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Finite Sample Comparison of Parametric, Semiparametric, and Wavelet Estimators of Fractional Integration*

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Abstract

In this paper we compare through Monte Carlo simulations the finite sample properties of estimators of the fractional differencing parameter, d. This involves frequency domain, time domain, and wavelet based approaches and we consider both parametric and semiparametric estimation methods. The estimators are briefly introduced and compared, and the criteria adopted for measuring finite sample performance are bias and root mean squared error. Most importantly, the simulations reveal that 1) the frequency domain maximum likelihood procedure is superior to the time domain parametric methods, 2) all the estimators are fairly robust to conditionally heteroscedastic errors, 3) the local polynomial Whittle and bias reduced log-periodogram regression estimators are shown to be more robust to short-run dynamics than other semiparametric (frequency domain and wavelet) estimators and in some cases even outperform the time domain parametric methods, and 4) without sufficient trimming of scales the wavelet based estimators are heavily biased.

JEL Classification: C14, C15, C22.

Keywords: Bias, finite sample distribution, fractional integration, maximum likelihood, Monte Carlo simulation, parametric estimation, semiparametric estimation, wavelet.

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1 Introduction

The past two decades have witnessed an increasing interest in fractionally integrated processes as a convenient way of describing the long memory properties of many time series. There is now a broad range of applications in e.g. finance and macroeconomics, see Baillie (1996), Henry & Zaffaroni (2003), or the references below for some examples. Fractionally integrated processes are characterized by a hyperbolically decaying autocorrelation function (contrary to the faster exponential decay which characterizes traditional autoregressive moving average (ARMA) models), thus suggesting distant observations to be highly correlated.

There have been many studies to provide a theoretical motivation for fractional integration and long memory, for instance models based on aggregation have been suggested by Robinson (1978) and Granger (1980), error duration models by Parke (1999), and regime switching models by Diebold & Inoue (2001). In empirical studies, fractional integration and long memory have been found relevant in many areas in macroeconomics and finance. Some examples of applications are Diebold & Rudebusch (1989, 1991) and Sowell (1992*b*) for various GDP measures, Gil-Alana & Robinson (1997) for the extended Nelson-Plosser data set, Hassler & Wolters (1995) and Baillie, Chung & Tieslau (1996) for inflation data, Diebold, Husted & Rush (1991) and Baillie (1996) for real exchange rate data, and Andersen, Bollerslev, Diebold & Ebens (2001) and Andersen, Bollerslev, Diebold & Labys (2001) for financial volatility series. See Baillie (1996) or Henry & Zaffaroni (2003) for a survey.

In this paper we consider several estimation methods for fractionally integrated ARMA models, including parametric, semiparametric, frequency domain, time domain, and wavelet methods. The methods are compared in an extensive Monte Carlo study using several data generating processes with different forms of short-run dynamics including the possibility of errors that exhibit autoregressive conditional heteroskedasticity (ARCH). The criteria we adopt for measuring the finite sample performance of the estimators are bias and root mean squared error (RMSE).

Our results show that among the parametric methods the frequency domain maximum likelihood procedure is superior with respect to both bias and RMSE. However, our results also show that the (sometimes quite severe) bias of the parametric time domain procedures is alleviated when larger sample sizes (e.g. 512) are considered.

Furthermore, according to our results all the methods under consideration are rather robust to the presence of ARCH effects, which are not parameterized, in the sense that the finite sample biases and RMSEs do not increase much compared to the case with white noise errors.

Among the semiparametric (frequency domain and wavelet) methods our results clearly demonstrate the usefulness of the bias reduced log-periodogram regression and local polynomial Whittle estimators of Andrews & Guggenberger (2003) and Andrews & Sun (2004), respectively. In several cases these two methods even outperform the correctly specified time domain parametric methods. Furthermore, when other methods are very heavily biased due to contamination from short-run dynamics, these estimators show a much lower bias at the expense of an increase in their RMSE. The bias reduction is due to their modelling of the logarithm of the spectral density of the short-run component by a polynomial instead of a constant. Finally, without sufficient trimming of scales the wavelet based methods are heavily biased when short-run dynamics is introduced.

Recent surveys on fractional integration and long memory are Robinson (1994, 2003), Baillie (1996), and the book by Beran (1994). However, since none of these really cover all the methods considered in the present study (some of which are very recent), we first briefly describe the fractionally integrated ARMA model and provide an introduction to the estimation methods considered in our Monte Carlo study with emphasis on the more recent methods. We shall not present all the mathematical assumptions underlying each estimation procedure, but rather describe the methods and their applicability in general, and also briefly discuss and compare the asymptotic distributions of the various estimators.

Previously, Monte Carlo studies of fractional integration estimators have also been conducted by Hauser (1997) who considers the early semiparametric methods like the rescaled range statistic, by Cheung & Diebold (1994) and Hauser (1999) who consider parametric maximum likelihood estimators, and by Tse, Ahn & Tieng (2002) who consider wavelet based estimators. However, in our Monte Carlo study we consider all three types of estimators including recently developed methods, and in particular we attempt to cover all estimators typically applied in empirical work and compare them with respect to finite sample bias and RMSE within the same model setup.

The remainder of the paper is organized as follows. In the next section we present the autoregressive fractionally integrated moving average (ARFIMA) model and estimation methods which are divided into groups of parametric, semiparametric, and wavelet based estimators. Section 3 presents the results of the Monte Carlo study in terms of the finite sample biases and RMSEs of the estimators in section 2, and section 4 offers some concluding remarks. Additional tables of simulation results are given in a separate appendix to this paper, which is available from the authors' websites.

2 Estimation of Fractional Integration

In this section we describe the class of autoregressive fractionally integrated moving average (ARFIMA) processes, introduced by Granger & Joyeux (1980) and Hosking (1981), and review the estimators that we consider in the Monte Carlo study and their properties.

A process is labelled an ARFIMA(p, d, q) process if its d'th difference is a stationary and invertible ARMA(p, q) process. Here, d may be any real number such that $-1/2 < d < 1/2$ (to ensure stationarity and invertibility). For a precise statement, y_t is an ARFIMA(p, d, q) if

$$\phi(L)(1-L)^d (y_t - \mu) = \theta(L)\varepsilon_t, \tag{1}$$

where $\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p$ and $\theta(z) = 1 + \theta_1 z + ... + \theta_q z^q$ are lag polynomials of order p and q, respectively, in the lag operator L $(Lx_t = x_{t-1})$ with roots strictly outside the unit circle, ε_t is $iid(0, \sigma^2)$, and $(1-L)^d$ is defined by its binomial expansion

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)} L^j \tag{2}$$

using the gamma function, $\Gamma(\cdot)$.

The parameter d determines the (long) memory of the process. If $d > -1/2$ the process is invertible and possesses a linear (Wold) representation, and if $d < 1/2$ it is covariance stationary. If $d = 0$ the spectral density is bounded at the origin and the process has only weak dependence (short memory). Furthermore, if $d > 0$ the process is said to have long memory since the autocorrelations die out at a hyperbolic rate (and indeed are no longer absolutely summable) in contrast to the much faster exponential rate in the weak dependence case, whereas if $d < 0$ the process is said to be anti-persistent (Mandelbrot (1982)), and has mostly negative autocorrelations. The case $0 \leq d < 1/2$ has proved particularly relevant for many applications in finance and economics, c.f. the references given in the introduction above, as well as hydrology, geology, and many other fields.

The autocorrelation function of the process in (1) satisfies

$$\rho_k \sim c_\rho k^{2d-1},\ 0 < c_\rho < \infty, \quad \text{as } k \to \infty, \tag{3}$$

which decays at a hyperbolic rate, c.f. Granger & Joyeux (1980) and Hosking (1981). The symbol "$\sim$" means that the ratio of the left and right hand sides tends to one in the limit. Equivalently, the behavior of the autocorrelations at large lags can be stated in the frequency domain at small frequencies.

Thus, defining the spectral density function of y_t, $f_y(\lambda)$, as

$$\gamma_k = \int_{-\pi}^{\pi} f_y(\lambda) e^{i\lambda k} d\lambda, \tag{4}$$

where γ_k is the k'th autocovariance of y_t, it can be shown that the spectral density of the ARFIMA(p,d,q) process (1) is given by

$$\begin{aligned} f_y(\lambda) &= \frac{\sigma^2}{2\pi} \left|1 - e^{i\lambda}\right|^{-2d} \frac{\left|\theta\left(e^{i\lambda}\right)\right|^2}{\left|\phi\left(e^{i\lambda}\right)\right|^2} \\ &= \frac{\sigma^2}{2\pi} \left(2 \sin \lambda/2\right)^{-2d} \frac{\left|\theta\left(e^{i\lambda}\right)\right|^2}{\left|\phi\left(e^{i\lambda}\right)\right|^2}. \end{aligned} \tag{5}$$

Now, the approximation (3) can be restated in the frequency domain as (see Granger & Joyeux (1980), Hosking (1981), or Beran (1994, p. 53))

$$f_y(\lambda) \sim g\left|\lambda\right|^{-2d},\ 0 < g < \infty, \quad \text{as } \lambda \to 0. \tag{6}$$

Very general conditions under which (3) and (6) are equivalent are given by Yong (1974) and Zygmund (2002, Chapter V.2). For a thorough exposition of long memory processes and ARFIMA models the reader is referred to e.g. the book by Beran (1994).

In the following subsections we describe several estimation methods for the ARFIMA model (1) that have appeared in the literature. First, we present the parametric methods which are (approximate or exact) likelihood methods in the time domain or frequency domain. Second, we describe the semiparametric log-periodogram regression and local Whittle methods and some of their extensions. Finally, wavelet based estimation methods are considered.

2.1 Parametric Estimators

Four different parametric maximum likelihood estimators (MLEs) are described in the following: The exact time domain MLE, modified profile likelihood estimator, conditional time domain MLE, and frequency domain MLE. The time domain estimators are based on the likelihood function of the ARFIMA(p, d, q) model with or without conditioning on initial observations, and the frequency domain estimator is based on Whittle's approximation to the likelihood function in the frequency domain.

2.1.1 Maximum Likelihood in the Time Domain

The exact Gaussian maximum likelihood objective function for the model (1) is (when $-1/2 < d < 1/2$)

$$L_E\left(d, \phi, \theta, \sigma^2, \mu\right) = -\frac{T}{2}\ln|\Omega| - \frac{1}{2}\left(Y - \mu l\right)' \Omega^{-1}\left(Y - \mu l\right), \tag{7}$$

where $l = (1, ..., 1)'$, $Y = (y_1, ..., y_T)'$, ϕ and θ are the parameters of $\phi(L)$ and $\theta(L)$, μ is the mean of Y, and Ω is the variance matrix of Y, which is a complicated function of d and the remaining parameters of the model. Sowell (1992a) derived an efficient procedure for solving this function in terms of hypergeometric functions. However, an important limitation is that the roots of the autoregressive polynomial cannot be multiple.

Gathering the parameters in the vector $\gamma = \left(d, \phi', \theta', \sigma^2, \mu\right)'$, the exact maximum likelihood (EML) estimator is obtained by maximizing the likelihood function (7) with respect to γ. Sowell (1992a) showed that the EML estimator of d is $\sqrt{T}$-consistent and asymptotically normal, i.e.

$$\sqrt{T}\left(\hat{d}_{EML} - d\right) \to_d N\left(0, \left(\pi^2/6 - C\right)^{-1}\right), \tag{8}$$

where $C = 0$ when $p = q = 0$ and $C > 0$ otherwise. The variance of the EML estimator may be derived as the $(1,1)$'th element of the inverse of the matrix

$$\frac{1}{4\pi}\int_0^{2\pi} \frac{\partial \ln f_y(\lambda)}{\partial \gamma} \frac{\partial \ln f_y(\lambda)}{\partial \gamma'} d\lambda.$$

Although the time and frequency domain (see below) maximum likelihood estimators are asymptotically equivalent, their finite sample properties differ, and a small Monte Carlo study carried out by Sowell (1992b) shows that the time domain estimator has better finite sample properties than the frequency domain estimator when the mean of the process is known. However, Cheung & Diebold (1994) show that the finite sample efficiency of the discrete Whittle frequency domain MLE (see (11) below) relative to time domain EML rises dramatically when the mean is unknown and has to be estimated.

The modified profile likelihood (MPL) estimator is based on a correction of the parameters of interest (here d, ϕ, θ) for second-order effects due to nuisance parameters (here σ^2, μ). Thus, the idea is to reduce the bias by applying a transformation that makes (d, ϕ, θ) orthogonal to (σ^2, μ), see Cox & Reid (1987) and An & Bloomfield (1993). The modified profile log-likelihood function is given as (without constants)

$$L_M(d, \phi, \theta; \hat{\mu}) = -\left(\frac{1}{2} - \frac{1}{T}\right) \ln|R| - \frac{1}{2} \ln\left(l' R^{-1} l\right) - \left(\frac{T-3}{2}\right) \ln\left[T^{-1}(Y - \hat{\mu} l)' R^{-1} (Y - \hat{\mu} l)\right], \tag{9}$$

where $R = \Omega/\sigma^2$ and $\hat{\mu} = \left(l' R^{-1} l\right)^{-1} l' R^{-1} Y$. The asymptotic distribution of the MPL estimator is unchanged compared to the EML estimator on which it is based, and hence it also satisfies (8).

Imposing the initialization $y_t = 0, t \leq 0$, the model (1) is valid for any value of d and is a type II fractional process in the terminology of Marinucci & Robinson (1999). The objective function corresponding to this DGP considered by Chung & Baillie (1993), Beran (1995), Tanaka (1999), and Nielsen (2004) is

$$L_C(d, \phi, \theta, \mu) = -\frac{T}{2} \ln\left[\sum_{t=1}^{T} \left(\frac{\phi(L)}{\theta(L)} (1-L)^d (y_t - \mu)\right)^2\right], \tag{10}$$

and we call the estimator that maximizes (10) the conditional maximum likelihood (CML) estimator. Maximizing L_C is equivalent to minimizing the usual (conditional) sum of squares and hence this estimator is also referred to as the CSS estimator by some authors, e.g. Chung & Baillie (1993) and Beran (1995). The CML estimator has the same asymptotic distribution (8) as the EML estimator for any value of d and is computationally much less demanding.

Note also that the parametric estimators are asymptotically efficient in the classical sense when the model is Gaussian and correctly specified.

2.1.2 Maximum Likelihood in the Frequency Domain

An alternative approximate MLE of the ARFIMA(p, d, q) model follows the idea of Whittle (1951), who noted that for stationary models the covariance matrix Ω can be diagonalized by transforming the model into the frequency domain. Fox & Taqqu (1986) showed that (when

$d \in (-1/2, 1/2)$) the log-likelihood can then be approximated by

$$L_F\left(d, \phi, \theta, \sigma^2\right) = -\sum_{j=1}^{\lfloor T/2 \rfloor} \left[\ln f_y\left(\lambda_j\right) + \frac{I\left(\lambda_j\right)}{f_y\left(\lambda_j\right)} \right], \tag{11}$$

where $\lambda_j = 2\pi j/T$ are the Fourier frequencies, $I(\lambda) = \frac{1}{2\pi T}\left|\sum_{t=1}^{T} y_t e^{it\lambda}\right|^2$ is the periodogram of y_t, $f_y(\lambda)$ is the spectral density of y_t given in (5), and $\lfloor x \rfloor$ denotes the largest integer that is not greater than x. Note that the FML estimator is invariant to the presence of a non-zero mean, i.e. $\mu \neq 0$, since $j = 0$ (the zero-frequency) is left out of the summation in (11).

The approximate frequency domain maximum likelihood (FML) estimator is defined as the maximizer of (11) and was proposed by Fox & Taqqu (1986), who also proposed a continuously integrated version of (11). Dahlhaus (1989) also assumed Gaussianity and considered the exact likelihood function in the frequency domain. The FML estimator has the same asymptotic properties as the EML estimator, i.e. $\sqrt{T}$-consistency and asymptotic normality, and when the process is Gaussian, asymptotic efficiency. Finally, Giraitis & Surgailis (1990) relax the Gaussianity assumption and analyze the Whittle estimate for linear processes, showing that it is $\sqrt{T}$-consistent and asymptotically normal but no longer efficient, while Hosoya (1997) extends the previous analysis to a multivariate framework.

2.2 Semiparametric Estimators

The semiparametric frequency domain estimators are based on the approximation (6) to the spectral density. Two classes of semiparametric estimators have become very popular in empirical work, the log-periodogram regression method suggested by Geweke & Porter-Hudak (1983) and the local Whittle approach suggested by Künsch (1987). In the following we describe these two estimators and some of the many extensions and improvements that have appeared in the literature. Some earlier work on the (adjusted) rescaled range, or "R/S statistic", by Hurst (1951) and Mandelbrot & Wallis (1969) or its modified version to allow for weak dependence by Lo (1991) is not considered here. Instead, the reader is referred to Hauser (1997).

The semiparametric estimators enjoy robustness to short-run dynamics since they use only information from the periodogram ordinates in the vicinity of the origin. Indeed, the short-run dynamics in the model, i.e. the autoregressive and moving average polynomials $\phi(\cdot)$ and $\theta(\cdot)$ in our model (1), does not even have to be specified. The drawback is that only $\sqrt{m}$-consistency is achieved, where $m = m(T)$ is a user-chosen bandwidth parameter, in comparison to $\sqrt{T}$-consistency (and efficiency) in the parametric case. Thus, the semiparametric approach is much less efficient than the parametric one since it requires at least $m/T \to 0$.

2.2.1 Log-Periodogram Regression

Probably the most commonly applied semiparametric estimator is the log-periodogram regression (LPR) estimator introduced by Geweke & Porter-Hudak (1983) and analyzed in detail by Robinson (1995*b*). Taking logs in (6) and inserting sample quantities we get the approximate regression relationship

$$\ln(I(\lambda_j)) = \text{constant} - 2d \ln(\lambda_j) + \text{error}. \tag{12}$$

The LPR estimator is defined as the OLS estimator in the regression (12) using $j = 1, ..., m$, where $m = m(T)$ is a bandwidth number which tends to infinity as $T \to \infty$ but at a slower rate than T. Note that the estimator is invariant to a non-zero mean since $j = 0$ is left out of the regression.

