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Fair Collective Choice Rules: Their Origin and Relationship

Yukihiro Nishimura
Yokohama National University and Queen's University

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

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Yukihiro Nishimura
Department of Economics, Yokohama National University, 79-3 Tokiwadai, Hodogaya-ku, Yokohama, 240-8501, Japan.*†

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Abstract

This paper provides a conceptual framework on fair collective choice rules that synthesizes the studies of Goldman and Sussangkarn (1978) and Suzumura (1981) on the one hand and Tadenuma (2002, 2005) on the other. We show that both frameworks have the following binary relation as a common origin: an allocation x is at least as good as an allocation z if (i) x Pareto dominates z, or (ii) x equity-dominates z. Its transitive-closure and the strict relation derive different ranking criteria, but remarkably, with respect to the maximal elements, they have a set-inclusive relationship.

Keywords: Welfare Economics, Social Choice, Efficiency, Equity, No-Envy

1 Introduction

There has been a large literature which pointed out a fundamental conflict between equity and efficiency. In the context of equity as *no-envy*, the conflict was first shown by Kolm (1972), Feldman and Kirman (1974) and Pazner and Schmeidler (1974). On the other hand, it has been a vital problem in welfare economics and social choice theory to construct ranking and choice criteria based on Pareto efficiency accompanied with a suitable notion of equity. The New Welfare Economics bases its analysis on ordinally non-comparable utilities, and attempts have been made in constructing choice rules that extend the Pareto

*Tel: +81-45-339-3569. Fax: +81-45-339-3574. E-mail: ynishimu@ynu.ac.jp

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criterion. In re-examining the Kaldor-Hicks-Scitovsky Hypothetical Compensation Tests and their drawback of cyclicity, Little (1957) emphasized the need of incorporating distributional value judgment.[1] The concept of no-envy satisfies the classic framework of ordinal non-comparability, and the social choice literature formalized several collective choice rules that rationalize certain binary relations based on Pareto efficiency and no-envy.[2]

Goldman and Sussangkarn (1978) found that the set of Pareto efficient and envy-free allocations are rationalized by the transitive closure of the following binary relation: an allocation x is at least as good as an allocation z (hereafter referred to as $xR^{*}y$) if (i) x Pareto dominates z, or (ii) x equity-dominates z. As such, R^{*} is a weak form of the ranking based on equity-as-no-envy and Pareto efficiency. Suzumura (1981) formalized in a general social-choice framework, and showed that the choice rule by Goldman and Sussangkarn (1978) (hereafter 'GS rule') satisfies important equity and efficiency properties. Tadenuma (1998), on the other hand, introduced two contrasting principles to socially rank allocations: the *efficiency-first principle* ranks an allocation x higher than z if (i) x Pareto dominates z or (ii) x and z are Pareto-noncomparable and x is equitable whereas z is not; the equity-first principle reverses the order of application of the two criteria. Tadenuma (2002, 2005) extended the analysis on equity-as-no-envy and egalitarian-equivalence respectively, and Nishimura (2000) characterized these two rules from the following *general rule for weighing equity and efficiency* (hereafter 'a weighing rule'): x is socially better than z if x is more efficient or more equitable than z, and the other criterion does not make the opposite suggestion. The weighing rule accommodates standpoints in classic welfare economics in that Pareto Principle is not sufficient for decision-making without taking an equity criterion into account (e.g., Little (1957)).

Tadenuma (1998) correctly noted that Tadenuma's (1998, 2002, 2005) framework is

[1] A number of works followed his provocative argument, focusing on the logical consistency of the ranking rule. Especially, Sen (1963) pointed out that a key for consistent value judgment is transitivity of the criterion. See Suzumura (1980) for an excellent review of a large literature and a formal analysis that investigated the consistency of all proposed criteria.

