (December 1986), 894–920.

[8] LOPOMO, G. The english auction is optimal among simple sequential auctions. *Journal of Economic Theory 82*, 1 (1998), 144 – 166.

[9] LOPOMO, G. Optimality and robustness of the english auction. *Games and Economic Behavior 36*, 2 (August 2001), 219–240.

[10] MASKIN, E. S., AND RILEY, J. G. Optimal auctions with risk averse buyers. *Econometrica 52*, 6 (November 1984), 1473–1518.

[11] MATTHEWS, S. A. Selling to risk averse buyers with unobservable tastes. *Journal of Economic Theory 30*, 2 (August 1983), 370–400.

[12] MCAFEE, R. P., AND RENY, P. J. Correlated information and mechanism design. *Econometrica 60*, 2 (March 1992), 395–421.

[13] MILGROM, P., AND WEBER, R. J. The value of information in a sealed-bid auction. *Journal of Mathematical Economics 10*, 1 (June 1982), 105–114.

[14] MILGROM, P. R., AND WEBER, R. J. A theory of auctions and competitive bidding. *Econometrica 50*, 5 (September 1982), 1089–1122.

[15] MYERSON, G. Optimal auction design. *Mathematics of Operations Research 6* (1981), 58–73.

[16] REECE, D. K. Competitive bidding for offshore petroleum leases. *Bell Journal of Economics 9*, 2 (Autumn 1978), 369–384.

[17] WILSON, R. Competitive bidding with disparate information. *Management Science 15*, 7 (March 1969), 446–448.

[18] WILSON, R. A bidding model of perfect competition. *Review of Economic Studies 44*, 3 (October 1977), 511–18.

[19] WILSON, R. B. Incentive efficiency of double auctions. *Econometrica 53*, 5 (September 1985), 1101–15.

[20] ZHANG, J. Optimal auction design under uncertainties: The role of return policies. *Working Paper, Queen's University* (2010).