Under suitable regularity conditions, including y_t being Gaussian (later relaxed by Velasco (2000)) and a restriction on the bandwidth, Robinson (1995*b*) derived the asymptotically normal limit distribution for the LPR estimator when d is in the stationary and invertible range $(-1/2, 1/2)$. The proof by Robinson (1995*b*) also employed trimming of the very lowest frequencies as suggested by Künsch (1986), but following recent research, e.g. Hurvich, Deo & Brodsky (1998), and the original suggestion of Geweke & Porter-Hudak (1983) the trimming is not necessary and has been largely ignored in empirical work. We shall follow this practice in our implementation of the estimator. Recently, Kim & Phillips (1999) and Velasco (1999*b*) demonstrated that the range of consistency is $d \in (-1/2, 1]$ and the range of asymptotic normality is $d \in (-1/2, 3/4)$.

To reduce the asymptotic order of the bias, which can be severe in finite samples, see Agiakloglou, Newbold & Wohar (1993), Andrews & Guggenberger (2003) have suggested to replace the constant in (12) by the polynomial $\sum_{r=0}^{R} \xi_r \lambda_j^{2r}$. Thus, the bias is reduced by modelling the logarithm of the spectral density of the short-run dynamics in the vicinity of the origin by a polynomial instead of a constant. We set $R = 1$ in our implementation of the bias reduced log-periodogram regression (BRLPR) estimator.

The limiting distribution of the LPR and BRLPR estimators for $d \in (-1/2, 1/2)$ is given by Robinson (1995*b*) and Andrews & Guggenberger (2003) as

$$\sqrt{m}\left(\hat{d}_R - d\right) \rightarrow_d N\left(0, \frac{\pi^2}{24}c_R\right), \tag{13}$$

where $c_0 = 1$ $(R = 0)$ corresponds to the LPR estimator and $c_1 = 2.25$ $(R = 1)$ corresponds to the BRLPR estimator. For other values of R see Andrews & Guggenberger (2003). Thus, the variance of the BRLPR estimator is increased only by a multiplicative constant, but it achieves a reduction in the asymptotic order of magnitude of the bias.

Another variant of LPR designed to model the short-run component in (12) in a more flexible way is the pooled log-periodogram regression (PLPR) estimator by Shimotsu & Phillips (2002*b*). This procedure allows the short-run component to vary across frequency bands and at the same time utilizes information in the larger frequencies. The pooled estimator utilizes (12) for the bands $B_0, \ldots, B_L$ (LPR uses only B_0) and is given by

$$\widehat{d}_{PLPR} = \frac{\sum_{i=0}^{L} \sum_{\{j:\lambda_j \in B_i\}} \left(Y_{ji} - \overline{Y}_{.i}\right)\left(X_{ji} - \overline{X}_{.i}\right)}{\sum_{i=0}^{L} \sum_{\{j:\lambda_j \in B_i\}} \left(X_{ji} - \overline{X}_{.i}\right)^2}, \tag{14}$$

where

$$\begin{aligned} \overline{Y}_{.i} &= \frac{1}{m} \sum_{\{j:\lambda_j \in B_i\}} Y_{ji} = \frac{1}{m} \sum_{\{j:\lambda_j \in B_i\}} \ln I(\lambda_j), \\ \overline{X}_{.i} &= \frac{1}{m} \sum_{\{j:\lambda_j \in B_i\}} X_{ji} = -\frac{1}{m} \sum_{\{j:\lambda_j \in B_i\}} \ln\left(4\sin^2(\lambda_j/2)\right), \end{aligned}$$

and

$$B_i = \begin{cases} \left\{\lambda_j \left| \kappa_i - \frac{\pi}{2M} < \lambda_j < \kappa_i + \frac{\pi}{2M} \right.\right\}, & \kappa_i = \frac{(2i+1)\pi}{2M}, i = 1, \ldots M-1, \\ \left\{\lambda_j \left| 0 < \lambda_j < \frac{\pi}{M} \right.\right\}, & \kappa_0 = 0, i = 0, \end{cases}$$

are the frequency bands which have width π/M. Thus, M is a parameter that determines the total number of distinct bands, $M = T/(2m)$, and the procedure uses L bands with $L \rightarrow \infty$ and $L/M \rightarrow 0$. Note that the estimator still uses frequencies only in the vicinity of the origin because $mL/T \rightarrow 0$. The easiest way to compute (14) and simultaneously derive inference, is to run the simple least squares model

$$Y_{ji} - \overline{Y}_{.i} = d\left(X_{ji} - \overline{X}_{.i}\right) + \varepsilon_{ji}, \tag{15}$$

i.e. the approach is analogous to the treatment of fixed effects in panel data regression.

When $d \in (-1/2, 1/2)$ the PLPR estimator is asymptotically distributed according to

$$\sqrt{m}\left(\hat{d}_{PLPR} - d\right) \rightarrow_d N\left(0, \frac{\pi^2}{24\left(1+\Xi\right)}\right), \tag{16}$$

where $\Xi > 0$ is a constant, see Shimotsu & Phillips (2002*b*). Thus, the asymptotic variance in (16) is smaller than that of the LPR estimator in (13) at the expense of a potential increase in the asymptotic bias (from using larger frequencies).

2.2.2 Local Whittle Approach

The other class of semiparametric frequency domain estimators we consider follows the local Whittle approach suggested by Künsch (1987). The local Whittle (LW) estimator was analyzed by Robinson (1995*a*) (who called it a Gaussian semiparametric estimator) and is attractive because of its likelihood interpretation, nice asymptotic properties, and very mild assumptions. The LW estimator is defined as the maximizer of the (local Whittle likelihood) function

$$Q\left(g, d\right) = -\frac{1}{m}\sum_{j=1}^{m}\left[\ln\left(g\lambda_j^{-2d}\right) + \frac{I\left(\lambda_j\right)}{g\lambda_j^{-2d}}\right]. \tag{17}$$

One drawback compared to log-periodogram estimation is that numerical optimization is needed. However, the assumptions underlying this estimator are weaker than those of the LPR estimator, and Robinson (1995*a*) showed that when $d \in (-1/2, 1/2)$,

$$\sqrt{m}(\hat{d}_{LW} - d) \rightarrow_d N(0, 1/4). \tag{18}$$

Thus, the asymptotic distribution is extremely simple, facilitating easy asymptotic inference, and in particular the estimator is more efficient than the LPR estimator. The ranges of consistency and asymptotic normality for the LW estimator have been shown by Velasco (1999*a*) and Phillips & Shimotsu (2004) to be the same as those of the LPR estimator.

An exact local Whittle (ELW) estimator has been proposed by Shimotsu & Phillips (2002*a*) which avoids some of the approximations in the derivation of the LW estimator and is valid for any value of d. The ELW estimator replaces the objective function (17) by the function

$$Q_E\left(g, d\right) = -\frac{1}{m}\sum_{j=1}^{m}\left[\ln\left(g\lambda_j^{-2d}\right) + \frac{I_{\Delta^d y}\left(\lambda_j\right)}{g}\right], \tag{19}$$

where $I_{\Delta^d y}(\lambda) = \frac{1}{2\pi T} \left| \sum_{t=1}^{T} \left(\Delta^d y_t\right) e^{it\lambda} \right|^2$ is the periodogram of $\Delta^d y_t$. The ELW estimator satisfies (18) for any value of d and is thus not confined to any particular range of d values, but it is however confined to zero-mean processes. In our implementation we use the feasible ELW (FELW) estimator by Shimotsu (2002) which allows for a non-zero mean.

Andrews & Sun (2004) propose a generalization of the local Whittle estimator in the spirit of the BRLPR estimator. Instead of approximating the spectral density of the short-run component in a shrinking neighborhood of frequency zero by a constant, they approximate its logarithm by a polynomial. This leads to the following likelihood function,

$$Q_R(g,d,\beta) = -\frac{1}{m}\sum_{j=1}^{m}\left[\ln\left(g\lambda_j^{-2d}\exp\left(-\sum\nolimits_{r=1}^{R}\xi_r\lambda_j^{2r}\right)\right) + \frac{I(\lambda_j)}{g\lambda_j^{-2d}\exp\left(-\sum_{r=1}^{R}\xi_r\lambda_j^{2r}\right)}\right]. \tag{20}$$

The maximization of (20) yields the local polynomial Whittle (LPW) estimator of d for $d \in (-1/2, 1/2)$. As shown in Andrews & Sun (2004) this method increases the asymptotic variance of d in (18) by the multiplicative constant c_R (as in the BRLPR estimator (13) above), but simultaneously reduces the order of magnitude of the asymptotic bias. As with the BRLPR estimator we use $R = 1$ in our implementation of the LPW estimator.

For both the log-periodogram regression method and the local Whittle approach we are left with a choice of bandwidth parameter, m. Results on optimal (mean squared error minimizing) choice of bandwidth for the log-periodogram regression have been derived by Hurvich et al. (1998) and results for the local Whittle approach have been derived by Henry & Robinson (1996). In both cases the optimal bandwidth is found to be a multiple of $T^{0.8}$, where the multiplicative constant depends on the smoothness of the spectral density near the origin, i.e. on the short-run dynamics of the process. In particular, Hurvich et al. (1998) argued that performance gains can be obtained by considering larger bandwidths than the $\sqrt{T}$ originally suggested by Geweke & Porter-Hudak (1983). However, generally the optimal bandwidths have not been applied much in practice so we use two different (arbitrarily chosen) bandwidths, $m = \lfloor T^{0.5} \rfloor$ and $m = \lfloor T^{0.65} \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x, in our implementation below.

2.3 Wavelet Estimators

An orthogonal wavelet is defined as any function $\psi(t)$, whose collection of dilations (scales), j, and translations, k,

$$\psi_{j,k}(t) \equiv 2^{-j/2}\psi\left(2^{-j}t - k\right), \quad j,k \in \mathbb{Z} = \{0, \pm1, \pm2, \ldots\}, \tag{21}$$

form an orthonormal basis of $\mathbb{L}^2$, the space of all square integrable functions on the extended real line. Any continuous function which decreases rapidly to zero as $t \to \pm\infty$ and oscillates $\left(\int \psi(t)dt = 0\right)$ qualifies as a wavelet.

A function $y_t \in \mathbb{L}^2$ with $t = 0, 1, \ldots, 2^p - 1$, where $p \in \mathbb{Z}$ can be expanded into a wavelet series,

$$y_t = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} w_{j,k}\psi_{j,k}(t)\, dt, \tag{22}$$

with coefficients

$$w_{j,k} = 2^{j/2} \int y(t)\, \psi_{j,k}(t)dt. \tag{23}$$

By design the wavelets strength rests in its ability to simultaneously localize a process in time and scale. At high scales, the wavelet has a small centralized time support enabling it to focus in on short lived time phenomena like a singularity point. At low scales, the wavelet has a large time support allowing it to identify long periodic behavior. By moving from low to high scales, the wavelet zooms in on the behavior of a process at a particular point in time, identifying singularities, jumps, and cusps. Alternatively, the wavelet can zoom out to reveal the long, smooth features of a series. In our implementation we use the Haar and Daubechies (1988) wavelets which are most commonly applied in the literature, e.g. the references cited below.

2.3.1 Wavelet OLS Estimator

Using the logarithmic decay of the autocovariance function of a long memory process, Jensen (1999) showed that a log-linear relationship (suggested by McCoy & Walden (1996) and Johnstone & Silverman (1997)) exists between the variance of the wavelet coefficient from the long memory process and its scale, which can be used to estimate d by least squares regression. Leaving out high level wavelet coefficients results in robustness to the short-run dynamics similar to the LPR estimator above, see McCoy & Walden (1996) and Tse et al. (2002).

In particular, Jensen (1999) shows that for $d \in (-1/2, 1/2)$,

$$w_{j,k} \to_d N\left(0, \sigma^2 2^{-2jd}\right) \quad \text{as } j \to 0, \tag{24}$$

when y_t is a fractionally integrated noise process, i.e. when $p = q = 0$. If we define the variance of $w_{j,k}$ as $R(j)$ the intuitive log-linear relationship

$$\ln R(j) = \ln \sigma^2 - d \ln 2^{2j} \tag{25}$$

arises. To estimate d through (25), an estimate of the variance is required. Jensen (1999) proposes

$$\widehat{R}(j) = 2^{-j} \sum_{k=0}^{2^j - 1} w_{j,k}^2, \quad j = 0, ..., p-1, \tag{26}$$

and the relationship (25) thus gives rise to the regression

$$\ln \hat{R}(j) = \text{constant} - d \ln 2^{2j} + \text{error}, \quad j = J, ..., p-1-K, \tag{27}$$

which can be estimated by ordinary least squares yielding the wavelet OLS (WOLS) estimator. The WOLS estimator is consistent and asymptotically normal when $d \in (-1/2, 1/2)$, see Jensen (1999). The trimming of the lowest J scales was suggested by Jensen (1999) to avoid boundary effects, and the trimming of the highest K scales was suggested by McCoy & Walden (1996) and Tse et al. (2002) (for the wavelet MLE, see below) since (24) is valid for small j only.

2.3.2 Maximum Likelihood in Scale and Space (Wavelet MLE)

An alternative to the (approximate) ML estimators described above is to use an approximate Wavelet ML (WML) estimator. Following the arguments of McCoy & Walden (1996) and Johnstone & Silverman (1997), see also Jensen (1998, 2000), we assume that (24) is satisfied, where σ^2 depends on other parameters of the model but does not vary with j.

It follows that, ignoring wavelet coefficients $j > p-1-K$, the approximate wavelet likelihood function is given by

$$L_W(d, \sigma^2) = -\frac{1}{2} \sum_{j=0}^{p-1-K} \left[\left(2^j - 1\right) \ln \left(\sigma^2 2^{-2jd}\right) + \sum_{k=0}^{2^j - 1} \frac{w_{j,k}^2}{\sigma^2 2^{-2jd}} \right] \tag{28}$$

and the WML estimator is obtained by maximizing L_W. Since (24) is only valid for small j, we follow McCoy & Walden (1996) and Tse et al. (2002) and leave out the K largest scales in the likelihood function (28) to achieve robustness to the possible presence of short-run dynamics in the same sense as the semiparametric frequency domain estimators.

3 Finite Sample Comparison

In this section we investigate the finite sample bias and root mean squared error (RMSE) of the estimation methods outlined in section 2 above. The objective of this exercise is to shed light on which estimator is most accurate in practical application with realistic sample sizes. In the next subsections we first present the Monte Carlo setup and subsequently the results.

3.1 Monte Carlo Setup

For each Monte Carlo DGP we generated $1,000$ artificial time series with 128, 256, and 512 observations by premultiplying a vector of *i.i.d.* standard normal variates by the Choleski decomposition of the autocovariance matrix of the desired process, i.e. the stationary type I fractionally integrated process in the terminology of Marinucci & Robinson (1999), see also Beran (1994, pp. 215-217). The simulations were made using Gauss v3.6 and Ox v3.3 with the Arfima package, see Doornik (2001) and Doornik & Ooms (2001). The sample sizes were chosen as powers of two in order to avoid contaminating the results with biases introduced by the effects of padding used in Fourier and wavelet transforms when the sample size is not a power of two. Furthermore, they were chosen to reflect realistic empirical samples from macroeconomic or financial data, see the examples of empirical references given in the introduction. Although financial samples based on high frequency data sets may some times be many times larger than the sample sizes considered here, most often empirical analyses are based on some aggregated measures such as monthly realized volatility/variance in which case the sample sizes considered here are very relevant.

We consider four different data generating processes (DGPs) in our Monte Carlo study. The first one is the simple ARFIMA(0,d,0) model,

$$(1-L)^d (y_t - \mu) = \varepsilon_t, \quad \varepsilon_t \sim i.i.d.N(0,\sigma^2), \tag{29}$$

where the parameter values $\mu = 0$ and $\sigma^2 = 1$ are chosen for the simulations (note that these values are not enforced in the estimation, i.e. even though $\mu = 0$, the parameter is still estimated in the parametric time domain models). For the parameter of interest, d, we consider the values $\{-0.25, 0, 0.25, 0.4\}$. Here, the case $d = 0$ corresponds to estimating d when in fact data is not (fractionally) integrated.

The next two models we consider are the ARFIMA(1,d,0) and ARFIMA(0,d,1) models given by

$$\begin{aligned} (1-\phi L)(1-L)^d (y_t - \mu) &= \varepsilon_t, \quad \varepsilon_t \sim i.i.d.N(0,\sigma^2), & (30) \\ (1-L)^d (y_t - \mu) &= (1+\theta L)\varepsilon_t, \quad \varepsilon_t \sim i.i.d.N(0,\sigma^2), & (31) \end{aligned}$$

where again $\mu = 0$ and $\sigma^2 = 1$. For ϕ and θ we use the values $\{-0.4, 0, 0.4, 0.8\}$, where $\phi = 0$ and $\theta = 0$ correspond to the cases where an autoregressive or moving average term is estimated even though it is not present in the data. For the fractional integration parameter d, we choose the same values as in the simpler model (29).

Thus, the two DGPs (30) and (31) are more complicated than (29), introducing short-run dynamics into the model. It is important to note that for the parametric estimation procedures, (29) is very different from (30) with $\phi = 0$ and from (31) with $\theta = 0$. The DGPs are of course the same, but in the former case it is assumed known that $\phi = \theta = 0$ whereas in the latter two cases ϕ or θ is estimated. Obviously, estimating ϕ or θ when it is not present (i.e. overfitting the model) may introduce a finite sample bias into the estimate of the parameter of interest, d. Thus, for the parametric models the cases with $\phi = 0$ or $\theta = 0$ correspond to a weak form of misspecification where the model is overspecified and irrelevant short-run dynamics is estimated.

On the other hand, for the semiparametric and wavelet estimation procedures the short-run dynamics is not specified. That is, there is no need to specify whether or not ϕ and θ are estimated and thus the DGP (29) and the DGPs (30) and (31) with $\phi = \theta = 0$ will yield the same results. Hence, for the semiparametric and wavelet methods we do not report the results for (30) with $\phi = 0$ and (31) with $\theta = 0$.