[2] In an early work, Varian (1974) provided insightful discussions on this issue.

independent of Suzumura (1981) in terms of the domain of social-choice problems and the logical relationships between axioms under consideration. Having acknowledged this fact, we show that both frameworks have R^* as a common origin (Theorems 1 and 2). Its transitive-closure and the strict relation derive different ranking criteria (Examples 1 and 2), but remarkably, with respect to the maximal elements, they have a set-inclusive relationship, so that one can be regarded as a refinement of the other (Theorem 3).

2 The Model

In an economy there are n agents. Each agent $i \in N \equiv \{1, ..., n\}$ has a preference relation represented by a utility function $u_i : X \to I\!R$ defined over a set X. Each allocation determines each agent's consumption bundle. Let $S^* \subset X^n$ be the set of feasible allocations in an economy: any $x \in S^*$ is a list of $(x_1, ..., x_n)$, where x_i is a consumption bundle of agent i $(i = 1, ..., n)$.

We examine the problem of ranking allocations among a set of allocations, $S \subset S^*$, based on Pareto efficiency and equity as no-envy. For allocations $x, z \in X^n$, $x \succ_P z$ if and only if $u_i(x_i) \geq u_i(z_i)$ for all $i = 1, ..., n$ with strict inequality for at least one i: we call these situations that x *Pareto dominates* z. An allocation $x \in S$ is *Pareto efficient* in S if and only if there is no $z \in S$ such that $z \succ_P x$. Let $P(S)$ be the set of Pareto efficient allocations in S. An allocation $x \in X^n$ is *envy-free (or no-envy)* if and only if $u_i(x_i) \geq u_i(x_j)$ for all $i \in N$ and $j \in N$. Let $F(S)$ be the set of envy-free allocations in S. For all $x, z \in S$, $x \succ_{F_E} z$ if and only if $x \in F(S)$ and $z \notin F(S)$.[3]

For a binary relation R, let $P(R)$ be a strict ordering of R: $xP(R)z$ iff xRz and not zRx. The set of *maximal elements* can be defined as $M_R(S) \equiv \{x \in S | \nexists z \in S, \ zRx\}$. For binary relations R_1 and R_2, R_1 *includes* R_2 iff xR_2z implies xR_1z for all x and z.

[3]Tadenuma (2002) considered the refined ordering based on Feldman-Kirman's (1974) cardinal measure, whereas Tadenuma (1998, 2007) considered the choice problem that corresponds to F_E.

3 Ranking Allocations Based on Equity and Efficiency

3.1 R^* and GS rule

Goldman and Sussangkarn (1978) explored the construction of social orderings based on the following binary-relation R^*:

$$xR^*z \iff x \succ_P z \text{ or } x \succ_{F_E} z. \tag{1}$$

Namely, an allocation x is at least as good as an allocation z if (i) x Pareto dominates z, or (ii) x equity-dominates z. In constructing social orderings based on equity-as-no-envy and Pareto efficiency, this is a natural starting point.

Goldman and Sussangkarn (1978) and Suzumura (1981) considered the *transitive closure* of R^*, $T(R^*)$.

$$xT(R^*)z \iff \exists z^1, z^2, ..., z^t \in S \text{ s.t. } xR^*z^1, z^1R^*z^2, \ ..., \ z^tR^*z.$$

Let $GS(S) \equiv \{x \in S | \forall z \in S, \ xT(R^*)z \text{ or } \neg zT(R^*)x\}$ (hereafter referred to as the GS rule). It is easy to see the following:

$$GS(S) = M_{P(T(R^*))}(S). \tag{2}$$

Namely, *the GS rule is the maximal elements of the strict ordering of the transitive closure of* R^*. Goldman and Sussangkarn (1978, Theorem 4) showed that, if $P(S) \cap F(S) \neq \emptyset$, then $GS(S) = P(S) \cap F(S)$. Suzumura (1981) called this property as *Fairness Extension*: if there are Pareto efficient and equitable allocations, then they should be all selected. Suzumura (1981, Theorems 1 and 3) also introduced a complementary condition of *Fairness Inclusion*[4], and proved that the GS rule is the smallest choice rule that satisfies the Fairness Inclusion.

