Queen's Economics Department Working Paper No. 1159

OPTION PRICING WITH V. G. MARTINGALE COMPONENTS

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10-2008

Mathematical Finance, Vol. 1, No. 4 (October 1991), 39–55

OPTION PRICING WITH V. G. MARTINGALE COMPONENTS[1]

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European call options are priced when the uncertainty driving the stock price follows the V. G. stochastic process (Madan and Seneta 1990). The incomplete markets equilibrium change of measure is approximated and identified using the log return mean, variance, and kurtosis. An exact equilibrium interpretation is also provided, allowing inference about relative risk aversion coefficients from option prices. Relative to Black-Scholes, V. G. option values are higher, particularly so for out of the money options with long maturity on stocks with high means, low variances, and high kurtosis.

KEYWORDS: option pricing, martingales, V. G. process

1. INTRODUCTION

Option pricing with Brownian motion as the martingale process describing the evolution of the underlying uncertainty is fairly well developed. The basic geometric Brownian motion model with constant drift and diffusion coefficients employed for valuing European call options in a constant interest rate setting (Black and Scholes 1973) has now been extended to allow for stochastic volatility (Wiggins 1987, Scott 1987, and Hull and White 1987), stochastic interest rates (Ho and Lee 1986, Heath, Jarrow, and Morton 1989), and the valuation of American put options by numerical methods (Brennan and Schwartz 1977, Parkinson 1977) and analytic approximation (Johnson 1983, Barone-Adesi and Whaley 1987, Jamshidian 1989, Barone-Adesi and Elliott 1989, and Carr, Jarrow, and Myneni 1989).

Madan and Seneta (1990) introduced an alternative martingale process, termed the V. G. (variance gamma), as a candidate for replacing the role of Brownian motion as a model for the underlying uncertainty driving stock market returns. This process is defined as $N(t) = b(G(t))$, where $b(t)$ is standard Brownian motion and $G(t)$ is the process of independent gamma increments with mean t and variance vt. Hence, the process may be viewed as Brownian motion evaluated at a random time change. It is a pure jump process with jump magnitudes concentrated near the origin (Madan and Seneta 1990). The distribution of $N(1)$ is long-tailed relative to the normal distribution and has a kurtosis of $3(1 + v)$. Conditional on $G(t)$ the distribution of $N(t)$ is normal with variance $G(t)$; hence, the nomenclature variance gamma.

[1] Earlier versions of this paper were presented at the Australian Graduate School of Management workshop in economic theory and at a seminar at the University of Maryland. More recently the paper was presented at the University of Alberta, the World Congress of the Econometrics Society Barcelona 1990, and at a seminar at Queen's University. We would like to thank the seminar participants for various comments and suggestions and, in particular, Robert Elliott, Stuart Turnbull, and the referees for considerable assistance. Errors remain our responsibility.
Manuscript received March 1990; final revision received May 1991.

This alternative was sought primarily to obtain a process consistent with the observation (Bookstaber and McDonald 1987, Madan and Seneta 1987, Praetz 1972, Press 1967, Fama 1965, and Mandelbrot 1963) that the local movement of log prices is long-tailed relative to the normal, while the movement over larger time intervals approaches normality. Properties of Brownian motion that are desirable and were sought to be maintained in the alternative models for the driving uncertainty include finite moments of at least the lower orders, sample path continuity, and independent identically distributed (i.i.d.) increments over nonoverlapping intervals of equal length.

Sample path continuity, however, essentially requires that the process be a stochastic integral with respect to Brownian motion, in which case the local motion is normal. Hence this property was abandoned and pure jump processes were considered. Unlike Jones (1984), the interest was not in accommodating by arbitrage the occasional large jumps, expanding of necessity the number of spanning assets required, but rather in correcting the local behavior of the driving uncertainty. The V. G. model was therefore contrasted with other i.i.d. pure jump processes, like the process of independent symmetric stable increments (Mandelbrot 1963) and the Press (1967) compound events model, in a study conducted by Madan and Seneta (1987). The results were supportive of the V. G. process, which attained minimum chi-squared in 12 out of 19 cases.