Finally, we consider the ARFIMA(0,d,0)-ARCH(1) model of, e.g., Baillie et al. (1996) and Ling & Li (1997),

$$(1-L)^d (y_t - \mu) = u_t, \quad u_t = h_t^{1/2}\varepsilon_t, \quad h_t = \alpha + \beta u_{t-1}^2, \quad \varepsilon_t \sim i.i.d.N(0,1), \tag{32}$$

where $\mu = 0$ as before. For the conditional variance parameters we consider the values $\{0.4, 0.8\}$ for β, and the values for α are chosen such that the unconditional variance is unity (i.e.

$\alpha = 1 - \beta$) to match model (29). For the fractional integration parameter d, we choose the same values as in the simpler model (29).

Unlike the models (30) and (31), the model in (32) is not completely parameterized by our parametric methods. It thus corresponds to a weak form of model misspecification where the ARCH part of the model is left unspecified and white noise errors are assumed for the estimation. However, consistency of the parametric methods in section 2.1 relies only on the errors being martingale differences and thus even though the ARCH part of the model is misspecified they are still consistent, although probably inefficient compared to a fully parameterized method that takes the conditional heteroskedasticity into account. In effect, the DGP (32) extends the simpler DGP (29) by introducing errors that have conditional heteroskedasticity and hence fat tails, thereby relaxing one of the more restrictive assumptions of the previous DGPs.

In Tables 1-21 the results of our Monte Carlo study are presented. Tables 1-7 display the results for the simple DGP (29), and Tables 8-14 and 15-21 display the results for the more complicated DGPs (30) and (31), respectively. Finally, the results for the DGP (32) in which the errors exhibit ARCH are in fact very similar to those in Tables 1-7 for the ARFIMA(0,d,0) model. Hence, to conserve space, the tables with the results for the ARFIMA(0,d,0)-ARCH(1) DGP (32) are presented in a separate appendix, which is available from the authors' websites.

For each DGP, the first table (i.e. Tables 1, 8, and 15) presents the results for the parametric methods of section 2.1. The next three tables (i.e. Tables 2-4, 9-11, and 16-18) present the results for the semiparametric approaches of section 2.2, and the last three tables for each DGP (i.e. Tables 5-7, 12-14, and 19-21) present the results for the wavelet methods of section 2.3. To present the results of the tables in the most comprehensible way, we have marked in bold font the cases with the lowest biases and the cases with the lowest RMSEs across each class of estimator (parametric, semiparametric, and wavelet) and for each DGP and parameter value.

3.2 Monte Carlo Results for Parametric Estimators

Consider first the Monte Carlo results for the parametric methods. Recall that these estimation methods use all available information, both in terms of utilizing all observations but also in terms of parameterizing the true DGP of the series at hand (except for the cases with $\phi = 0$, $\theta = 0$, or with ARCH). Thus, it is interesting to see how well these methods perform compared to the semiparametric and wavelet methods when handling the contamination caused by the presence of an AR or MA parameter as the latter estimation methods do not parameterize the short-run dynamics nor do they use all available observations.

Furthermore, we expect the parametric time domain estimators to be systematically negatively biased compared to the parametric frequency domain estimator, FML. This is caused by the fact that the methods differ in the treatment of the mean, i.e. of the frequency zero in the periodogram. While this frequency is excluded from the FML estimator, it is implicitly included in the time domain estimators through the autocovariance function. As we consider zero-mean processes in the Monte Carlo study, the periodogram is zero at frequency zero, however, for $d > 0$ the spectral density approaches infinity as the frequency approaches zero. Thus, the time domain estimators will try to model the upward slope of the true spectral density (and thus of the periodogram) for low frequencies, but at the same time have to take into account estimating the mean which is at frequency zero. Consequently, we expect these estimators to suffer from a negative bias, see also Cheung & Diebold (1994) and Hauser (1999).

Turning to the results in Table 1 we find, as expected, that the time domain estimators generally exhibit a negative bias, which becomes more pronounced when adding short-run noise in Tables 8 and 15. This downward bias is especially high when the AR coefficient is 0 or .4, but the estimation methods seem fairly robust towards positive MA noise and curiously also towards strong, positive AR noise (i.e. $\phi = .8$). The phenomenon that autoregressive coefficients of moderate size are most troublesome for the parametric estimation methods has previously been noted from a theoretical viewpoint by Nielsen (2004, p. 131). Thus, the time domain estimators are very sensitive to the inclusion of short-run dynamics. Among the time domain estimators we generally find the CML estimator to possess the lowest bias. Furthermore, for relatively small sample sizes, i.e. for $T \leq 256$, Table 8 shows that the time domain estimators suffer from a rather severe negative bias (of the order -.05 to -.20) when mistaking the true DGP of the series at hand to be an ARFIMA$(1, d, 0)$ when it is actually an ARFIMA$(0, d, 0)$. Fortunately, the bias is not as severe for ARFIMA$(0, d, 1)$ processes, see Table 15.

The results in the separate appendix, which illustrate the perhaps more empirically realistic ARFIMA$(0, d, 0)$-ARCH(1) scenario (32) where the errors are conditionally heteroskedastic, show that the biases for the parametric estimators are only slightly more negative compared to the case of white noise errors. However, as the ARCH effect increases (β increases) the estimators generally become a little more biased with slightly higher RMSEs. Hence, the parametric estimators seem robust towards ARCH innovations which is in accordance with the theory where the innovations need only be martingale differences for the estimators to be consistent.

Compared to the time domain estimators, the frequency domain estimator, FML, is vastly superior with respect to bias. It does not suffer from any of the above mentioned problems, i.e. it is robust towards both AR and MA noise, ARCH innovations, and it does not possess noticeable bias when one wrongfully overfits the true DGP. In addition to the lower bias, the FML estimator also obtains an improvement in the RMSE, especially in the ARFIMA$(0, d, 0)$ and ARFIMA$(1, d, 0)$ cases (except with $\phi = 0.8$).

Thus, the FML estimator is superior with respect to both bias and RMSE compared to parametric time domain estimators.

3.3 Monte Carlo Results for Semiparametric Estimators

We next turn to the results for the semiparametric methods described above in section 2.2.

Contrary to the parametric methods the semiparametric methods utilize only frequencies in a shrinking neighborhood of frequency zero. The number of frequencies used is governed by the bandwidth m, and in this Monte Carlo study we focus on $m = \lfloor T^{0.5} \rfloor$ and $m = \lfloor T^{0.65} \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. When no short-run dynamics is present in the data it should be preferable to use the larger bandwidth, but except for the bias correction (local polynomial) methods the opposite would typically be the case when short-run dynamics is present.

In the ARFIMA$(0, d, 0)$ and ARFIMA$(0, d, 0)$-ARCH(1) cases the biases are generally very low as evident from Tables 2-4 and the corresponding tables in the separate appendix. I.e., the estimators seem almost unbiased in the case of ARCH innovations indicating that the theoretical robustness towards such innovations carries over to practice. This is also supported by the fact that the biases are independent of the size of β (the ARCH parameter).

Comparing the LW estimator with the modifications by Shimotsu & Phillips (2002a) and Shimotsu (2002) (FELW) and Andrews & Sun (2004) (LPW), we find the accuracy of the FELW estimator not to be noticeably different neither in bias nor in RMSE, see Table 3. However, this does not apply for the LPW estimator in Table 4 as this estimator exhibits higher bias and RMSE. Of course the increase in RMSE was expected in light of the asymptotic variance of the estimator, see section 2.2.2. Similarly, the modifications of the LPR estimator by Shimotsu & Phillips (2002*b*) (PLPR) has approximately the same accuracy as the LPR estimator (i.e., although the PLPR estimator has smaller asymptotic variance than the LPR estimator this does not seem to carry over to practice), see Table 2, but as expected from (13) the BRLPR estimator by Andrews & Guggenberger (2003) in Table 4 has a higher RMSE.

When introducing short-run dynamics we generally find the estimators to be biased because the low frequencies are contaminated by the higher frequencies of the spectral density, especially in the case of positive AR noise. In the ARFIMA$(1, d, 0)$ case in Tables 9-11 the biases increase dramatically when the short-run noise becomes more persistent. The methods handle negative AR noise quite well, but except for the local polynomial methods LPW and BRLPR, it is still crucial to use a smaller bandwidth as the biases (for all ϕ) and even RMSEs (for $\phi = .8$) decrease noticeably when m is reduced from $\lfloor T^{0.65} \rfloor$ to $\lfloor T^{0.5} \rfloor$. On the contrary, with the proper choice of bandwidth ($m = \lfloor T^{0.5} \rfloor$) the estimation methods seem more robust towards MA noise, see Tables 16-18 where the biases are fairly small regardless of the size of the MA parameter, although the lowest biases are obtained for positive values. However, this is expected since MA noise affects the short-run part of the spectral density, i.e. the higher frequencies, and thus contaminates the long-run part less than the AR noise does.

For the ARFIMA$(1, d, 0)$ series the FELW and PLPR estimators are again very similar to their original LW and LPR counterparts with respect to both bias and RMSE (Tables 9 and 10). On the other hand, in the presence of strong autoregressive noise the usefulness of the LPW and BRLPR estimators is clearly revealed in Table 11. Approximating the logarithm of the short-run component of the spectral density by a polynomial instead of a constant seems very much justified when the short-run noise is persistent since the bias of the LPW estimator is dramatically less than the LW and FELW estimators. As shown by Andrews & Sun (2004) this reduction does not come without a sacrifice as the variance increases by a multiplicative constant (in our case with $R = 1$, the constant is $c_1 = 2.25$), which is also observed from the RMSEs in Table 11. For the BRLPR estimator, the increase in the RMSE compared to the LPR estimator is not as pronounced as the increase in RMSE of the LPW estimator compared to the LW estimator, but the bias improvement is also smaller.

Contrary to the case with AR noise, when focusing on short-run MA contamination our results in Table 18 give no special justification of the LPW estimator. However, the BRLPR estimator still performs favorably compared to the LPR and PLPR estimators in Table 16.

As mentioned above it is generally preferable to use a smaller bandwidth when short-run dynamics is present in the data. This is actually not the case for the LPW and BRLPR estimators in the ARFIMA$(0, d, 1)$ case where the opposite is true, see Table 18. That is, the LPW and BRLPR estimators are very robust to MA noise because of the way they approximate the spectral density of the short-run noise by a polynomial, and it is thus possible to choose a higher bandwidth (in the presence of MA noise) without incurring a large increase in bias.

In sum, the results for the semiparametric estimators reveal the need for the LPW and BRLPR estimators when persistent AR noise is present in the data. With the exception of the FML estimator, we find that the semiparametric methods perform better than the parametric methods in several cases. Thus, the semiparametric procedures may be preferred because of their simplicity, i.e. we do not need to know the true DGP of the investigated series to consistently estimate the long memory parameter.

3.4 Monte Carlo Results for Wavelet Estimators

Finally, we turn to the results for the wavelet methods described in section 2.3.

As a counterpart to the semiparametric LPR estimator we have the WOLS procedure. From Tables 5 and 6 and the corresponding tables in teh separate appendix we note that for the ARFIMA$(0, d, 0)$ and ARFIMA$(0, d, 0)$-ARCH(1) cases the biases are similar and fairly low but still higher than for most of the other estimators. Thus, the WOLS estimator is relatively robust towards ARCH effects in the innovations and the biases remain fairly independent of the size of β. We typically find that the WOLS estimator is negatively biased using both the Haar wavelet (Table 5) and the Daubechies4 wavelet (Table 6). Other variants of the Daubechies wavelet have also been applied and the results are virtually indistinguishable from the Daubechies4 results presented. For the wavelet MLE in the ARFIMA$(0, d, 0)$ case (Table 7) the bias generally changes sign from negative to positive d. This suggests that the WML estimator cannot fully capture the extent of the true memory parameter, i.e. the bias is positive when d is negative and vice versa. Interestingly, this is not the case when the innovations are conditionally heteroskedastic, see the separate appendix. The most successful of the wavelet estimators seems to be the WML estimator with trimming of the highest $K = 2$ scales which obtains biases in line with the parametric time domain methods in some cases (white noise errors, ARCH errors, or weak serial correlation), and thus in particular careful trimming can render the WML estimator robust to ARCH errors. For the WOLS estimator it seems that there is something gained from trimming the lowest $J = 2$ scales to remove boundary effects, see Jensen (1999), when there is no short-run dynamics present in the data.

When the noise from the AR parameter is large ($\phi = .8$ in Tables 12 and 13) it is preferable to follow Tse et al. (2002) and trim the highest scales for the WOLS estimator (similar to choosing a smaller bandwidth in the semiparametric approach), but with moderate AR noise ($\phi = .4$) it is preferable not to trim at all. If trimming is not used when $\phi = .8$ the estimates become severely positively biased. Furthermore, the WOLS estimator seems almost useless in the presence of a negative AR parameter where the biases are very negative even if trimming is used. Thus, the procedure cannot distinguish short- and long-run dynamics in this case.

When introducing MA dynamics into the series (Tables 19 and 20) one observes a failure of the WOLS estimator to render reliable estimates of the long memory parameter. If $\theta > 0$, the method is fairly usable (if no trimming is employed) with biases in line with the parametric time domain procedures, but if $\theta < 0$ or if any kind of trimming is applied (of low or high scales) the WOLS estimator becomes heavily biased.

In the presence of short-run dynamics the trimming of the highest scales becomes very important for the WML estimator, see Tables 14 and 21. With sufficient trimming ($K = 4$), the biases in the ARFIMA(1,d,0) case and the ARFIMA(0,d,1) case with a positive MA parameter are comparable to those of the parametric time domain methods. However, the RMSEs are noticeable higher because the trimming of the highest scales entails a large decrease in the sample size effectively used in estimating d.

Generally, in terms of biases, the more smooth Daubechies wavelet filters are preferred to the Haar filter.

4 Conclusions

In this paper we have compared through Monte Carlo simulations the finite sample properties of estimators of the fractional differencing parameter, d, in ARFIMA models. We have considered methods in the frequency domain, time domain, and wavelet based approaches and both parametric and semiparametric estimation methods, and the methods were compared in terms of finite sample bias and RMSE.

Our results show that among the parametric methods the frequency domain maximum likelihood procedure is superior with respect to both bias and RMSE. However, our results also show that the (sometimes quite severe) bias of the parametric time domain procedures is alleviated when larger sample sizes (e.g. 512) are considered. For all the estimators under consideration we find the bias to improve and the RMSE to decrease as the sample size increases from 128 to 256 and 512.

Furthermore, according to our results all the methods under consideration are rather robust to the presence of ARCH effects, which are not parameterized, in the sense that the finite sample biases and RMSEs do not increase much compared to the case with white noise errors.

Among the semiparametric (frequency domain and wavelet) methods our results clearly demonstrate the usefulness of the bias reduced log-periodogram regression and local polynomial Whittle estimators of Andrews & Guggenberger (2003) and Andrews & Sun (2004), respectively. In several cases these methods even outperform the correctly specified time domain parametric methods. Furthermore, when other methods are very heavily biased due to contamination from short-run dynamics, these estimators show a much lower bias at the expense of an increase in their RMSE. The bias reduction is due to their modelling of the logarithm of the spectral density of the short-run component by a polynomial instead of a constant. Finally, without sufficient trimming of scales the wavelet based methods are heavily biased when short-run dynamics is introduced.

A natural next step towards a deeper understanding of the simulation findings presented here would be to study the higher-order asymptotic properties of the involved estimators. Some recent work has already been done in this direction. For example, Lieberman, Rousseau & Zucker (2003) and Andrews & Lieberman (2005) derive valid Edgeworth expansions for parametric MLEs of ARFIMA models, Lieberman & Phillips (2004) present an explicit second-order asymptotic expansion for the MLE in the ARFIMA(0,d,0) case, and Giraitis & Robinson (2003) derive Edgeworth expansions for the semiparametric local Whittle estimator. For more details on this course of study, we refer the reader to these articles.

References

Agiakloglou, C., Newbold, P. & Wohar, M. (1993), 'Bias in an estimator of the fractional difference parameter', *Journal of Time Series Analysis* **14**, 235–246.

An, S. & Bloomfield, P. (1993), 'Cox and Reid's modification in regression models with correlated errors', *Technical Report, North Carolina State University* .

Andersen, T. G., Bollerslev, T., Diebold, F. X. & Ebens, H. (2001), 'The distribution of realized stock return volatility', *Journal of Financial Economics* **61**, 43–76.

Andersen, T. G., Bollerslev, T., Diebold, F. X. & Labys, P. (2001), 'The distribution of exchange rate volatility', *Journal of the American Statistical Association* **96**, 42–55.

Andrews, D. W. K. & Guggenberger, P. (2003), 'A bias-reduced log-periodogram regression estimator for the long-memory parameter', *Econometrica* **71**, 675–712.

Andrews, D. W. K. & Lieberman, O. (2005), 'Valid Edgeworth expansions for the Whittle maximum likelihood estimator for stationary long-memory Gaussian time series', *Econometric Theory* **21**, 710–734.

Andrews, D. W. K. & Sun, Y. (2004), 'Adaptive local polynomial Whittle estimation of long-range dependence', *Econometrica* **72**, 569–614.

Baillie, R. T. (1996), 'Long memory processes and fractional integration in econometrics', *Journal of Econometrics* **73**, 5–59.

Baillie, R. T., Chung, C.-F. & Tieslau, M. A. (1996), 'Analysing inflation by the fractionally integrated ARFIMA-GARCH model', *Journal of Applied Econometrics* **11**, 23–40.

Beran, J. (1994), *Statistics for Long-Memory Processes*, Chapman-Hall, New York.

Beran, J. (1995), 'Maximum likelihood estimation of the differencing parameter for invertible short and long memory autoregressive integrated moving average models', *Journal of the Royal Statistical Society Series B* **57**, 659–672.

Cheung, Y.-W. & Diebold, F. X. (1994), 'On maximum-likelihood estimation of the differencing parameter of fractionally-integrated noise with unknown mean', *Journal of Econometrics* **62**, 301–316.

Chung, C.-F. & Baillie, R. T. (1993), 'Small sample bias in conditional sum of squares estimators of fractionally integrated ARMA models', *Empirical Economics* **18**, 791–806.

Cox, D. R. & Reid, N. (1987), 'Parameter orthogonality and approximate conditional inference (with discussion)', *Journal of the Royal Statistical Society Series B* **49**, 1–39.