[4] If a choice function selects $z \in S$, then any allocation $x \in S$ such xR^*z should also be selected.

3.2 Nishimura (2000) and Tadenuma (2002, 2005)

Nishimura (2000) examined a class of the social choice orderings called *the general rule for weighing equity and efficiency* (hereafter 'a weighing rule'), based on equity and efficiency relation. Define a relation $\succ_w$ such that:

$$x \succ_w z \iff \text{(a) } x \succ_P z \text{ and } \neg(z \succ_{F_E} x) \text{ or (b) } x \succ_{F_E} z \text{ and } \neg(z \succ_P x). \tag{3}$$

It basically suggests that x is socially better than z if x is more efficient or more equitable than z, *and* the other criterion does *not* make the opposite suggestion. The conditions subsume widely-acceptable standpoints in classic welfare economics, in that Pareto Principle is not sufficient for decision-making without taking an equity criterion into account (e.g., Little (1957)).[5] It is immediate to see that R^* includes $\succ_w$, but the converse may not necessarily hold: in the conflicting situations where $x \succ_P z$ and $z \succ_{F_E} x$, xR^*z and zR^*x (R^* recommends *indifference*), but $\succ_w$ reserves the judgment.

Other than R^*, two representative social choice rules that include $\succ_w$ are Tadenuma's (2002, 2005) *Efficiency-first principle* ($\succ_{PF_E}$) and *Equity-first principle* ($\succ_{F_EP}$). The efficiency-first principle ranks an allocation x higher than z ($x \succ_{PF_E} z$) if and only if (a) $x \succ_P z$ or (b) $x \not\succ_P z$, $z \not\succ_P x$ and $x \succ_{F_E} z$. As to the equity-first principle, $x \succ_{F_EP} z$ if and only if (a) $x \succ_{F_E} z$ or (b) $x \not\succ_{F_E} z$, $z \not\succ_{F_E} x$ and $x \succ_P z$.[6] Both of them include a relation $\succ_w$ in common, *and* give additional judgments in the conflicting situations where $x \succ_P z$ and $z \succ_{F_E} x$. Namely, each principle gives a priority to one criterion over the other lexicographically, as a natural resolution to the equity-efficiency trade-off.[7]

[5] Condition (a) is motivated from a standpoint that the Paretian relationship is not sufficient for allocation x to be better than z. Condition (b) is an extension of the classical relation in welfare economics to make an equity-comparison among Pareto noncomparable states.

[6] Condition (b) of $\succ_{PF_E}$ is called *P-conditional no-envy* by Tadenuma (1998, 2005, 2007). Condition (b) of $\succ_{F_EP}$ is a complementary condition of *F-conditional Pareto*. Tadenuma (1998, 2007) uses the relation $\succ_{F_E}$ as an equity relationship, whereas Tadenuma (2002) used the fairness ordering based on Feldman and Kirman (1974), and Tadenuma (2005) used the egalitarian-equivalence. The analysis of the present paper remains valid for any of these fairness orderings.

[7] As another example in an economic environment where $X = \mathbb{R}_+^{|M|}$, $M = \{1, ..., m\}$, one can consider

4 Main Theorems

For binary relations R_1 and R_2, $R_1 = R_2$ iff $xR_1z \iff xR_2z$ for all $x, z \in X^n$. The following theorems clarify a connection between R^* in Goldman and Sussangkarn (1978) and Nishimura (2000) and Tadenuma's (2002, 2005):

Theorem 1 $P(R^*) = \succ_w$.