Though other processes have been used in the literature for option pricing, they have been either Itô processes with sample path continuity (Cox and Ross 1976) or, in the presence of jump discontinuities, they have invoked a risk-neutral valuation result (Merton 1976, Cox and Ross 1976, Jarrow and Rudd 1982). The complex issue, noted in Brennan and Schwartz (1978), of identifying the change of measure in the jump context is taken up here. As observed in Madan, Milne, and Shefrin (1989), this leads to relevance of the particular change of measure via a dependence of the option price on the mean rate that reflects the coefficient of relative risk aversion in certain interpretations developed below.

The pricing of European options for a pure jump driving uncertainty can not be accomplished using the cost of a hedging strategy. As Naik and Lee (1990) observe, a self-financing continuous trading strategy in the underlying asset and a riskless bond that replicates the payoff of the option do not exist. Naik and Lee obtain a necessarily incomplete markets equilibrium value for the option by solving a one-individual equilibrium model, employing a constant relative risk aversion utility function for the individual.

Many authors have observed under a variety of circumstances that discounted asset price processes in an equilibrium, with complete or incomplete markets, are a martingale under an equivalent change of measure (Harrison and Kreps 1979, Harrison and Pliska 1981, Huang 1985, Duffie 1988, and Back 1991). With completeness the equivalent change of measure is unique, while with incompleteness we have its existence but not in general its uniqueness. Duffie derives the result in an equilibrium with initial and final consumption for square-integrable semimartingale price processes and a predictable discounting process that pays at maturity and has a price process strictly bounded above and below. Back allows intermediate consumption and obtains the result for square-integrable semimartingale price processes but employs a continuous discounting process. The absence of arbitrage opportunities is not quite sufficient for the derivation of this result, and the interested reader is referred to Back and Pliska (1991) for further discussion.

Employing the Duffie and Back result, the change of measure density process is first parameterized using results from Jacod and Shiryayev (1987). Relying on the fact, noted earlier, that most of the jumps for our process are near the origin, we construct a one-parameter approximation by a Taylor series expansion to the function that recharges the jump magnitudes. The single parameter is then identified using the martingale condition for the discounted asset price process, as it is in modern versions of Black-Scholes theory. The option pricing formula derived as a discounted expected value under the identified change of measure can then be viewed as a first-order approximation to an incomplete markets equilibrium price.

Alternatively, one can follow Naik and Lee (1990) and set up a representative agent (Lucas 1978) pure exchange economy and derive the option pricing formula as an exact equilibrium price for this economy. We note for completeness that Naik and Lee (1990) derive an option price for a driving martingale component that is a mixture of a diffusion and the Press (1967) normal compound Poisson process that was also studied in Madan and Seneta (1987).

Section 2 introduces the V. G. stochastic process. Section 3 parameterizes the equivalent martingale measure, and the density process for the change of measure is then identified in Section 4. The exact equilibrium price interpretation following Naik and Lee (1990) is presented in Section 5. The option pricing formula is then derived in Section 6 along with a closed-form approximation valid for large maturity dates. Comparisons with Black-Scholes are presented in Section 7. Section 8 concludes the paper.

2. THE STOCHASTIC BASIS

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let the time span be $[0, T]$, where T is a positive real number. Suppose that both standard Brownian motion $b = (b(t), t \in [0, T])$ and $G = (G(t), t \in [0, T])$, the right-continuous process of independent gamma increments with mean t and variance vt, $t \in [0, T]$, are defined on the probability space. Let $N = \{N(t) = b(G(t)), t \in [0, T]\}$ be the V. G. process, and let $\mathbf{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the right-continuous filtration generated by N. (A filtration is an increasing family of sub-σ-fields of $\mathcal{F}$). We assume that $\mathcal{F}_0$ contains all the P-null sets and that $\mathcal{F}_T = \mathcal{F}$. Since $G(0) = 0$ a.s., $\mathcal{F}_0$ is almost trivial containing just sets of measure zero or one.