Dahlhaus, R. (1989), 'Efficient parameter estimation for self-similar processes', *Annals of Statistics* **17**, 1749–1766.

Daubechies, I. (1988), 'Orthonormal bases of compactly supported wavelets', *Communications on Pure and Applied Mathematics* **41**, 909–996.

Diebold, F. X., Husted, S. & Rush, M. (1991), 'Real exchange rates under the gold standard', *Journal of Political Economy* **99**, 1252–1271.

Diebold, F. X. & Inoue, A. (2001), 'Long memory and regime switching', *Journal of Econometrics* **105**, 131–159.

Diebold, F. X. & Rudebusch, G. D. (1989), 'Long memory and persistence in aggregate output', *Journal of Monetary Economics* **24**, 189–209.

Diebold, F. X. & Rudebusch, G. D. (1991), 'Is consumption too smooth? Long memory and the Deaton paradox', *Review of Economics and Statistics* **73**, 1–9.

Doornik, J. A. (2001), *Ox: An Object-Oriented Matrix Language*, 4th edn, Timberlake Consultants Press, London.

Doornik, J. A. & Ooms, M. (2001), 'A package for estimating, forecasting and simulating arfima models: Arfima package 1.01 for Ox', *Working Paper, Nuffield College, Oxford* .

Fox, R. & Taqqu, M. S. (1986), 'Large-sample properties of parameter estimates for strongly dependent stationary gaussian series', *Annals of Statistics* **14**, 517–532.

Geweke, J. & Porter-Hudak, S. (1983), 'The estimation and application of long memory time series models', *Journal of Time Series Analysis* **4**, 221–238.

Gil-Alana, L. A. & Robinson, P. M. (1997), 'Testing of unit root and other non-stationary hypotheses in macroeconomic time series', *Journal of Econometrics* **80**, 241–268.

Giraitis, L. & Robinson, P. M. (2003), 'Edgeworth expansions for the semiparametric Whittle estimator of long memory', *Annals of Statistics* **31**, 1325–1375.

Giraitis, L. & Surgailis, D. (1990), 'A central limit theorem for quadratic forms in strongly dependent linear variables and its application to asymptotic normality of Whittle's estimate', *Probability Theory and Related Fields* **86**, 87–104.

Granger, C. W. J. (1980), 'Long memory relationships and the aggregation of dynamic models', *Journal of Econometrics* **14**, 227–238.

Granger, C. W. J. & Joyeux, R. (1980), 'An introduction to long memory time series models and fractional differencing', *Journal of Time Series Analysis* **1**, 15–29.

Hassler, U. & Wolters, J. (1995), 'Long memory in inflation rates: International evidence', *Journal of Business and Economic Statistics* **13**, 37–45.

Hauser, M. A. (1997), 'Semiparametric and nonparametric testing for long memory: A Monte Carlo study', *Empirical Economics* **22**, 247–271.

Hauser, M. A. (1999), 'Maximum likelihood estimators for ARMA and ARFIMA models: A Monte Carlo study', *Journal of Statistical Planning and Inference* **80**, 229–255.

Henry, M. & Robinson, P. M. (1996), Bandwidth choice in Gaussian semiparametric estimation of long range dependence, *in* P. M. Robinson & M. Rosenblatt, eds, 'Athens Conference on Applied Probability and Time Series Analysis, Volume II: Time Series Analysis, In Memory of E. J. Hannan', Springer, New York, pp. 220–232.

Henry, M. & Zaffaroni, P. (2003), The long range dependence paradigm for macroeconomics and finance, *in* P. Doukhan, G. Oppenheim & M. S. Taqqu, eds, 'Theory and Applications of Long-Range Dependence', Birkhäuser, Boston, pp. 417–438.

Hosking, J. R. M. (1981), 'Fractional differencing', *Biometrika* **68**, 165–176.

Hosoya, Y. (1997), 'A limit theory for long-range dependence and statistical inference on related models', *Annals of Statistics* **25**, 105–137.

Hurst, H. E. (1951), 'Long-term storage capacity of reservoirs', *Transactions of American Civil Engineers* **116**, 770–779.

Hurvich, C. M., Deo, R. S. & Brodsky, J. (1998), 'The mean squared error of Geweke and Porter-Hudak's estimator of the memory paramater of a long memory time series', *Journal of Time Series Analysis* **19**, 19–46.

Jensen, M. J. (1998), 'An approximate wavelet MLE of short- and long-memory parameters', *Studies in Nonlinear Dynamics and Econometrics* **3**, 239–253.

Jensen, M. J. (1999), 'Using wavelets to obtain a consistent ordinary least squares estimator of the long-memory parameter', *Journal of Forecasting* **18**, 17–32.

Jensen, M. J. (2000), 'An alternative maximum likelihood estimator of long-memory processes using compactly supported wavelets', *Journal of Economic Dynamics and Control* **24**, 361–387.

Johnstone, J. M. & Silverman, B. W. (1997), 'Wavelet threshold estimators for data with correlated noise', *Journal of the Royal Statistical Society Series B* **59**, 319–351.

Kim, C. S. & Phillips, P. C. B. (1999), 'Log periodogram regression in the nonstationary case', *Mimeo, Yale University* .

Künsch, H. R. (1986), 'Discrimination between monotonic trends and long-range dependence', *Journal of Applied Probability* **23**, 1025–1030.

Künsch, H. R. (1987), Statistical aspects of self-similar processes, *in* Y. Prokhorov & V. V. Sazanov, eds, 'Proceedings of the First World Congress of the Bernoulli Society', VNU Science Press, Utrecht, pp. 67–74.

Lieberman, O. & Phillips, P. C. B. (2004), 'Expansions for the distribution of the maximum likelihood estimator of the fractional difference parameter', *Econometric Theory* **20**, 464–484.

Lieberman, O., Rousseau, J. & Zucker, D. M. (2003), 'Valid Edgeworth expansions for the maximum likelihood estimator of the parameter of a stationary, Gaussian, strongly dependent series', *Annals of Statistics* **31**, 586–612.

Ling, S. & Li, W. K. (1997), 'On fractionally integrated autoregressive moving-average time series models with conditional heteroskedasticity', *Journal of the American Statistical Association* **92**, 1184–1194.

Lo, A. W. (1991), 'Long term memory in stock market prices', *Econometrica* **59**, 1279–1313.

Mandelbrot, B. B. (1982), *The Fractal Geometry of Nature*, W. H. Freeman and Company, New York.

Mandelbrot, B. B. & Wallis, T. R. (1969), 'Robustness of the rescaled range R/S in the measurement of noncyclic long run statistical dependence', *Water Resources Research* **5**, 967–988.

Marinucci, D. & Robinson, P. M. (1999), 'Alternative forms of fractional Brownian motion', *Journal of Statistical Planning and Inference* **80**, 111–122.

McCoy, E. J. & Walden, A. T. (1996), 'Wavelet analysis and synthesis of stationary long-memory processes', *Journal of Computational and Graphical Statistics* **5**, 26–56.

Nielsen, M. Ø. (2004), 'Efficient likelihood inference in nonstationary univariate models', *Econometric Theory* **20**, 116–146.

Parke, W. R. (1999), 'What is fractional integration?', *Review of Economics and Statistics* **81**, 632–638.

Phillips, P. C. B. & Shimotsu, K. (2004), 'Local Whittle estimation in nonstationary and unit root cases', *Annals of Statistics* **32**, 656–692.

Robinson, P. M. (1978), 'Statistical inference for a random coefficient autoregressive model', *Scandinavian Journal of Statistics* **5**, 163–168.

Robinson, P. M. (1994), Time series with strong dependence, *in* C. A. Sims, ed., 'Advances in Econometrics', Cambridge University Press, Cambridge, pp. 47–95.

Robinson, P. M. (1995*a*), 'Gaussian semiparametric estimation of long range dependence', *Annals of Statistics* **23**, 1630–1661.

Robinson, P. M. (1995*b*), 'Log-periodogram regression of time series with long range dependence', *Annals of Statistics* **23**, 1048–1072.

Robinson, P. M. (2003), Long-memory time series, *in* P. M. Robinson, ed., 'Time Series With Long Memory', Oxford University Press, Oxford, pp. 4–32.

Shimotsu, K. (2002), 'Exact local Whittle estimation of fractional integration with unknown mean and time trend', *Department of Economics Discussion Paper No. 543, University of Essex* .

Shimotsu, K. & Phillips, P. C. B. (2002*a*), 'Exact local Whittle estimation of fractional integration', *Forthcoming in Annals of Statistics* .

Shimotsu, K. & Phillips, P. C. B. (2002*b*), 'Pooled log periodogram regression', *Journal of Time Series Analysis* **23**, 57–93.

Sowell, F. B. (1992*a*), 'Maximum likelihood estimation of stationary univariate fractionally integrated time series models', *Journal of Econometrics* **53**, 165–188.

Sowell, F. B. (1992*b*), 'Modeling long run behavior with the fractional ARIMA model', *Journal of Monetary Economics* **29**, 277–302.

Tanaka, K. (1999), 'The nonstationary fractional unit root', *Econometric Theory* **15**, 549–582.

Tse, Y. K., Ahn, V. V. & Tieng, Q. (2002), 'Maximum likelihood estimation of the fractional differencing parameter in an ARFIMA model using wavelets', *Mathematics and Computers in Simulation* **59**, 153–161.

Velasco, C. (1999*a*), 'Gaussian semiparametric estimation of non-stationary time series', *Journal of Time Series Analysis* **20**, 87–127.

Velasco, C. (1999*b*), 'Non-stationary log-periodogram regression', *Journal of Econometrics* **91**, 325–371.

Velasco, C. (2000), 'Non-Gaussian log-periodogram regression', *Econometric Theory* **16**, 44–79.

Whittle, P. (1951), *Hypothesis Testing in Time Series Analysis*, Almquist and Wiksells, Uppsala.

Yong, C. H. (1974), *Asymptotic Behaviour of Trigonometric Series*, Chinese University of Hong Kong, Hong Kong.

Zygmund, A. (2002), *Trigonometric Series*, third edn, Cambridge University Press, Cambridge.

Table 1: Parametric Estimators - ARFIMA(0,d,0)

		EML		MPL		CML		FML	
d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.25	128	-.0290	.0852	-.0290	.0852	-.0284	.0834	**-.0053**	**.0833**
	256	-.0162	.0553	-.0162	.0553	-.0157	.0550	**-.0027**	**.0537**
	512	-.0093	.0375	-.0093	.0375	-.0089	.0373	**-.0012**	**.0370**
0	128	-.0279	.0818	-.0279	.0818	-.0280	.0830	**-.0013**	**.0791**
	256	-.0147	.0535	-.0147	.0535	-.0148	.0540	**-.0006**	**.0520**
	512	-.0084	.0365	-.0084	.0365	-.0084	.0366	**-.0008**	**.0357**
.25	128	-.0342	.0821	-.0342	.0821	-.0264	.0834	**.0013**	**.0801**
	256	-.0198	.0555	-.0198	.0555	-.0156	.0557	**-.0007**	**.0540**
	512	-.0097	.0365	-.0097	.0365	-.0074	.0365	**.0008**	**.0360**
.45	128	-.0656	.0889	-.0656	.0889	-.0327	.0857	**-.0022**	**.0814**
	256	-.0367	.0559	-.0367	.0559	-.0149	.0556	**.0014**	**.0548**
	512	-.0200	**.0361**	-.0200	**.0361**	-.0066	.0370	**.0032**	.0377

Table 2: Semiparametric I - ARFIMA (0,d,0)

		LPR ($m = \lfloor T^{0.5} \rfloor$)		LPR ($m = \lfloor T^{0.65} \rfloor$)		PLPR ($m = \lfloor T^{0.5} \rfloor$)		PLPR ($m = \lfloor T^{0.65} \rfloor$)	
d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.25	128	.0130	.2864	.0056	.1745	.0126	.2767	**.0022**	.1735
	256	.0089	.2158	.0033	.1258	.0054	.2044	**.0012**	.1248
	512	.0115	.1692	.0071	.0973	.0111	.1628	.0069	.0957
0	128	.0047	.2742	.0048	.1641	.0078	.2624	.0051	.1651
	256	.0012	.2086	.0013	.1243	.0003	.2004	.0004	.1235
	512	-.0019	.1689	-.0015	.0968	-.0034	.1623	-.0015	.0954
.25	128	.0207	.2766	.0090	.1712	.0191	.2605	.0118	.1708
	256	.0189	.2087	.0041	.1266	.0206	.2034	.0053	.1247
	512	.0124	.1769	.0056	.0958	.0095	.1702	.0074	.0956
.45	128	.0247	.2822	.0084	.1604	.0215	.2733	.0156	.1617
	256	.0211	.2136	.0085	.1275	.0223	.2023	.0129	.1262
	512	.0203	.1752	.0123	.0994	.0208	.1653	.0154	.0994

Table 3: Semiparametric II - ARFIMA (0,d,0)

		LW ($m = \lfloor T^{0.5} \rfloor$)		LW ($m = \lfloor T^{0.65} \rfloor$)		FELW ($m = \lfloor T^{0.5} \rfloor$)		FELW ($m = \lfloor T^{0.65} \rfloor$)	
d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.25	128	-.0249	.2461	-.0105	**.1454**	-.0356	.2574	-.0029	.1467
	256	-.0104	.1810	-.0050	.1050	-.0155	.1811	-.0023	**.1044**
	512	-.0061	.1421	-.0004	**.0810**	-.0125	.1473	**.0002**	.0816
0	128	-.0196	.2362	-.0083	**.1362**	-.0146	.2382	.0088	.1381
	256	-.0217	.1740	-.0084	**.1024**	-.0216	.1771	**-.0002**	.1035
	512	-.0203	.1419	-.0079	.0785	-.0193	.1384	-.0032	**.0781**
.25	128	-.0130	.2424	-.0048	**.1442**	**-.0040**	.2454	.0188	.1489
	256	**.0022**	.1745	-.0032	**.1021**	.0068	.1805	.0074	.1039
	512	-.0078	.1428	**-.0015**	.0793	-.0064	.1443	.0039	**.0788**
.45	128	-.0160	.2370	-.0091	**.1317**	**-.0039**	.2318	.0250	.1399
	256	**.0000**	.1785	.0003	**.0999**	.0080	.1727	.0217	.1092
	512	.0036	.1425	**.0027**	**.0801**	.0120	.1436	.0175	.0919

Table 4: Semiparametric III - ARFIMA (0,d,0)

d	T	LPW ($m = \lfloor T^{0.5} \rfloor$) Bias	RMSE	LPW ($m = \lfloor T^{0.65} \rfloor$) Bias	RMSE	BRLPR ($m = \lfloor T^{0.5} \rfloor$) Bias	RMSE	BRLPR ($m = \lfloor T^{0.65} \rfloor$) Bias	RMSE
-.25	128	-.1536	.8243	-.0540	.2770	.0177	.5925	.0090	.3097
	256	-.0662	.3807	-.0208	.1863	.0134	.4094	.0121	.2229
	512	-.0486	.2990	-.0101	.1265	.0199	.3199	.0125	.1517
0	128	-.1408	.5803	-.0526	.2847	-.0186	.5867	**.0023**	.3036
	256	-.0710	.3587	-.0294	.1802	.0053	.4169	.0041	.2138
	512	-.0439	.2892	-.0227	.1306	**.0001**	.3289	-.0011	.1546
.25	128	-.1395	.8602	-.0360	.2765	.0450	.5859	.0222	.3168
	256	-.0647	.6790	-.0176	.1798	-.0045	.4139	.0124	.2142
	512	-.0555	.2962	-.0136	.1322	.0120	.3303	.0076	.1610
.45	128	-.1272	.8882	-.0351	.2644	.0338	.5788	.0312	.3080
	256	-.0594	.4673	-.0096	.1807	.0354	.4106	.0205	.2154
	512	-.0383	.3803	-.0037	.1350	.0223	.3034	.0184	.1649

Table 5: Haar Wavelet OLS - ARFIMA (0,d,0)

		$J = K = 0$		$J = 2, K = 0$		$J = 0, K = 2$	
d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.25	128	-.0902	.2141	.0204	.1245	-.1894	.3981
	256	-.0573	.1503	.0175	.0962	-.1228	.2623
	512	-.0504	.1248	.0185	.0735	-.1048	.2067
0	128	-.1254	.2180	-.0422	.1307	-.1985	.3830
	256	-.1026	.1762	-.0322	.0922	-.1562	.2879
	512	-.0829	.1401	-.0260	.0705	-.1212	.2150
.25	128	-.1477	.2372	-.0792	.1555	-.2069	.3960
	256	-.1212	.1917	-.0583	.1090	-.1628	.2942
	512	-.1096	.1714	-.0553	.0929	-.1424	.2521
.45	128	-.1616	.2475	-.0999	.1619	-.2069	.3990
	256	-.1253	.1943	-.0824	.1259	-.1519	.2888
	512	-.1082	.1611	-.0610	.0952	-.1302	.2282

Table 6: Daubechies4 Wavelet OLS - ARFIMA (0,d,0)

		$J = K = 0$		$J = 2, K = 0$		$J = 0, K = 2$	
d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.25	128	-.0991	.2118	-.0117	.1277	-.1793	.3835
	256	-.0765	.1639	-.0088	.0979	-.1354	.2790
	512	-.0670	.1396	-.0058	.0698	-.1159	.2255
0	128	-.1212	.2183	-.0436	.1433	-.1880	.3793
	256	-.1023	.1832	-.0372	.0997	-.1532	.2987
	512	-.0983	.1574	-.0287	.0733	-.1452	.2427
.25	128	-.1042	.2098	-.0650	.1460	-.1365	.3593
	256	-.0913	.1736	-.0479	.1077	-.1204	.2733
	512	-.0803	.1441	-.0404	.0812	-.1029	.2141
.45	128	**.0037**	.1844	-.0758	.1457	.0788	.3462
	256	**.0017**	.1426	-.0573	.1095	.0482	.2431
	512	**-.0059**	.1275	-.0433	.0838	.0236	.2009

Table 7: Wavelet MLE - ARFIMA(0,d,0)