Proof: Suppose that $xP(R^*)z$. This implies $[x \succ_P z$ or $x \succ_{F_E} z]$ and not $[z \succ_P x$ or $z \succ_{F_E} x]$. Consider first the case that $x \succ_P z$. By the nature of the Pareto ranking, $z \succ_P x$ would not happen. To prevent zR^*x to happen, $z \succ_{F_E} x$ should not happen. Consider next the case that $x \succ_{F_E} z$. By the same logic, $z \succ_P x$ should not happen. Taking these two consequences together, we conclude that $x \succ_w z$. Examination of (3) immediately leads to conclude that $x \succ_w z$ implies $xP(R^*)z$. *Q.E.D.*

For binary relations R_1 and R_2, let $R_1 \cup R_2$ be a binary relation that indicates $xR_1 \cup R_2z \iff xR_1z$ or xR_2z for all $x, z \in X^n$.

Theorem 2 *(i)* $\succ_P \cup P(R^*) = \succ_{PF_E}$. *(ii)* $\succ_{F_E} \cup P(R^*) = \succ_{F_EP}$. *(iii)* $\succ_{PF_E} \cup \succ_{F_EP} = R^*$.

Theorem 2.(i) and 2.(ii) show a dual property: $\succ_{PF_E}$ (resp. $\succ_{F_EP}$) is characterized as an efficient (resp. equitable) part of the weighing rule. Theorem 2.(iii) means the union composes R^*.

Proof: From Theorem 1, we refer to (3) in examining $P(R^*)$. Suppose that $x \succ_P \cup P(R^*)z$. Then, condition (a) of $\succ_{PF_E}$ is automatically satisfied by $\succ_P$. To see that condition (b) of $\succ_{PF_E}$ is satisfied, suppose that $x \not\succ_P z$ and $z \not\succ_P x$. Then only condition (b) of (3) can be used for ranking x and z. It implies that $x \succ_{F_E} z$ has to hold. Therefore, we conclude that $x \succ_P \cup P(R^*)z$ implies $x \succ_{PF_E} z$. Suppose next $x \succ_{PF_E} z$. Then condition

the following *Conditional Equity-first Principle*: For some reference level $\bar{x} \in X$, *if* $\min\{x_i^l, z_i^l\} < \bar{x}^l$ *for some* $l \in M$ *and* $i \in N$, *then* $x \succ z$ (efficiency is given priority when some person's consumption bundle is below some poverty line). Otherwise, $z \succ x$ (equity is given priority otherwise).

(a) of $\succ_{PF_E}$ implies $x \succ_P z$ and condition (a) of (3). Suppose that $x \succ_{F_E} z$ and $\neg(z \succ_P x)$. There are two possibilities: $x \succ_P z$ or $\neg(x \succ_P z)$. Applying conditions (a) and (b) of $\succ_{PF_E}$ respectively, we conclude that x is ranked higher than z, which is condition (b) of (3). This completes part (i) of the theorem. Part (ii) of the theorem is shown exactly the same way. To show (iii), $\succ_{PF_E} \cup \succ_{F_E P} = \succ_P \cup \succ_{F_E} \cup P(R^*) \ = \ R^* \cup P(R^*) \ = \ R^*$. *Q.E.D.*

If we define a common relationship by an operator $\cap$, then, as a dual version of Theorem 2.(iii), $\succ_{PF_E} \cap \succ_{F_E P} = \ P(R^*)$.

These two theorems and the previous results on GS rule clarify the common origin as well as the difference of the GS rule and the weighing rule. Both have R^* ranking as a basic relationship. The GS rule extends the transitive-closure of R^*, whereas the weighing rule and Tadenuma's (2002, 2005) two rules are derived from the strict relationship, $P(R^*)$. Our discussion so far is summarized as follows:

Corollary 1

$$P(T(R^*)) \ \Rightarrow \ T(R^*) \ \Leftarrow \ R^* \ \iff \succ_{PF_E} \cup \succ_{F_E P} \Leftarrow \succ_{PF_E} \cap \succ_{F_E P} \iff \ P(R^*) \ \iff \succ_w,$$

where, for example, $P(T(R^*)) \ \Rightarrow \ T(R^*)$ *indicates a fact that* $T(R^*)$ *includes* $P(T(R^*))$.

Notice that $P(T(R^*))$ is the base binary relation of $GS(S)$ according to (2), so that Corollary 1 clarifies the relationship between GS rule and Tadenuma's (2002, 2005) rules.

The following examples show that $P(T(R^*))$ and $P(R^*)$ do not have logical relationships:

Example 1 Suppose that $S = \{x, x', z\}$, $F(S) = \{x, z\}$, $x \in P(S)$, $x' \succ_P z$ and $\neg(x \succ_P z)$. $x \succ_{F_E} x'$ and $x' \succ_P z$, so $xT(R^*)z$. On the other hand, neither zR^*x nor $x'R^*x$, so that $\neg(zT(R^*)x)$, i.e., $xP(T(R^*))z$. However, by (3), $xP(R^*)z$ does not hold.

Example 2 Suppose that $S = \{x, x', z\}$, $F(S) = \{z\}$ and $x \succ_P x'$, $x' \succ_P z$. By (3), $xP(R^*)x'$. But $x' \succ_P z$ and $z \succ_{F_E} x$ imply $x'T(R^*)x$, so $xP(T(R^*))x'$ does not hold.

However, with respect to the *maximal elements*, there are the following notable relationships.

Theorem 3 *(i) When $P(S) \cap F(S) \neq \emptyset$, $GS(S) \subset M_{P(R^*)}(S)$ and $GS(S) \subset M_{\succ_{PF_E}}(S) \cup M_{\succ_{F_E P}}(S)$. (ii) When $P(S) \cap F(S) = \emptyset$, $M_{P(R^*)}(S) \subset GS(S)$ and $M_{\succ_{PF_E}}(S) \cup M_{\succ_{F_E P}}(S) \subset GS(S)$.*

Proof: In the Appendix we show that $M_{P(R^*)}(S) = M_{\succ_{PF_E}}(S) \cup M_{\succ_{F_E P}}(S)$. For (i), $GS(S) = P(S) \cap F(S)$ as introduced above, and $M_{\succ_{PF_E}}(S) = P(S) \cap F(S)$ by Tadenuma (Proposition 3.(i)). For (ii), it can be shown that, if $P(S) \cap F(S) = \emptyset$, $GS(S) = F(S) \cup \{x | \exists z \in F(S),\ x \succ_P z\}$. Also, $M_{\succ_{F_E P}}(S) = P(F(S))$ and $M_{\succ_{PF_E}}(S) = \{x \in S | x \in P(S) \text{ and } x \succ_P z\ \forall z \in F(S)\}$ (Tadenuma (2002, Propositions 3.(ii)[8] and 5)). Clearly, both are subsets of $GS(S)$. *Q.E.D.*

The following examples show that the converse set inclusions in (ii) and (iii) do not hold:

[$P(S) \cap F(S) \neq \emptyset$ and $GS(S) \not\supset M_{P(R^*)}(S)$]: In Example 1, $GS(S) = \{x\}$. On the other hand, $M_{\succ_{PF_E}}(S) = \{x\}$ and $M_{\succ_{F_E P}}(S) = \{x, z\}$, so $M_{P(R^*)} = \{x, z\}$.

The fact that there is no *envy-free* allocation that Pareto dominates z will make it survive in the weighing rule (in other words, $M_{P(R^*)}(S)$ does not satisfy the Fairness Extension), whereas the GS rule satisfies the Fairness Extension.

[$P(S) \cap F(S) = \emptyset$ and $M_{P(R^*)}(S) \not\supset GS(S)$]: In Example 2, $GS(S) = \{x, x', z\}$. On the other hand, $M_{\succ_{PF_E}}(S) = \{x\}$ and $M_{\succ_{F_E P}}(S) = \{z\}$, so $M_{P(R^*)} = \{x, z\}$.