		Haar ($K=0$)		Haar ($K=2$)		Daub4 ($K=0$)		Daub4 ($K=2$)	
d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.25	128	.0611	.0958	**-.0001**	.1907	.0372	**.0833**	-.0125	.1966
	256	.0643	.0808	.0178	.1175	.0395	**.0636**	**-.0001**	.1185
	512	.0633	.0723	.0233	.0820	.0398	**.0524**	**.0032**	.0767
0	128	**-.0041**	**.0721**	-.0270	.2821	-.0060	.0795	-.0283	.3019
	256	**-.0037**	**.0462**	-.0173	.1590	-.0038	.0470	-.0142	.1760
	512	**-.0020**	.0330	-.0121	.1429	-.0030	**.0329**	-.0136	.1746
.25	128	-.0473	.0893	-.0358	.1973	-.0266	**.0796**	**.0026**	.2347
	256	-.0436	.0672	-.0233	.1172	-.0269	**.0579**	**.0039**	.1136
	512	-.0428	.0553	-.0157	.0753	-.0287	**.0452**	**.0018**	.0745
.45	128	-.0740	.1051	-.0400	.1890	.0059	**.0897**	.1227	.2437
	256	-.0668	.0846	-.0202	.1189	-.0103	**.0609**	.0813	.1515
	512	-.0635	.0729	-.0154	.0808	-.0224	**.0488**	.0520	.1020

Table 8: Parametric Estimators - ARFIMA(1,d,0)

			EML		MPL		CML		FML	
ϕ	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.40	-.25	128	-.0474	.1210	-.0474	.1210	-.0486	**.1174**	**-.0145**	.1177
		256	-.0205	.0721	-.0205	.0721	-.0216	.0720	**-.0004**	**.0712**
		512	-.0108	**.0466**	-.0108	**.0466**	-.0112	**.0466**	**.0008**	**.0466**
	0	128	-.0672	.1375	-.0672	.1375	-.0666	.1363	**-.0252**	**.1187**
		256	-.0356	.0817	-.0356	.0817	-.0354	.0812	**-.0146**	**.0750**
		512	-.0195	.0519	-.0195	.0519	-.0188	.0508	**-.0083**	**.0488**
	.25	128	-.0674	.1297	-.0674	.1297	-.0614	.1384	**-.0114**	**.1098**
		256	-.0330	.0783	-.0330	.0783	-.0287	.0793	**-.0049**	**.0734**
		512	-.0157	.0495	-.0157	.0495	-.0131	.0497	**-.0002**	**.0479**
	.45	128	-.1011	.1359	-.1011	.1359	-.0613	.1293	**-.0134**	**.1122**
		256	-.0573	.0799	-.0573	.0799	-.0303	.0768	**-.0050**	**.0714**
		512	-.0322	.0502	-.0322	.0502	-.0149	.0496	**-.0004**	**.0480**
0	-.25	128	-.0949	.2185	-.0949	.2185	-.0912	.2193	**-.0352**	**.1979**
		256	-.0439	.1190	-.0439	.1190	-.0429	.1196	**-.0148**	**.1161**
		512	-.0221	.0723	-.0221	.0723	-.0216	.0729	**-.0044**	**.0652**
	0	128	-.1299	.2680	-.1299	.2680	-.1390	.2866	**-.0410**	**.2006**
		256	-.0603	.1453	-.0603	.1453	-.0619	.1498	**-.0202**	**.1143**
		512	-.0277	.0712	-.0277	.0712	-.0278	.0717	**-.0093**	**.0653**
	.25	128	-.1838	.3367	-.1838	.3367	-.1597	.3254	**-.0482**	**.2168**
		256	-.0718	.1683	-.0718	.1683	-.0614	.1683	**-.0201**	**.1394**
		512	-.0291	.0770	-.0291	.0770	-.0232	.0760	**-.0035**	**.0661**
	.45	128	-.2231	.3469	-.2231	.3469	-.1460	.3158	**-.0462**	**.2214**
		256	-.0998	.1693	-.0998	.1693	-.0546	.1576	**-.0112**	**.1138**
		512	-.0493	.0800	-.0493	.0800	-.0210	.0730	**-.0017**	**.0699**
.40	-.25	128	-.1763	.2909	-.1761	.2906	-.1561	.2743	**-.0526**	**.2462**
		256	-.1177	.2233	-.1177	.2233	-.1078	.2157	**-.0451**	**.1881**
		512	-.0679	.1609	-.0679	.1609	-.0625	.1547	**-.0282**	**.1427**
	0	128	-.2201	.3113	-.2201	.3113	-.1683	.2807	**-.0581**	**.2369**
		256	-.1533	.2505	-.1533	.2505	-.1207	.2261	**-.0513**	**.1871**
		512	-.0843	.1686	-.0843	.1686	-.0660	.1501	**-.0364**	**.1382**
	.25	128	-.2602	.3374	-.2602	.3374	-.1763	.3000	**-.0558**	**.2365**
		256	-.1792	.2667	-.1797	.2672	-.1304	.2393	**-.0546**	**.1933**
		512	-.1092	.1907	-.1092	.1907	-.0829	.1744	**-.0399**	**.1444**
	.45	128	-.3490	.3994	-.3490	.3994	-.1438	.2956	**-.0521**	**.2524**
		256	-.2349	.3001	-.2349	.3001	-.1015	.2267	**-.0474**	**.1986**
		512	-.1355	.1993	-.1355	.1993	-.0579	.1608	**-.0221**	**.1362**
.80	-.25	128	**-.0243**	**.1455**	**-.0243**	**.1455**	.0249	.1889	.0326	.1988
		256	-.0161	**.1304**	-.0161	**.1304**	**.0093**	.1501	.0244	.1679
		512	**-.0044**	**.1056**	**-.0044**	**.1056**	.0081	.1140	.0202	.1273
	0	128	-.0303	.1377	-.0303	.1377	**-.0102**	**.1104**	.0489	.2067
		256	-.0251	.1194	-.0251	.1194	**-.0086**	**.0954**	.0318	.1661
		512	-.0151	.0970	-.0151	.0970	**-.0042**	**.0787**	.0223	.1277
	.25	128	-.0727	**.1350**	-.0727	**.1350**	.1386	.2887	**.0114**	.1987
		256	-.0517	**.1101**	-.0517	**.1101**	.0911	.2239	**.0182**	.1689
		512	-.0317	**.0894**	-.0317	**.0894**	.0499	.1597	**.0139**	.1292
	.45	128	-.1227	**.1444**	-.1227	**.1444**	.1940	.3145	**-.0355**	.2160
		256	-.0931	**.1145**	-.0931	**.1145**	.1633	.2744	**-.0287**	.1747
		512	-.0653	**.0855**	-.0653	**.0855**	.1268	.2215	**-.0127**	.1376

Table 9: Semiparametric I - ARFIMA (1,d,0)

			LPR ($m = \lfloor T^{0.5} \rfloor$)		LPR ($m = \lfloor T^{0.65} \rfloor$)		PLPR ($m = \lfloor T^{0.5} \rfloor$)		PLPR ($m = \lfloor T^{0.65} \rfloor$)	
ϕ	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.40	-.25	128	**.0044**	.2893	-.0320	.1793	-.0220	.2798	-.0611	.1857
		256	**.0068**	.2115	-.0163	.1256	-.0150	.2012	-.0437	.1308
		512	.0113	.1679	-.0048	.0969	**-.0016**	.1616	-.0262	.0986
	0	128	-.0160	.2716	-.0412	.1697	-.0445	.2665	-.0682	.1785
		256	-.0137	.2104	-.0248	.1258	-.0343	.2032	-.0508	.1312
		512	**-.0004**	.1768	-.0153	.0948	-.0143	.1697	-.0359	.0985
	.25	128	**.0010**	.2862	-.0436	.1770	-.0247	.2794	-.0655	.1835
		256	**.0049**	.2170	-.0198	.1307	-.0140	.2079	-.0424	.1345
		512	**.0027**	.1604	-.0092	.0951	-.0112	.1538	-.0293	.0970
	.45	128	**.0129**	.2685	-.0326	.1715	-.0144	.2607	-.0508	.1748
		256	.0098	.1993	-.0179	.1294	-.0113	.1907	-.0376	.1314
		512	.0116	.1703	-.0043	.0954	**-.0022**	.1616	-.0225	.0971
.40	-.25	128	.0648	.2797	.1453	.2222	.1001	.2790	.1611	.2323
		256	.0339	.2119	.0948	.1599	.0621	.2093	.1144	.1700
		512	.0136	.1676	.0588	.1101	.0359	.1649	.0833	.1238
	0	128	.0699	.2689	.1461	.2177	.1027	.2722	.1654	.2317
		256	.0348	.2174	.0979	.1572	.0609	.2141	.1207	.1715
		512	.0214	.1666	.0646	.1142	.0441	.1644	.0884	.1287
	.25	128	.0673	.2831	.1387	.2185	.1005	.2785	.1621	.2333
		256	.0348	.2188	.0919	.1578	.0651	.2101	.1182	.1729
		512	.0279	.1630	.0689	.1162	.0488	.1629	.0957	.1328
	.45	128	.0619	.2781	.1434	.2232	.0995	.2815	.1701	.2412
		256	.0335	.2123	.1008	.1608	.0658	.2147	.1266	.1771
		512	.0252	.1695	.0654	.1163	.0468	.1691	.0913	.1316
.80	-.25	128	.4108	.4920	.5807	.6040	.4536	.5243	.5949	.6178
		256	.2729	.3401	.4716	.4880	.3236	.3768	.4932	.5086
		512	.1666	.2348	.3862	.3976	.2227	.2732	.4145	.4248
	0	128	.3922	.4750	.5758	.5994	.4391	.5099	.5965	.6193
		256	.2751	.3453	.4736	.4896	.3286	.3860	.4981	.5134
		512	.1606	.2320	.3807	.3915	.2196	.2711	.4100	.4195
	.25	128	.3922	.4802	.5652	.5885	.4384	.5121	.5889	.6112
		256	.2691	.3433	.4665	.4842	.3214	.3799	.4938	.5104
		512	.1538	.2278	.3773	.3893	.2105	.2652	.4071	.4178
	.45	128	.3872	.4752	.5387	.5654	.4252	.5014	.5634	.5889
		256	.2604	.3398	.4539	.4709	.3126	.3757	.4808	.4967
		512	.1585	.2381	.3739	.3860	.2149	.2748	.4041	.4146

Table 10: Semiparametric II - ARFIMA (1,d,0)

			LW ($m = \lfloor T^{0.5} \rfloor$)		LW ($m = \lfloor T^{0.65} \rfloor$)		FELW ($m = \lfloor T^{0.5} \rfloor$)		FELW ($m = \lfloor T^{0.65} \rfloor$)	
ϕ	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.40	-.25	128	-.0335	.2493	-.0494	**.1548**	-.0499	.2609	-.0465	**.1548**
		256	-.0132	.1812	-.0261	.1087	-.0206	.1811	-.0257	**.1073**
		512	-.0067	.1416	-.0127	**.0819**	-.0147	.1461	-.0133	.0826
	0	128	-.0501	.2426	-.0582	.1494	-.0541	.2505	-.0446	**.1483**
		256	-.0337	.1849	-.0357	.1087	-.0355	.1838	-.0286	**.1061**
		512	-.0182	.1472	-.0220	.0776	-.0188	.1440	-.0181	**.0767**
	.25	128	-.0284	.2491	-.0588	.1497	-.0245	.2483	-.0402	**.1445**
		256	-.0152	.1749	-.0288	.1099	-.0143	.1766	-.0207	**.1082**
		512	-.0114	.1351	-.0185	.0759	-.0105	.1351	-.0135	**.0741**
	.45	128	-.0227	.2366	-.0514	.1478	-.0148	.2297	-.0244	**.1451**
		256	-.0146	.1716	-.0326	**.1089**	**-.0072**	.1702	-.0165	.1121
		512	-.0105	.1439	-.0148	**.0785**	-.0080	.1427	-.0060	.0833
.40	-.25	128	.0313	.2387	.1324	**.1938**	.0258	.2421	.1463	.2059
		256	.0150	.1753	.0883	**.1368**	.0101	.1817	.0939	.1418
		512	**-.0016**	.1410	.0556	**.0958**	-.0070	.1437	.0576	.0971
	0	128	.0344	.2341	.1347	**.1929**	.0432	.2431	.1567	.2104
		256	.0132	.1842	.0906	**.1366**	.0146	.1859	.1013	.1442
		512	.0053	.1373	.0571	**.0962**	.0057	.1392	.0628	.1000
	.25	128	.0394	.2402	.1307	**.1936**	.0548	.2427	.1645	.2237
		256	**.0096**	.1825	.0846	**.1343**	.0184	.1852	.1005	.1487
		512	**.0065**	.1388	.0600	**.0970**	.0115	.1415	.0665	.1012
	.45	128	.0334	.2345	.1315	**.1934**	.0552	.2328	.1685	.2166
		256	.0185	.1760	.0913	**.1380**	.0291	.1727	.1142	.1517
		512	.0142	.1392	.0604	**.0991**	.0241	.1428	.0772	.1132
.80	-.25	128	.3950	.4625	.5928	.6104	.4076	.4796	.6301	.6498
		256	.2620	.3151	.4981	.5108	.2639	.3183	.5133	.5265
		512	.1542	.2095	.4090	.4176	.1538	**.2094**	.4161	.4247
	0	128	.3812	.4445	.5953	.6137	.4049	.4663	.6423	.6605
		256	.2589	.3128	.4975	.5092	.2678	.3225	.5282	.5408
		512	.1460	.2071	.4044	.4126	.1496	.2107	.4178	.4272
	.25	128	.3763	.4506	.5810	.5996	.4026	.4692	.6303	.6491
		256	.2554	.3141	.4929	.5056	.2750	.3319	.5214	.5338
		512	.1474	**.2019**	.4029	.4116	.1578	.2140	.4199	.4281
	.45	128	.3653	.4345	.5526	.5718	.3956	.4578	.6200	.6381
		256	.2541	.3110	.4804	.4920	.2600	.3107	.5153	.5268
		512	.1526	.2105	.3997	.4078	.1574	.2078	.4148	.4232

Table 11: Semiparametric III - ARFIMA (1,d,0)

ϕ	d	T	LPW ($m = \lfloor T^{0.5} \rfloor$) Bias	RMSE	LPW ($m = \lfloor T^{0.65} \rfloor$) Bias	RMSE	BRLPR ($m = \lfloor T^{0.5} \rfloor$) Bias	RMSE	BRLPR ($m = \lfloor T^{0.65} \rfloor$) Bias	RMSE
-.40	-.25	128	-.1267	.5805	-.0498	.2794	.0191	.5972	.0125	.3127
		256	-.0620	.3795	-.0178	.1860	.0192	.4049	.0152	.2187
		512	-.0362	.3083	-.0080	.1263	.0228	.3183	.0148	.1513
	0	128	-.1336	.5561	-.0684	.2888	-.0083	.5649	**-.0035**	.2928
		256	-.0901	.3610	-.0413	.1893	-.0137	.4002	**-.0048**	.2166
		512	-.0321	.3055	-.0192	.1317	.0075	.3217	.0040	.1600
	.25	128	-.1177	.6445	-.0398	.2683	.0156	.6065	.0117	.3121
		256	-.0767	.3694	-.0192	.1782	.0119	.4241	.0126	.2215
		512	-.0612	.3003	-.0114	.1247	-.0056	.3116	.0074	.1504
	.45	128	-.1089	.6174	-.0352	.2530	.0477	.5697	.0308	.2959
		256	-.0725	.3413	-.0196	.1790	.0098	.3830	.0143	.2142
		512	-.0636	.2996	-.0125	.1333	.0127	.3161	.0136	.1595
.40	-.25	128	-.1210	.6135	-.0242	.2661	**-.0030**	.5867	.0295	.3063
		256	-.0816	.4016	-.0126	.1839	**-.0012**	.4120	.0167	.2170
		512	-.0598	.3166	-.0134	.1624	.0078	.3189	.0098	.1587
	0	128	-.0991	.5391	**-.0287**	.3196	.0336	.5612	.0425	.2916
		256	-.0674	.3561	-.0132	.1830	**.0091**	.4092	.0221	.2184
		512	-.0535	.3043	-.0082	.1294	**-.0026**	.2982	.0150	.1551
	.25	128	-.1421	.9145	**-.0124**	.2610	.0468	.5798	.0463	.3042
		256	-.0571	.8135	-.0143	.1868	.0301	.4280	.0233	.2222
		512	-.0409	.2761	-.0076	.1420	.0225	.3063	.0248	.1567
	.45	128	-.1607	1.0604	**-.0149**	.2597	.0247	.5548	.0396	.2998
		256	-.0742	.5706	**-.0091**	.1822	.0376	.4058	.0190	.2177
		512	-.0421	.2711	**-.0064**	.1480	.0082	.3028	.0146	.1574
.80	-.25	128	**-.0025**	1.0194	.3339	**.4258**	.1944	.6033	.3744	.4785
		256	**-.0044**	.5101	.2373	**.2953**	.0946	.4054	.2587	.3317
		512	**-.0162**	.4533	.1160	.2310	.0370	.3013	.1766	.2325
	0	128	**.0103**	.6355	.3192	**.4037**	.1603	.6029	.3520	.4596
		256	**-.0208**	.3943	.2324	**.2907**	.0748	.4276	.2566	.3313
		512	**-.0225**	.2715	.1211	**.1724**	.0329	.3097	.1654	.2241
	.25	128	**-.0758**	1.0032	.3169	**.4126**	.1492	.5819	.3547	.4697
		256	**-.0252**	.4122	.2246	**.2933**	.0690	.4159	.2490	.3300
		512	-.0329	.2916	.0809	.2096	**.0279**	.3071	.1661	.2312
	.45	128	**-.0501**	.9960	.3188	**.4122**	.1575	.5869	.3597	.4694
		256	**-.0080**	.3982	.2342	**.2941**	.0775	.4151	.2506	.3345
		512	**-.0136**	.2667	.0905	**.1979**	.0371	.3112	.1673	.2355

Table 12: Haar Wavelet OLS - ARFIMA (1,d,0)

ϕ	d	T	$J=K=0$ Bias	$J=K=0$ RMSE	$J=2,K=0$ Bias	$J=2,K=0$ RMSE	$J=0,K=2$ Bias	$J=0,K=2$ RMSE
-.40	-.25	128	-.2002	.2783	-.1416	**.1884**	-.2439	.4242
		256	-.1522	.2065	-.1146	.1497	-.1728	.2907
		512	-.1336	.1802	-.0919	.1162	-.1509	.2421
	0	128	-.2400	.3032	-.2186	.2517	-.2423	.4110
		256	-.2034	.2526	-.1780	.2001	-.2019	.3218
		512	-.1708	.2112	-.1453	.1610	-.1644	.2550
	.25	128	-.2641	.3237	-.2705	.3014	-.2409	.4158
		256	-.2131	.2553	-.2033	.2232	-.1878	.2992
		512	-.1789	.2150	-.1610	.1748	-.1546	.2431
	.45	128	-.2560	.3168	-.2787	.3099	-.2139	.4007
		256	-.2209	.2626	-.2114	.2306	-.1917	.3049
		512	-.1836	.2214	-.1674	.1814	-.1547	.2480
.40	-.25	128	.0686	.1948	.2426	.2715	-.1042	.3473
		256	.0654	.1593	.2003	.2212	-.0694	.2520
		512	.0526	.1275	.1706	.1845	-.0626	.1921
	0	128	**.0229**	**.1896**	.1866	.2229	-.1352	.3692
		256	.0271	.1471	.1537	.1775	-.0972	.2597
		512	.0195	.1174	.1262	.1439	-.0836	.1999
	.25	128	**-.0038**	**.1816**	.1417	.1888	-.1458	.3569
		256	**-.0056**	.1498	.1157	.1482	-.1209	.2785
		512	**.0052**	.1130	.0911	.1146	-.0812	.1937
	.45	128	-.0360	.1959	.1090	**.1644**	-.1751	.3912
		256	**-.0216**	.1520	.0875	.1269	-.1254	.2802
		512	**-.0182**	.1164	.0703	.1007	-.1020	.2057
.80	-.25	128	.3769	.4191	.6109	.6237	.1631	.3723
		256	.3273	.3568	.5452	.5521	.1309	**.2729**
		512	.2898	.3142	.4765	.4816	.1157	.2233
	0	128	.3261	.3836	.5617	.5767	.1073	.3878
		256	.2890	.3220	.4973	.5052	.0979	**.2587**
		512	.2547	.2780	.4261	.4320	.0881	**.1959**
	.25	128	.2861	**.3400**	.5095	.5254	.0820	.3434
		256	.2414	.2844	.4449	.4540	.0535	**.2587**
		512	.2080	.2436	.3834	.3898	**.0422**	.2045
	.45	128	.2315	**.3058**	.4561	.4738	.0230	.3594
		256	.2025	**.2550**	.3982	.4081	**.0217**	.2593
		512	.1806	.2149	.3417	.3489	**.0269**	**.1832**

Table 13: Daubechies4 Wavelet OLS - ARFIMA (1,d,0)

			$J=K=0$		$J=2,K=0$		$J=0,K=2$	
ϕ	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.40	-.25	128	-.2092	.2864	-.2072	.2440	-.1954	.4046
		256	-.1671	.2221	-.1594	.1871	-.1507	.2881
		512	-.1447	.1906	-.1258	.1439	-.1326	.2355
	0	128	-.2424	.3074	-.2396	.2690	-.2249	.4107
		256	-.2036	.2540	-.1847	.2043	-.1863	.3162
		512	-.1688	.2079	-.1457	.1609	-.1497	.2434
	.25	128	-.2176	.2834	-.2644	.2957	-.1526	.3605
		256	-.1739	.2207	-.1951	.2160	-.1194	.2560
		512	-.1488	.1896	-.1572	.1731	-.1058	.2118
	.45	128	-.1023	.2084	-.2646	.2952	**.0648**	.3344
		256	-.0917	.1797	-.2034	.2238	**.0264**	.2598
		512	-.0725	.1405	-.1554	.1720	**.0206**	.1906
.40	-.25	128	.0646	**.1877**	.2413	.2726	-.1165	.3438
		256	.0508	.1552	.1991	.2193	-.0998	.2639
		512	.0405	.1328	.1598	.1737	-.0844	.2154
	0	128	.0324	.1898	.2044	.2396	-.1378	.3643
		256	.0221	.1486	.1668	.1918	-.1213	.2740
		512	.0209	.1226	.1327	.1489	-.0943	.2119
	.25	128	.0540	.1963	.1746	.2199	-.0709	.3495
		256	.0337	.1522	.1433	.1703	-.0779	.2598
		512	.0321	.1193	.1151	.1331	-.0579	.1894
	.45	128	.1427	.2413	.1545	.2024	.1116	.3744
		256	.1161	.1846	.1281	.1595	.0762	.2529
		512	.0919	.1527	.0993	.1238	.0497	.1974
.80	-.25	128	.3767	.4191	.6461	.6586	.1293	**.3625**
		256	.3343	.3674	.5698	.5771	.1176	.2795
		512	.2964	.3171	.4926	.4975	.1062	.2067
	0	128	.3555	.4051	.6171	.6301	.1135	**.3734**
		256	.3139	.3480	.5447	.5516	.1019	.2731
		512	.2700	.2956	.4648	.4698	.0822	.2076
	.25	128	.3620	.4069	.5730	.5888	.1643	.3774
		256	.3148	.3451	.5071	.5154	.1300	.2697
		512	.2718	.2954	.4364	.4418	.1033	.2105
	.45	128	.4420	.4831	.5336	.5491	.3494	.4961
		256	.3773	.4048	.4777	.4865	.2624	.3587
		512	.3194	.3437	.4130	.4188	.1982	.2804

Table 14: Wavelet MLE - ARFIMA(1,d,0)

ϕ	d	T	Haar ($K=2$) Bias	Haar ($K=2$) RMSE	Haar ($K=4$) Bias	Haar ($K=4$) RMSE	Daub4 ($K=2$) Bias	Daub4 ($K=2$) RMSE	Daub4 ($K=4$) Bias	Daub4 ($K=4$) RMSE
-.40	-.25	128	-.0626	.2004	-.1432	.6810	-.0511	.2049	**-.0466**	.6570
		256	-.0440	**.1261**	-.0621	.3354	-.0409	.1270	**-.0277**	.3584
		512	-.0349	.0860	-.0423	.1928	-.0366	**.0857**	**-.0123**	.1881
	0	128	-.0921	.2292	-.1532	.6914	**-.0713**	**.2008**	-.1094	.6667
		256	-.0732	.1382	-.0739	.3279	**-.0496**	**.1250**	-.0642	.3342
		512	-.0651	.1239	-.0443	.1897	-.0403	**.0829**	**-.0300**	.1894
	.25	128	-.0808	.2036	-.0867	.6377	-.0280	**.1881**	**.0120**	.6362
		256	-.0648	.1316	-.0535	.3302	-.0248	**.1152**	**-.0073**	.3243
		512	-.0548	.0903	-.0344	.1854	-.0247	**.0757**	**-.0045**	.1854
	.45	128	-.0729	**.1946**	-.0900	.6346	.0941	.2248	.3651	.7383
		256	-.0626	**.1332**	-.0721	.3169	.0565	.1462	.2076	.4040
		512	-.0509	**.0908**	-.0413	.1981	.0335	.0949	.1312	.2483
.40	-.25	128	.1037	.2136	-.0592	.6577	.0838	.2003	**-.0553**	.6702
		256	.1100	.1595	.0119	.3238	.0869	**.1439**	**-.0322**	.3377
		512	.1082	.1308	.0231	.1818	.0854	**.1133**	**-.0057**	.1852
	0	128	.0781	.2039	-.0856	.6567	.0751	.2070	-.0822	.6640
		256	.0810	.1402	**-.0196**	.3311	.0751	**.1354**	-.0407	.3314
		512	.0780	.1079	**-.0043**	.1877	.0750	**.1042**	-.0177	.1856
	.25	128	.0459	.2319	-.0564	.6330	.0867	.2251	.0293	.6687
		256	.0532	**.1337**	-.0295	.3255	.0748	.1394	.0132	.3384
		512	.0571	**.0925**	-.0105	.1772	.0717	.1017	.0143	.1913
	.45	128	**.0284**	.1800	-.1201	.6696	.1823	.2762	.3734	.7656
		256	.0357	**.1170**	-.0495	.3305	.1346	.1852	.2234	.4226
		512	.0396	**.0862**	-.0197	.1844	.1059	.1355	.1296	.2539
.80	-.25	128	.4492	.4851	.1155	.6486	.4550	.4951	**.0390**	.6423
		256	.4261	.4414	.1352	.3553	.4342	.4509	**.0841**	.3407
		512	.4100	.4172	.1319	.2271	.4195	.4268	**.0958**	**.2052**
	0	128	.3998	.4348	.0779	.6909	.4215	.4612	**.0462**	.6536
		256	.3810	.3957	.1038	.3464	.4091	.4252	**.0776**	.3349
		512	.3632	.3705	.1015	.2139	.3911	.3992	**.0814**	.2000
	.25	128	.3458	.3902	**.0299**	.6219	.4135	.4567	.1536	.6565
		256	.3262	.3456	**.0528**	.3257	.3874	.4064	.1191	.3490
		512	.3154	.3245	.0654	**.1956**	.3711	.3796	.0947	.2114
	.45	128	.2938	.3397	**-.0092**	.6435	.4772	.5146	.4878	.8281
		256	.2814	.3016	.0262	.3145	.4090	.4269	.2968	.4559
		512	.2743	.2843	.0484	.1886	.3690	.3779	.1986	.2897

Table 15: Parametric Estimators - ARFIMA(0,d,1)

θ	d	T	EML Bias	EML RMSE	MPL Bias	MPL RMSE	CML Bias	CML RMSE	FML Bias	FML RMSE
-.40	-.25	128	-.0903	**.2059**	-.0903	**.2059**	-.0846	.2066	**.0056**	.2331
		256	-.0479	**.1519**	-.0477	.1522	-.0454	.1542	**.0199**	.1823
		512	-.0265	**.1065**	-.0265	**.1065**	-.0248	.1074	**.0167**	.1314
	0	128	-.1354	**.2224**	-.1354	**.2224**	-.1372	.2254	**-.0094**	.2394
		256	-.0798	**.1566**	-.0798	**.1566**	-.0830	.1587	**-.0036**	.1777
		512	-.0408	**.1029**	-.0408	**.1029**	-.0420	.1054	**.0001**	.1196
	.25	128	-.1624	**.2258**	-.1624	**.2258**	-.1443	.2378	**-.0032**	.2403
		256	-.0949	**.1475**	-.0949	**.1475**	-.0808	.1583	**.0062**	.1756
		512	-.0476	**.0967**	-.0476	**.0967**	-.0378	.1066	**.0108**	.1256
	.45	128	-.1945	**.2303**	-.1945	**.2303**	-.1301	.2323	**.0004**	.2322
		256	-.1201	**.1508**	-.1201	**.1508**	-.0616	.1618	**.0149**	.1773
		512	-.0732	**.0976**	-.0732	**.0976**	-.0296	.1083	**.0193**	.1302
0	-.25	128	-.0587	**.1381**	-.0587	**.1381**	-.0578	.1409	**-.0009**	.1606
		256	-.0344	**.0905**	-.0344	**.0905**	-.0346	.0907	**-.0051**	.0908
		512	-.0177	**.0629**	-.0177	**.0629**	-.0172	.0624	**-.0010**	.0634
	0	128	-.0723	**.1489**	-.0723	**.1489**	-.0718	.1532	**-.0036**	.1666
		256	-.0415	**.1008**	-.0415	**.1008**	-.0410	.1041	**-.0086**	.1032
		512	-.0204	.0648	-.0204	.0648	-.0204	.0652	**-.0032**	**.0647**
	.25	128	-.0838	**.1428**	-.0838	**.1428**	-.0639	.1507	**.0053**	.1667
		256	-.0491	**.0946**	-.0491	**.0946**	-.0390	.0966	**-.0060**	.0958
		512	-.0231	**.0601**	-.0231	**.0601**	-.0175	.0609	**.0002**	.0603
	.45	128	-.1139	**.1441**	-.1139	**.1441**	-.0527	.1445	**.0117**	.1613
		256	-.0698	**.0937**	-.0698	**.0937**	-.0270	.0981	**.0055**	.0983
		512	-.0399	**.0597**	-.0399	**.0597**	-.0123	.0620	**.0065**	.0633
.40	-.25	128	-.0364	.1099	-.0364	.1099	-.0337	**.1076**	**-.0027**	.1112
		256	-.0205	.0702	-.0205	.0702	-.0191	**.0692**	**-.0019**	.0695
		512	-.0107	.0470	-.0107	.0470	-.0097	**.0465**	**-.0002**	.0468
	0	128	-.0485	.1114	-.0485	.1114	-.0453	.1121	**-.0089**	**.1073**
		256	-.0245	.0743	-.0245	.0743	-.0230	.0744	**-.0039**	**.0728**
		512	-.0130	.0484	-.0130	.0484	-.0122	.0484	**-.0017**	**.0478**
	.25	128	-.0593	.1110	-.0593	.1110	-.0435	.1118	**-.0082**	**.1072**
		256	-.0324	.0725	-.0324	.0725	-.0240	.0726	**-.0051**	**.0702**
		512	-.0170	.0479	-.0170	.0479	-.0126	.0478	**-.0021**	**.0469**
	.45	128	-.0825	.1096	-.0825	.1096	-.0309	**.1061**	**.0035**	.1075
		256	-.0504	.0721	-.0504	.0721	-.0175	.0719	**.0019**	**.0712**
		512	-.0277	**.0457**	-.0277	**.0457**	-.0071	.0471	**.0047**	.0476
.80	-.25	128	-.0283	.0925	-.0283	.0925	-.0195	**.0896**	**-.0034**	.0922
		256	-.0157	.0605	-.0157	.0605	-.0113	**.0591**	**-.0012**	.0595
		512	-.0091	.0400	-.0091	.0400	-.0067	**.0394**	**-.0007**	.0397
	0	128	-.0365	.0916	-.0365	.0916	-.0244	**.0838**	**-.0082**	.0877
		256	-.0196	.0599	-.0196	.0599	-.0150	**.0546**	**-.0044**	.0582
		512	-.0095	.0387	-.0095	.0387	-.0078	**.0358**	**-.0012**	.0379
	.25	128	-.0466	.0913	-.0466	.0913	-.0259	.0892	**-.0105**	**.0869**
		256	-.0234	.0601	-.0234	.0601	-.0127	.0594	**-.0024**	**.0582**
		512	-.0119	.0392	-.0119	.0392	-.0063	.0388	**-.0003**	**.0383**
	.45	128	-.0690	.0940	-.0690	.0940	-.0183	**.0886**	**.0002**	.0987
		256	-.0401	.0600	-.0401	.0600	-.0088	.0595	**.0015**	**.0591**
		512	-.0219	**.0393**	-.0219	**.0393**	**-.0028**	.0410	.0034	.0417

Table 16: Semiparametric I - ARFIMA (0,d,1)

			LPR ($m = \lfloor T^{0.5} \rfloor$)		LPR ($m = \lfloor T^{0.65} \rfloor$)		PLPR ($m = \lfloor T^{0.5} \rfloor$)		PLPR ($m = \lfloor T^{0.65} \rfloor$)	
θ	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.40	-.25	128	**-.0157**	.2719	-.1166	.2005	-.0604	.2654	-.1402	.2150
		256	-.0193	.2262	-.0827	.1539	-.0476	.2234	-.1079	.1671
		512	**.0003**	.1764	-.0506	.1088	-.0202	.1697	-.0774	.1223
	0	128	-.0596	.2859	-.1472	.2258	-.0962	.2867	-.1695	.2418
		256	-.0327	.2121	-.0978	.1632	-.0627	.2085	-.1202	.1761
		512	-.0232	.1781	-.0638	.1173	-.0440	.1752	-.0884	.1314
	.25	128	-.0551	.2927	-.1420	.2170	-.0873	.2887	-.1586	.2278
		256	-.0236	.2029	-.0919	.1531	-.0547	.2007	-.1134	.1665
		512	-.0110	.1712	-.0590	.1115	-.03551	.1661	-.0816	.1239
	.45	128	-.0469	.2872	-.1369	.2201	-.0768	.2834	-.1486	.2276
		256	-.0184	.2151	-.0855	.1551	-.0492	.2138	-.1054	.1663
		512	**.0029**	.1631	-.0547	.1103	-.0187	.1563	-.0763	.1211
.40	-.25	128	.0189	.2712	.0494	.1718	.0456	.2673	.0694	.1786
		256	.0027	.2197	.0268	.1301	.0218	.2102	.0492	.1358
		512	.0081	.1716	.0188	.0982	.0213	.1664	.0390	.1024
	0	128	**.0060**	.2559	.0398	.1676	.0332	.2507	.0645	.1741
		256	**.0001**	.2096	.0206	.1274	.0203	.2060	.0461	.1327
		512	-.0055	.1674	.0093	.0963	.0070	.1636	.0305	.0983
	.25	128	.0207	.2690	.0425	.1689	.0543	.2649	.0740	.1795
		256	.0143	.2044	.0194	.1227	.0337	.1984	.0469	.1279
		512	.0046	.1641	.0091	.0924	.0175	.1577	.0318	.0961
	.45	128	.0287	.2729	.0463	.1708	.0537	.2677	.0784	.1831
		256	.0162	.2195	.0301	.1293	.0387	.2123	.0592	.1367
		512	.0130	.1756	.0200	.0983	.0239	.1670	.0427	.1037
.80	-.25	128	.0197	.2845	.0560	.1760	.0588	.2796	.0951	.1928
		256	.0220	.2069	.0347	.1270	.0484	.2033	.0744	.1417
		512	.0052	.1719	.0203	.0984	.0256	.1655	.0542	.1082
	0	128	.0128	.2774	.0575	.1766	.0564	.2745	.0960	.1933
		256	**-.0116**	.2179	.0228	.1292	.0219	.2093	.0629	.1404
		512	-.0097	.1728	.0100	.0979	.0112	.1650	.0455	.1064
	.25	128	.0163	.2752	.0478	.1759	.0579	.2658	.0914	.1921
		256	.0083	.2044	.0268	.1257	.0383	.1992	.0708	.1392
		512	.0034	.1703	.0208	.1001	.0252	.1643	.0584	.1121
	.45	128	.0251	.2666	.0624	.1774	.0673	.2651	.1098	.1992
		256	.0184	.2109	.0399	.1342	.0479	.2086	.0850	.1525
		512	.0259	.1641	.0285	.1011	.0447	.1638	.0655	.1159