The x' belongs to $GS(S)$ from the fact that x' Pareto dominates an equitable state z. On the other hand, the standpoint of Nishimura (2000) and Tadenuma (2002, 2005) is that, as long as that x' is Pareto dominated by another feasible allocation x, there is no reason for x' to belong to the choice set. A property of $GS(S)$ in Example 2 — no finer selection

[8] As noted in Tadenuma (2002, footnote 8), his Proposition 3.(ii) is originally shown in Nishimura (2000).

from S — is reflected in its violation of the *Chernoff's Axiom*, as shown in Suzumura (1981); similarly, from Example 1 one can see the violation of the complementary axiom of the *Superset Axiom*.[9] On the other hand, one can show that $M_{\succ_{F_E P}}(S)$ satisfies the Chernoff's Axiom and the Superset Axiom, and $M_{\succ_{PF_E}}(S)$ satisfies the Chernoff's Axiom (Nishimura (2000, Lemma 2)). See also Nishimura (2007) and Tadenuma (2002, 2007) for other results for $M_{\succ_{F_E P}}(S)$ and $M_{\succ_{PF_E}}(S)$.[10]

Studies including Kolm (1972), Feldman and Kirman (1974) and Pazner and Schmeidler (1974) show fundamental trade-offs in the sense that $P(S) \cap F(S) = \emptyset$ in many situations. In such situations, $M_{R^*}(S) = M_{T(R^*)}(S) = P(S) \cap F(S) = \emptyset$, so that, following Corollary 1, it is vital to see the class of allocations that $GS(S)$ and $M_{P(R^*)}(S)$ specify. $GS(S)$ selects *all* envy-free allocations and *all* allocations that Pareto dominates *at least one* envy-free allocation. Such a choice set is too large and does not effectively narrow down the candidates for a social choice. Theorem 3.(ii) is informative in that Nishimura (2000) and Tadenuma's (2002, 2005) criteria are *refinements* of the GS rule. Nishimura (2007) applied the analysis in production economies where individuals have different productivities, in both first-best and second-best environments.

5 Concluding Remarks

The present paper is part of a larger project in welfare economics and social choice theory to construct ranking and choice criteria based on Pareto efficiency accompanied with a suitable notion of equity. The concept of no-envy satisfies the classic framework of ordinal

[9] A choice function $C(\cdot)$ satisfies the Chernoff's Axiom if, for all S_1 and S_2 with $S_1 \subset S_2$, $S_1 \cap C(S_2) \subset C(S_1)$ has to hold. In Example 2, set $S_1 = \{x, x'\}$ and $S_2 = S$. Then $x' \notin GS(S_1)$, so that the Chernoff's Axiom is violated (Suzumura (1981, Example 4)). A choice function satisfies the Superset Axiom if, for all S_1 and S_2 with $S_1 \subset S_2$ and $C(S_2) \subset C(S_1)$, $C(S_1) = C(S_2)$ has to hold. Suzumura (1981, Example 3), which is our Example 1, showed GS rule's violation of the Superset Axiom for $S_1 = \{x, z\}$ and $S_2 = S$.

[10] For example, Tadenuma (2002, Proposition 4) and Nishimura (2007, Theorem 1) showed that $M_{\succ_{PF_E}}(S^*)$ could be empty in exchange economies and production economies, respectively. Tadenuma (2007) examined the Chernoff's Axiom and the *Path Independence* of the social choice correspondences satisfying the efficiency-first and the equity-fiest principles.

non-comparability, and it is possible to construct a binary relation based on no-envy. It is therefore important to provide a conceptual framework that synthesizes the previous studies. Recent studies show the usefulness for applications to economic environments, including Tadenuma (2002, 2005, 2007) and Nishimura (2007).

Appendix

In facilitating the proof of Theorem 3, we prove here that $M_{P(R^*)}(S) = M_{\succ_{PF_E}}(S) \cup M_{\succ_{F_E P}}(S)$. We begin with showing $M_{P(R^*)}(S) \subset M_{\succ_{PF_E}}(S) \cup M_{\succ_{F_E P}}(S)$. For any $z \in S$ we will divide into four cases by whether it is in $P(S)$ or not, and whether it is in $F(S)$ or not. Then, we will examine conditions for those allocations to be included to $M_{P(R^*)}(S)$.