Table 17: Semiparametric II - ARFIMA (0,d,1)

θ	d	T	LW ($m = \lfloor T^{0.5} \rfloor$) Bias	RMSE	LW ($m = \lfloor T^{0.65} \rfloor$) Bias	RMSE	FELW ($m = \lfloor T^{0.5} \rfloor$) Bias	RMSE	FELW ($m = \lfloor T^{0.65} \rfloor$) Bias	RMSE
-.40	-.25	128	-.0497	.2398	-.1322	**.1889**	-.0620	.2426	-.1327	.1903
		256	-.0365	.1921	-.0910	.1398	-.0463	.1901	-.0914	**.1384**
		512	-.0186	.1507	-.0589	**.0997**	-.0296	.1484	-.0621	.1006
	0	128	-.0826	.2469	-.1624	.2157	-.0891	.2543	-.1531	**.2096**
		256	-.0542	.1821	-.1052	.1489	-.0568	.1840	-.0989	**.1451**
		512	-.0330	.1481	-.0673	.1027	-.0358	.1485	-.0637	**.1006**
	.25	128	-.0809	.2581	-.1593	.2099	-.0773	.2495	-.1410	**.1949**
		256	-.0465	.1823	-.1039	.1452	-.0492	.1811	-.0966	**.1398**
		512	-.0273	.1456	-.0651	.0998	-.0301	.1453	-.0606	**.0967**
	.45	128	-.0757	.2527	-.1553	.2097	-.0702	.2457	-.1321	**.1965**
		256	-.0404	.1892	-.0984	.1448	-.0388	.1843	-.0868	**.1406**
		512	-.0117	.1344	-.0606	**.0974**	-.0075	.1366	-.0549	.0981
.40	-.25	128	-.0102	.2352	.0370	**.1391**	-.0184	.2422	.0468	.1457
		256	-.0221	.1835	.0193	**.1035**	-.0281	.1890	.0245	.1052
		512	-.0087	.1430	.0119	.0812	-.0152	.1407	.0125	**.0806**
	0	128	-.0219	.2284	.0238	**.1374**	-.0175	.2322	.0429	.1435
		256	-.0210	.1740	.0109	**.1012**	-.0199	.1756	.0204	.1027
		512	-.0178	.1407	**.0038**	**.0762**	-.0177	.1392	.0088	.0771
	.25	128	-.0023	.2304	.0276	**.1374**	.0165	.2362	.0551	.1507
		256	-.0067	.1679	.0107	**.0974**	**-.0017**	.1695	.0224	.1009
		512	-.0147	.1395	**.0027**	**.0758**	-.0129	.1382	.0088	.0755
	.45	128	**-.0006**	.2387	.0307	**.1404**	.0133	.2316	.0685	.1533
		256	**-.0035**	.1852	.0223	**.1042**	.0087	.1828	.0493	.1197
		512	**-.0009**	.1470	.0168	**.0794**	.0070	.1492	.0320	.0927
.80	-.25	128	-.0126	.2454	.0470	**.1463**	-.0225	.2525	.0563	.1527
		256	**-.0007**	.1721	.0251	**.1024**	-.0060	.1774	.0292	.1056
		512	-.0081	.1434	.0137	**.0797**	-.0133	.1452	.0149	.0799
	0	128	-.0196	.2471	.0423	**.1486**	-.0184	.2567	.0609	.1574
		256	-.0278	.1884	.0147	**.1043**	-.0265	.1887	.0248	.1072
		512	-.0271	.1475	**.0042**	**.0798**	-.0263	.1471	.0094	.0801
	.25	128	-.0174	.2374	.0341	**.1454**	-.0038	.2410	.0628	.1619
		256	-.0148	.1781	.0166	**.1044**	-.0078	.1763	.0298	.1085
		512	-.0114	.1436	.0148	**.0804**	-.0081	.1415	.0207	.0811
	.45	128	**-.0015**	.2324	.0441	**.1482**	.0134	.2261	.0786	.1628
		256	**-.0031**	.1720	.0266	**.1074**	.0068	.1702	.0472	.1174
		512	**.0086**	.1366	.0207	**.0817**	.0178	.1394	.0336	.0907

Table 18: Semiparametric III - ARFIMA (0,d,1)

θ	d	T	LPW ($m = \lfloor T^{0.5} \rfloor$)		LPW ($m = \lfloor T^{0.65} \rfloor$)		BRLPR ($m = \lfloor T^{0.5} \rfloor$)		BRLPR ($m = \lfloor T^{0.65} \rfloor$)	
			Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.40	-.25	128	-.0899	.5671	-.1382	.8197	.0551	.5969	.0244	.3060
		256	-.0303	.3446	-.0292	.1991	.0037	.4116	**.0025**	.2252
		512	.0175	.3007	-.0283	.1250	.0055	.3096	.0086	.1595
	0	128	-.1504	.5823	-.1100	.5206	**-.0207**	.5899	-.0354	.3020
		256	-.0817	.3599	-.0495	.1880	**-.0082**	.4124	-.0157	.2188
		512	-.0604	.3066	-.0270	.1329	-.0180	.3250	**-.0132**	.1626
	.25	128	-.1527	.5617	-.0795	.3133	**-.0250**	.5847	-.0335	.3175
		256	-.0860	.3562	-.0429	.1861	**.0062**	.3943	-.0082	.2092
		512	-.0569	.3098	-.0230	.1347	.0155	.3103	**-.0030**	.1622
	.45	128	-.1726	.5547	-.0764	.2715	-.0392	.5877	**-.0250**	.3104
		256	-.0689	.3669	-.0319	.1870	.0215	.4084	**.0046**	.2146
		512	-.0776	.3362	-.0248	.1536	.0246	.3151	.0043	.1548
.40	-.25	128	-.1156	.7453	-.1045	.6435	**.0000**	.5800	.0015	.2960
		256	-.0517	.3726	-.0384	.1897	**-.0002**	.4319	-.0006	.2257
		512	-.0076	.2747	-.0202	.1461	.0045	.3103	**.0025**	.1592
	0	128	-.1232	.5294	-.0771	.4340	.0196	.5749	-.0072	.2862
		256	-.0626	.3604	-.0396	.1772	.0072	.4207	-.0021	.2145
		512	-.0572	.3082	-.0262	.1286	-.0146	.3113	-.0063	.1544
	.25	128	-.1339	.6152	-.0539	.2538	-.0002	.5855	**.0000**	.2926
		256	-.0800	.3744	-.0316	.1774	.0169	.4075	.0030	.2085
		512	-.0647	.2914	-.0197	.1571	.0038	.3183	.0039	.1548
	.45	128	-.1677	.7477	-.0382	.2654	.0119	.5735	.0196	.2993
		256	-.0659	.5834	-.0211	.1877	.0088	.4223	.0086	.2253
		512	-.0614	.3915	-.0121	.1139	.0172	.3167	.0112	.1684
.80	-.25	128	-.1101	.5728	-.0861	.6038	**.0025**	.6042	.0065	.3114
		256	-.0132	.3504	-.0195	.1783	.0432	.4031	.0164	.2127
		512	.0063	.2797	-.0209	.1570	.0157	.3059	**.0044**	.1583
	0	128	-.1184	.5825	-.0871	.4965	.0093	.5788	**-.0030**	.2970
		256	-.1029	.3736	-.0511	.1955	-.0307	.4054	-.0203	.2217
		512	-.0781	.3071	-.0332	.1386	-.0235	.3167	-.0108	.1629
	.25	128	-.1361	.6247	-.0646	.3387	.0057	.5738	**.0003**	.3085
		256	-.0731	.3743	-.0297	.1896	.0208	.4105	**.0036**	.2164
		512	-.0490	.3081	-.0187	.1314	-.0033	.3172	**-.0013**	.1555
	.45	128	-.1655	.8452	-.0400	.2862	.0208	.5598	.0176	.3041
		256	-.0608	.3653	-.0158	.1838	.0363	.4153	.0176	.2193
		512	-.0377	.2819	-.0125	.1416	.0323	.3190	.0202	.1601

Table 19: Haar Wavelet OLS - ARFIMA (0,d,1)

θ	d	T	$J=K=0$ Bias	$J=K=0$ RMSE	$J=2,K=0$ Bias	$J=2,K=0$ RMSE	$J=0,K=2$ Bias	$J=0,K=2$ RMSE
-.40	-.25	128	-.2243	.2962	-.1648	**.2068**	-.2652	.4361
		256	-.1941	.2422	-.1508	.1772	-.2198	.3245
		512	-.1690	.2067	-.1292	.1468	-.1846	.2613
	0	128	-.2936	.3448	-.2753	.3034	-.3008	.4437
		256	-.2474	.2865	-.2257	.2425	-.2447	.3457
		512	-.2095	.2431	-.1936	.2060	-.1970	.2764
	.25	128	-.3278	.3797	-.3297	.3539	-.3067	.4667
		256	-.2655	.3014	-.2649	.2794	-.2337	.3342
		512	-.2175	.2470	-.2182	.2298	-.1804	.2573
	.45	128	-.3344	.3862	-.3448	.3654	-.3005	.4637
		256	-.2592	.2966	-.2752	.2897	-.2071	.3158
		512	-.2191	.2514	-.2212	.2314	-.1734	.2595
.40	-.25	128	.0134	**.1828**	.1765	.2144	-.1513	.3669
		256	.0134	.1552	.1443	.1696	-.1145	.2807
		512	.0093	**.1160**	.1175	.1360	-.0945	.2040
	0	128	-.0403	.1896	.1073	**.1638**	-.1856	.3857
		256	-.0260	.1471	.0816	.1253	-.1327	.2741
		512	-.0292	.1150	.0701	.0954	-.1192	.2118
	.25	128	-.0610	.1918	.0628	**.1374**	-.1801	.3762
		256	-.0594	.1584	.0450	**.1011**	-.1566	.2931
		512	-.0576	.1398	.0316	.0764	-.1367	.2436
	.45	128	-.0868	.2142	-.2212	.2314	-.2017	.4059
		256	-.0700	.1691	.0319	.1300	-.1566	.2991
		512	-.0630	.1405	.0273	.0940	-.1323	.2368
.80	-.25	128	.0490	.1994	.2406	.2733	-.1475	.3841
		256	.0512	.1611	.1852	.2071	-.0947	.2730
		512	.0394	.1271	.1552	.1714	-.0815	.2060
	0	128	**.0014**	.1862	.1722	.2112	-.1723	.3828
		256	**-.0075**	.1501	.1294	.1567	-.1465	.2899
		512	**-.0092**	.1233	.0999	.1227	-.1205	.2274
	.25	128	-.0387	.1967	.1120	**.1734**	-.1914	.3968
		256	-.0313	.1486	.0838	.1263	-.1482	.2837
		512	-.0288	.1305	.0682	.0994	-.1227	.2339
	.45	128	-.0518	.1917	.0812	**.1508**	-.1810	.3789
		256	-.0486	.1508	.0582	**.1068**	-.1524	.2832
		512	-.0370	.1244	.0466	.0857	-.1155	.2198

Table 20: Daubechies4 Wavelet OLS - ARFIMA (0,d,1)

			$J=K=0$		$J=2,K=0$		$J=0,K=2$	
θ	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.40	-.25	128	-.2465	.3061	-.2514	.2823	-.2290	.3994
		256	-.2170	.2610	-.2179	.2384	-.1976	.3120
		512	-.1803	.2158	-.1790	.1937	-.1554	.2413
	0	128	-.3047	.3556	-.3087	.3336	-.2840	.4389
		256	-.2423	.2777	-.2467	.2628	-.2096	.3081
		512	-.2118	.2419	-.2012	.2121	-.1825	.2592
	.25	128	-.2848	.3400	-.3312	.3547	-.2193	.4049
		256	-.2351	.2820	-.2616	.2759	-.1758	.3131
		512	-.1950	.2307	-.2108	.2210	-.1414	.2405
	.45	128	-.1616	.2414	-.3366	.3600	**.0226**	.3269
		256	-.1359	.2043	-.2679	.2849	**.0033**	.2534
		512	-.1112	.1604	-.2120	.2236	**-.0007**	.1797
.40	-.25	128	**.0035**	.1846	.1693	.2104	-.1669	.3752
		256	**.0011**	**.1470**	.1244	.1544	-.1314	.2765
		512	.0040	.1210	.1011	.1234	-.0988	.2135
	0	128	-.0351	.1955	.1245	.1788	-.1933	.4046
		256	-.0265	.1497	.0889	.1255	-.1450	.2851
		512	-.0310	.1257	.0714	.0982	-.1292	.2313
	.25	128	-.0175	.1846	.0897	.1534	-.1287	.3619
		256	-.0171	.1399	.0662	.1124	-.1061	.2564
		512	-.0249	.1287	.0501	.0836	-.1009	.2235
	.45	128	.0923	.2026	.0764	**.1496**	.0936	.3381
		256	.0656	.1608	.0541	**.1049**	.0488	.2522
		512	.0539	.1325	.0442	.0850	.0345	.1934
.80	-.25	128	.0538	.1882	.2439	.2750	-.1468	.3594
		256	.0465	.1510	.1756	.1988	-.1066	.2635
		512	.0274	.1250	.1399	.1578	-.1013	.2177
	0	128	.0116	.1882	.1982	.2334	-.1815	.3858
		256	-.0035	.1482	.1450	.1726	-.1585	.2917
		512	.0039	.1169	.1089	.1282	-.1133	.2162
	.25	128	.0110	.1923	.1531	.1999	-.1417	.3753
		256	.0102	.1503	.1147	.1490	-.1069	.2722
		512	.0063	.1173	.0897	.1150	-.0886	.2038
	.45	128	.1315	.2233	.1338	.1877	.1095	.3502
		256	.0967	.1839	.0950	.1367	.0616	.2689
		512	.0841	.1425	.0763	.1050	.0516	.1888

Table 21: Wavelet MLE - ARFIMA(0,d,1)

θ	d	T	Haar ($K=2$)		Haar ($K=4$)		Daub4 ($K=2$)		Daub4 ($K=4$)	
			Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
-.40	-.25	128	-.0975	.2160	-.1748	.6737	-.1053	.2175	**-.0644**	.6654
		256	-.0882	**.1459**	-.1068	.3445	-.1044	.1566	**-.0458**	.3204
		512	-.0782	**.1072**	-.0765	.2051	-.0937	.1240	**-.0325**	.1996
	0	128	-.1648	.2998	-.1503	.6527	-.1438	**.2425**	**-.1348**	.6758
		256	-.1326	.1880	-.1084	.3459	-.1138	**.1636**	**-.0631**	.3270
		512	-.1139	.1386	-.0702	.1977	-.0994	**.1245**	**-.0435**	.1865
	.25	128	-.1515	**.2388**	-.1454	.6694	-.1114	.2928	**-.0460**	.6559
		256	-.1286	.1694	-.0868	.3333	-.0894	**.1461**	**-.0271**	.3896
		512	-.1100	.1324	-.0537	.1911	-.0808	**.1101**	**-.0171**	.1919
	.45	128	-.1526	.2502	-.1716	.6557	.0419	**.2195**	.3396	.7912
		256	-.1156	.1664	-.0697	.3358	.0086	**.1423**	.2076	.4128
		512	-.0979	.1238	-.0367	.1830	-.0109	**.0928**	.1307	.2399
.40	-.25	128	.0290	.3640	-.0567	.6422	.0106	.2215	-.0651	.6411
		256	.0490	.1610	-.0311	.3790	.0101	.2045	-.0387	.3463
		512	.0337	.1568	**.0001**	.2513	.0192	.1778	-.0058	.2001
	0	128	.0073	.1887	-.1013	.6577	**.0071**	.1773	-.1099	.6599
		256	.0204	**.1139**	-.0394	.3224	**.0099**	.1116	-.0560	.3272
		512	.0238	**.0777**	-.0224	.1828	**.0156**	.0759	-.0333	.1847
	.25	128	**.0018**	.1721	-.0935	.6346	.0290	.1782	.0102	.6624
		256	**.0013**	.1106	-.0582	.3341	.0176	.1151	.0051	.3171
		512	.0056	**.0735**	-.0297	.1898	.0164	.0743	**-.0014**	.1882
	.45	128	**-.0196**	.1904	-.1172	.6681	.1398	.2461	.3591	.7409
		256	**-.0032**	.1134	-.0574	.3383	.0957	.1578	.2060	.4070
		512	**.0019**	**.0787**	-.0357	.1883	.0686	.1119	.1236	.2456
.80	-.25	128	.0577	.1899	-.0538	.6956	**.0252**	**.1845**	-.0757	.6719
		256	.0717	.1325	.0052	.3534	.0364	**.1146**	**-.0041**	.3253
		512	.0701	.1039	.0059	.2121	.0355	**.0833**	**-.0035**	.1897
	0	128	.0152	.1906	-.0922	.6520	.0090	**.1859**	-.0893	.6312
		256	.0250	.1193	-.0617	.3314	.0129	**.1183**	-.0745	.3523
		512	.0281	.0823	-.0342	.1893	.0201	**.0757**	-.0328	.1888
	.25	128	-.0041	.1952	-.0972	.6595	.0207	.1895	**-.0022**	.6443
		256	.0075	**.1195**	-.0341	.3310	.0242	.1199	**.0069**	.3159
		512	.0143	.0809	-.0266	.1851	.0243	**.0798**	**.0074**	.1829
	.45	128	**-.0079**	.1915	-.1018	.6289	.1560	.2612	.3997	.7596
		256	**-.0006**	.1149	-.0470	.3161	.1060	.1716	.2252	.4160
		512	**.0089**	**.0787**	-.0171	.1854	.0765	.1139	.1445	.2502

Separate Appendix to "Finite Sample Comparison of Parametric, Semiparametric, and Wavelet Estimators of Fractional Integration"

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July 28, 2005

Abstract

This appendix contains Tables 22-28 of Nielsen, M.Ø., and P.H. Frederiksen, 2005, "Finite sample comparison of parametric, semiparametric, and wavelet estimators of fractional integration", working paper, Cornell University. The tables present the simulation results for the ARFIMA(0,d,0)-ARCH(1) DGP (32) of the paper. For an explanation of the notation, etc., please see the paper.