Case 1 *$z \in P(S) \cap F(S)$.*

By definition, $[P(S) \cap F(S) \cap M_{P(R^*)}(S)] \subset [P(S) \cap F(S)]$. Notice that $P(S) \cap F(S)$ is either empty or included in $M_{\succ_{PF_E}}(S)$, by Tadenuma (2002, Proposition 3). As the empty set is a subset of any set, we conclude:

$$[P(S) \cap F(S) \cap M_{P(R^*)}(S)] \subset M_{\succ_{PF_E}}(S) \cup M_{\succ_{F_E P}}(S). \tag{4}$$

Case 2 *$z \notin P(S)$ and $z \notin F(S)$.*

There exists $x \in P(S)$ which Pareto dominates z, which also dominates z according to $P(R^*)$ by (3). Therefore, no such z is included in $M_{P(R^*)}(S)$. We showed that

$$[F(S)^c \cap P(S)^c \cap M_{P(R^*)}(S)] = \emptyset. \tag{5}$$

Case 3 *$z \notin P(S)$ and $z \in F(S)$.*

For z to be in $M_{P(R^*)}(S)$, there is no x for which condition (a) of (3) is applicable. This means that

$$\forall x[x \in F(S) \rightarrow \neg(x \succ_P z)],$$

which means that $z \in M_{\succ_{F_E P}}(S)$, by Tadenuma (2002, Proposition 5). As $P(S) \cap F(S) \subset M_{\succ_{F_E P}}(S)$, the following holds:

$$\begin{aligned} F(S) \cap M_{P(R^*)}(S) &= [P(S) \cap F(S) \cap M_{P(R^*)}(S)] \cup [P(S)^c \cap F(S) \cap M_{P(R^*)}(S)] \\ &\subset [P(S) \cap F(S)] \cup [P(S)^c \cap F(S) \cap M_{P(R^*)}(S)] \qquad (6) \\ &\subset M_{\succ_{F_E P}}(S). \end{aligned}$$

Case 4 *$z \in P(S)$ and $z \notin F(S)$.*

For z to be in $M_{P(R^*)}(S)$, there is no x for which condition (b) of (3) is applicable. This means that

$$\forall x[x \not\succ_P z, z \not\succ_P x \to x \notin F(S), \text{ thus } x \in F(S) \to z \succ_P x],$$

i.e., z must beat any envy-free allocation by Pareto test, which means that $z \in M_{\succ_{PF_E}}(S)$ by Tadenuma (2002, Proposition 3.(ii)). As $P(S) \cap F(S) \subset M_{\succ_{PF_E}}(S)$, the following holds:

$$\begin{aligned} P(S) \cap M_{P(R^*)}(S) &= [P(S) \cap F(S) \cap M_{P(R^*)}(S)] \cup [P(S) \cap F(S)^c \cap M_{P(R^*)}(S)] \\ &\subset [P(S) \cap F(S)] \cup [P(S) \cap F(S)^c \cap M_{P(R^*)}(S)] \qquad (7) \\ &\subset M_{\succ_{PF_E}}(S). \end{aligned}$$

Combining (4), (5), (6), and (7), we show the desired result of $M_{P(R^*)}(S) \subset M_{\succ_{PF_E}}(S) \cup M_{\succ_{F_E P}}(S)$.

To show $M_{P(R^*)}(S) \supset M_{\succ_{PF_E}}(S) \cup M_{\succ_{F_E P}}(S)$, we make use of Tadenuma (2002, Propositions 3.(ii) and 5) that $M_{\succ_{F_E P}}(S) \cup M_{\succ_{PF_E}}(S) = P(F(S)) \cup \{x \in S | x \in P(S) \text{ and } x \succ_P z \ \forall z \in F(S)\}$. It is clear that for any $z \in M_{\succ_{F_E P}}(S) \cup M_{\succ_{PF_E}}(S)$, there is no $x \in S$ such that $xP(R^*)z$. *Q.E.D.*

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