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Table 22: Parametric Estimators - ARFIMA(0,d,0)-ARCH(1)

			EML		MPL		CML		FML	
β	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
.40	-.25	128	-.0281	.1041	-.0281	.1041	-.0281	**.1027**	**-.0039**	.1043
		256	-.0150	.0731	-.0150	.0731	-.0148	**.0725**	**-.0011**	.0729
		512	-.0088	.0494	-.0088	.0494	-.0084	**.0489**	**-.0012**	.0494
	0	128	-.0351	.1025	-.0351	.1025	-.0351	.1040	**-.0076**	**.1001**
		256	-.0162	.0675	-.0162	.0675	-.0162	.0681	**-.0016**	**.0671**
		512	-.0100	.0479	-.0100	.0479	-.0100	.0481	**-.0025**	**.0473**
	.25	128	-.0468	.1052	-.0468	.1052	-.0385	.1083	**-.0104**	**.1043**
		256	-.0250	.0709	-.0250	.0709	-.0207	.0719	**-.0058**	**.0705**
		512	-.0149	.0497	-.0149	.0497	-.0126	.0501	**-.0045**	**.0492**
	.45	128	-.0626	**.0965**	-.0626	**.0965**	-.0210	.1078	**.0086**	.1098
		256	-.0367	**.0639**	-.0367	**.0639**	-.0105	.0719	**.0067**	.0736
		512	-.0225	**.0444**	-.0225	**.0444**	-.0073	.0496	**.0024**	.0502
.80	-.25	128	-.0316	.1471	-.0316	.1471	-.0296	**.1445**	**-.0057**	.1499
		256	-.0160	.1165	-.0160	.1165	-.0145	**.1151**	**-.0017**	.1185
		512	-.0112	.0904	-.0112	.0904	-.0103	**.0896**	**-.0032**	.0907
	0	128	-.0388	**.1501**	-.0388	**.1501**	-.0365	.1550	**-.0085**	.1557
		256	-.0202	**.1191**	-.0202	**.1191**	-.0193	.1215	**-.0048**	.1220
		512	-.0091	**.0931**	-.0091	**.0931**	-.0087	.0943	**-.0001**	.0949
	.25	128	-.0473	**.1367**	-.0473	**.1367**	-.0356	.1488	**-.0071**	.1504
		256	-.0281	**.1073**	-.0281	**.1073**	-.0218	.1144	**-.0070**	.1143
		512	-.0156	**.0883**	-.0156	**.0883**	-.0123	.0922	**-.0041**	.0926
	.45	128	-.0845	**.1296**	-.0845	**.1296**	-.0394	.1494	**-.0096**	.1480
		256	-.0493	**.0887**	-.0493	**.0887**	-.0150	.1097	**.0021**	.1112
		512	-.0323	**.0685**	-.0323	**.0685**	-.0087	.0874	**.0007**	.0884

Table 23: Semiparametric I - ARFIMA (0,d,0)-ARCH(1)

β	d	T	LPR ($m = \lfloor T^{0.5} \rfloor$) Bias	RMSE	LPR ($m = \lfloor T^{0.65} \rfloor$) Bias	RMSE	PLPR ($m = \lfloor T^{0.5} \rfloor$) Bias	RMSE	PLPR ($m = \lfloor T^{0.65} \rfloor$) Bias	RMSE
.40	-.25	128	.0100	.2694	.0108	.1683	.0044	.2549	.0066	.1676
		256	.0080	.2100	.0020	.1336	.0076	.1999	**.0009**	.1320
		512	.0060	.1708	.0088	.1028	.0040	.1641	.0074	.1010
	0	128	.0021	.2707	-.0014	.1673	.0071	.2578	**-.0003**	.1659
		256	**.0005**	.2053	.0026	.1241	.0006	.1964	.0023	.1219
		512	-.0042	.1675	.0014	.0956	-.0045	.1611	.0013	.0942
	.25	128	-.0110	.2768	-.0125	.1658	-.0098	.2666	-.0097	.1655
		256	-.0101	.2126	-.0116	.1313	-.0078	.2045	-.0077	.1290
		512	**.0018**	.1668	-.0021	.0965	.0048	.1610	-.0020	.0950
	.45	128	.0117	.2726	**.0042**	.1659	.0149	.2641	.0123	.1645
		256	.0162	.2072	.0096	.1243	.0135	.2012	.0135	.1232
		512	.0085	.1697	.0070	.0947	.0094	.1643	.0092	.0933
.80	-.25	128	.0021	.2782	.0048	.1953	.0008	.2656	-.0009	.1930
		256	-.0023	.2017	**-.0009**	.1460	-.0022	.1960	-.0030	.1408
		512	**.0002**	.1700	.0020	.1129	.0002	.1608	.0009	.1092
	0	128	**-.0004**	.2735	-.0023	.1943	-.0020	.2609	**-.0004**	.1891
		256	.0038	.2085	-.0005	.1486	.0009	.1995	**-.0001**	.1450
		512	.0040	.1727	**.0015**	.1119	.0036	.1690	**.0015**	.1082
	.25	128	-.0014	.2754	-.0099	.1937	**-.0008**	.2615	-.0040	.1903
		256	-.0069	.2177	-.0044	.1545	-.0096	.2112	-.0027	.1486
		512	-.0027	.1658	-.0066	.1173	**-.0006**	.1594	-.0047	.1121
	.45	128	.0037	.2741	-.0113	.1918	**.0015**	.2616	-.0042	.1877
		256	**-.0005**	.2156	.0028	.1505	.0006	.2041	.0083	.1480
		512	.0086	.1710	**.0025**	.1108	.0097	.1623	.0040	.1074

Table 24: Semiparametric II - ARFIMA (0,d,0)-ARCH(1)

			LW ($m = \lfloor T^{0.5} \rfloor$)		LW ($m = \lfloor T^{0.65} \rfloor$)		FELW ($m = \lfloor T^{0.5} \rfloor$)		FELW ($m = \lfloor T^{0.65} \rfloor$)	
β	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
.40	-.25	128	-.0175	.2337	-.0040	**.1424**	-.0266	.2410	**.0024**	.1458
		256	-.0102	.1735	-.0046	**.1089**	-.0181	.1762	-.0032	.1110
		512	-.0128	.1395	-.0010	**.0815**	-.0178	.1413	**-.0004**	.0824
	0	128	-.0320	.2302	-.0175	**.1370**	-.0309	.2361	-.0019	.1391
		256	-.0167	.1744	-.0085	**.1024**	-.0165	.1754	-.0006	.1039
		512	-.0142	.1391	-.0066	**.0760**	-.0143	.1386	-.0021	**.0760**
	.25	128	-.0465	.2418	-.0233	**.1419**	-.0368	.2447	**.0000**	.1447
		256	-.0225	.1773	-.0162	.1083	-.0177	.1782	**-.0054**	**.1071**
		512	-.0149	.1395	-.0113	.0774	-.0134	.1421	-.0056	**.0764**
	.45	128	-.0192	.2262	-.0107	**.1385**	-.0069	.2273	.0276	.1462
		256	-.0099	.1782	-.0010	**.1054**	**-.0007**	.1806	.0201	.1144
		512	-.0090	.1416	**.0012**	**.0812**	-.0035	.1433	.0148	.0914
.80	-.25	128	-.0232	.2381	-.0104	**.1644**	-.0300	.2486	-.0049	.1689
		256	-.0136	.1699	-.0072	**.1199**	-.0179	.1743	-.0041	.1225
		512	-.0134	.1410	-.0033	**.0933**	-.0188	.1424	-.0028	.0943
	0	128	-.0277	.2367	-.0182	**.1666**	-.0272	.2451	-.0048	.1706
		256	-.0169	.1732	-.0132	**.1244**	-.0164	.1746	-.0042	.1265
		512	-.0105	.1372	-.0068	**.0932**	-.0116	.1399	-.0021	.0942
	.25	128	-.0233	.2304	-.0239	**.1643**	-.0101	.2383	-.0014	.1667
		256	-.0266	.1812	-.0178	**.1314**	-.0222	.1850	-.0064	.1321
		512	-.0193	.1397	-.0118	.0974	-.0170	.1446	-.0065	**.0973**
	.45	128	-.0185	.2381	-.0238	**.1668**	-.0053	.2304	.0085	.1698
		256	-.0111	.1798	-.0053	**.1224**	-.0016	.1730	.0149	.1300
		512	-.0063	.1403	-.0043	**.0893**	.0032	.1421	.0083	.0983

Table 25: Semiparametric III - ARFIMA (0,d,0)-ARCH(1)

			LPW ($m = \lfloor T^{0.5} \rfloor$)		LPW ($m = \lfloor T^{0.65} \rfloor$)		BRLPR ($m = \lfloor T^{0.5} \rfloor$)		BRLPR ($m = \lfloor T^{0.65} \rfloor$)	
β	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
.40	-.25	128	-.1290	.7423	-.0376	.2600	.0105	.5751	.0127	.3003
		256	-.0602	.3479	-.0205	.1812	.0122	.4048	.0106	.2198
		512	-.0580	.2793	-.0182	.1407	.0024	.2941	.0076	.1601
	0	128	-.1253	.5348	-.0607	.2639	.0015	.5819	-.0051	.2999
		256	-.0622	.3651	-.0265	.1806	.0105	.3928	.0046	.2131
		512	-.0560	.2753	-.0189	.1327	-.0058	.3065	**-.0002**	.1607
	.25	128	-.1439	.8190	-.0684	.2701	.0145	.5883	-.0061	.3022
		256	-.0743	.4059	-.0362	.1828	-.0143	.4105	-.0133	.2159
		512	-.0637	.3285	-.0250	.1311	-.0060	.3161	-.0059	.1536
	.45	128	-.1599	.9706	-.0361	.2549	-.0083	.6041	.0140	.2971
		256	-.0891	.5893	-.0154	.1820	.0138	.3973	.0252	.2148
		512	-.0255	.4888	-.0145	.1369	.0338	.3183	.0101	.1575
.80	-.25	128	-.1109	.5447	-.0466	.2702	-.0030	.5544	**.0005**	.3084
		256	-.0667	.3441	-.0220	.1740	.0018	.3798	.0025	.2066
		512	-.0490	.2644	-.0159	.1381	.0062	.2978	-.0011	.1606
	0	128	-.1226	.5194	-.0542	.3227	.0007	.5624	.0066	.3076
		256	-.0677	.3361	-.0250	.1798	-.0008	.4016	.0052	.2160
		512	-.0431	.2576	-.0161	.1389	-.0074	.3084	.0038	.1584
	.25	128	-.1387	.6261	-.0465	.2626	-.0130	.5619	-.0012	.3128
		256	-.0768	.4317	-.0320	.1869	.0161	.3894	**.0012**	.2278
		512	-.0693	.2756	-.0250	.1379	-.0150	.3043	-.0047	.1567
	.45	128	-.1364	.7163	-.0378	.2601	.0067	.5438	.0141	.3055
		256	-.0785	.5197	-.0229	.1853	.0043	.4097	.0012	.2239
		512	-.0488	.2919	-.0125	.1296	.0125	.3068	.0063	.1566

Table 26: Haar Wavelet OLS - ARFIMA (0,d,0)-ARCH(1)

			$J = K = 0$		$J = 2, K = 0$		$J = .K = 2$	
β	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE
.40	-.25	128	-.0823	.2125	.0142	.1402	-.1722	.3938
		256	-.0662	.1608	.0163	**.0982**	-.1365	.2788
		512	-.0484	.1251	.0171	.0754	-.1015	.2070
	0	128	-.1269	.2216	-.0436	**.1401**	-.1988	.3833
		256	-.1009	.1804	-.0342	**.0992**	-.1522	.2923
		512	-.0918	.1529	-.0289	.0770	-.1349	.2350
	.25	128	-.1563	.2502	-.0880	.1628	-.2118	.4088
		256	-.1326	.2007	-.0705	.1193	-.1759	.3070
		512	-.1130	.1647	-.0571	.0928	-.1452	.2359
	.45	128	-.1724	.2650	-.1026	.1699	-.2274	.4325
		256	-.1322	.2023	-.0815	.1274	-.1619	.2997
		512	-.1094	.1633	-.0685	.1015	-.1297	.2287
.80	-.25	128	-.0873	.2101	.0122	**.1624**	-.1762	.3731
		256	-.0619	.1698	.0151	.1202	-.1262	.2816
		512	-.0472	.1303	.0107	.0888	-.0956	.2078
	0	128	-.1200	.2272	-.0415	**.1685**	-.1875	.3810
		256	-.1023	.1842	-.0322	**.1189**	-.1557	.2904
		512	-.0863	.1564	-.0245	**.0896**	-.1275	.2369
	.25	128	-.1568	.2544	-.0922	.1854	-.2095	.4071
		256	-.1308	.2060	-.0729	.1400	-.1695	.3064
		512	-.1159	.1716	-.0571	.1072	-.1478	.2447
	.45	128	-.1703	.2542	-.1207	.1980	-.2010	.3884
		256	-.1389	.2095	-.0898	.1441	-.1665	.3040
		512	-.1179	.1742	-.0678	.1090	-.1390	.2409

Table 27: Daubechies4 Wavelet OLS - ARFIMA (0,d,0)-ARCH(1)

			$J = K = 0$		$J = 2, K = 0$		$J = .K = 2$	
β	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE
.40	-.25	128	-.1003	.2252	**.0005**	**.1349**	-.1905	.4085
		256	-.0761	.1657	-.0034	.0998	-.1377	.2799
		512	-.0713	.1454	-.0062	**.0737**	-.1228	.2325
	0	128	-.1258	.2219	-.0505	.1440	-.1970	.3851
		256	-.1048	.1807	-.0381	.1041	-.1586	.2898
		512	-.0912	.1477	-.0269	**.0736**	-.1334	.2248
	.25	128	-.1159	.2169	-.0726	**.1501**	-.1499	.3631
		256	-.1058	.1956	-.0568	**.1137**	-.1380	.3061
		512	-.0814	.1460	-.0422	.0809	-.1020	.2165
	.45	128	**.0044**	.1864	-.0776	**.1578**	.0786	.3427
		256	**-.0007**	.1467	-.0552	**.1124**	.0438	.2485
		512	**-.0066**	.1189	-.0484	.0887	.0232	.1874
.80	-.25	128	-.0989	.2218	-.0173	.1660	-.1724	.3819
		256	-.0822	.1789	-.0163	**.1193**	-.1383	.2905
		512	-.0684	.1459	-.0153	**.0883**	-.1106	.2270
	0	128	-.1321	.2410	-.0592	.1834	-.1971	.3985
		256	-.1034	.1837	-.0372	.1234	-.1513	.2847
		512	-.0920	.1550	-.0301	.0947	-.1332	.2311
	.25	128	-.1039	.2174	-.0702	**.1726**	-.1281	.3549
		256	-.0883	.1738	-.0565	**.1276**	-.1079	.2603
		512	-.0788	.1448	-.0447	**.0971**	-.0962	.2070
	.45	128	**.0082**	.1939	-.0859	**.1817**	.0981	.3591
		256	**.0006**	.1541	-.0624	**.1337**	.0525	.2607
		512	**-.0024**	.1207	-.0514	**.0982**	.0332	.1902

Table 28: Wavelet MLE - ARFIMA(0,d,0)-ARCH(1)

			Haar ($K=2$)		Haar ($K=4$)		Daub4 ($K=2$)		Daub4 ($K=4$)	
β	d	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
.40	-.25	128	.0094	.1921	-.0679	.6902	-.0027	.1931	-.0880	.6554
		256	.0139	.1254	-.0270	.3335	**.0032**	.1229	-.0307	.3280
		512	.0247	.0825	-.0061	.1807	**.0056**	.0802	-.0182	.1933
	0	128	**-.0308**	.2394	-.1123	.7189	-.0309	.1854	-.0675	.6532
		256	-.0157	.1331	-.0477	.3344	**-.0138**	.1247	-.0509	.3829
		512	-.0134	.1512	-.0249	.1861	**-.0119**	.1056	-.0259	.2034
	.25	128	-.0434	.1981	-.1124	.6516	**-.0123**	.1935	-.0329	.6366
		256	-.0269	.1239	-.0513	.3214	**-.0030**	.1239	-.0136	.3319
		512	-.0207	**.0791**	-.0367	.1897	**-.0036**	.0783	-.0051	.1874
	.45	128	-.0393	.1989	-.1115	.6747	.1232	.2400	.3907	.7305
		256	-.0235	.1282	-.0398	.3416	.0779	.1534	.2212	.3985
		512	-.0177	**.0842**	-.0262	.1917	.0483	.1007	.1236	.2421
.80	-.25	128	**.0055**	.2334	-.1012	.6475	-.0065	.2310	-.0693	.6389
		256	.0218	.1511	-.0280	.3377	**-.0021**	.1526	-.0250	.3482
		512	.0235	.1137	-.0072	.2069	**.0041**	.1045	-.0049	.2078
	0	128	-.0206	.2077	-.1098	.6465	**-.0158**	.2181	-.0595	.6600
		256	-.0137	.1496	-.0507	.3174	**-.0110**	.1472	-.0414	.3354
		512	-.0063	.1018	-.0268	.1959	**-.0028**	.1101	-.0197	.1901
	.25	128	-.0318	.2175	-.1064	.6712	**.0046**	.2062	-.0121	.6256
		256	-.0259	.1464	-.0537	.3421	**-.0036**	.1432	-.0039	.3295
		512	-.0190	.1112	-.0382	.1986	**-.0021**	.1049	-.0029	.1876
	.45	128	-.0331	.2189	-.0988	.6558	.1403	.2632	.3984	.7597
		256	-.0204	.1468	-.0464	.3474	.0877	.1907	.2162	.4282
		512	-.0147	.1035	-.0271	.2062	.0576	.1231	.1378	.2623