We use $\mathcal{O}$ (resp., $\mathcal{P}$) to denote the optional (resp., predictable) σ-field and π to denote the product measure on $\Omega \times [0, T]$ generated by P and Lebesgue measure. (The optional (resp., predictable) σ-field is the σ-field generated on $\Omega \times [0, T]$ by the adapted right-continuous (resp., left-continuous) processes. A process measurable with respect to $\mathcal{O}$ (resp., $\mathcal{P}$) is naturally adapted to $\mathbf{F}$ and is said to be an optional (resp., predictable) process.

It is well known that any process of independent increments like $N(t)$ with $N(0) = 0$, and without fixed times of discontinuity, has a distribution that is infinitely divisible. Furthermore, letting $t\phi(u)$ be the log characteristic function satisfying $E(e^{iuN(t)}) = e^{t\phi(u)}$, then

$$Y(t) = e^{iuN(t) - t\phi(u)} \tag{2.1}$$

is a martingale (Jacod and Shiryayev 1987, p. 75). In particular, for our specific V. G. process we have

$$\phi(u) = \frac{1}{v} \ln \left(\frac{1}{1 + vu^2/2} \right). \tag{2.2}$$

The role of Brownian motion b in the original Black-Scholes model may be replaced by the V. G. process N by postulating the following model for the stock price.

Assumption 2.1. The stock price $S = \{S(t), t \in [0, T]\}$ follows the optional semimartingale process

$$S(t) = S(0)e^{\mu t - t\phi(\sigma/i) + \sigma N(t)}, \tag{2.3}$$

where μ is the expected rate of return on the stock.

Equation (2.1) implies that $e^{\sigma N(t) - t\phi(\sigma/i)}$ is a martingale if $t\phi(\sigma/i) = (t/v)\ln[1/(1 - v\sigma^2/2)]$ is finite. This requires $v\sigma^2/2 < 1$ or $v < 2/\sigma^2$. This bounds the kurtosis that can be accommodated.

A better understanding of the process (2.3) may be obtained by considering its local movement as described by the stochastic differential of (2.3) and by writing the process S as a stochastic integral with respect to the integer-valued random measure associated with the process $N(t)$. Taking the stochastic differential, we obtain, on noting that $N(t)$ has no continuous martingale component,

$$\begin{aligned} S(t) = S(0) &+ \int_0^t \left(\mu - \phi\left(\frac{\sigma}{i}\right)\right) S(t_-)dt \\ &+ \sum_{0<s\leq t} S(t_-)(e^{\sigma \Delta N_s} - 1). \end{aligned} \tag{2.4}$$

We see from (2.4) that there is a continuous proportional drift of $\mu - \phi(\sigma/i)$ with proportional jumps of magnitude $e^{\sigma\Delta N_s} - 1$ synchronized with the jumps in N. Note that the proportional jumps always exceed -1, and this keeps the price process nonnegative.

Let μ^N be the optional integer-valued random measure on $[0, T] \times \mathbb{R}$, associated with the process $N(t)$ and defined by

$$\mu^N(\omega; dt, dx) = \sum_s \mathbf{1}_{\{\Delta N_s(\omega) \neq 0\}} \varepsilon_{(s, \Delta N_s(\omega))}(dt, dx), \tag{2.5}$$

where ε_a is the Dirac measure at the point a. The process S may also be written in terms of μ^N as

$$\begin{aligned} S(t_-) = S(0) &+ \int_0^t \left(\mu - \phi\left(\frac{\sigma}{i}\right)\right) S(t_-)\, dt \\ &+ \int_{[0,t]\times\mathbb{R}} S(t_-)(e^{\sigma x} - 1)\mu^N(\omega; ds, dx), \end{aligned} \tag{2.6}$$

and so a jump in N of magnitude x yields a jump in S of magnitude $S(t_-)(e^{\sigma x} - 1)$.

Madan and Seneta (1990) show that $N(t)$ can be approximated as the difference of two identical but independent increasing compound Poisson processes with arrival rates given by $\beta_n = \int_{1/n}^{\infty} e^{-z/(\sqrt{2/v})}(vz)^{-1}\, dz$ and conditional jump density given by $\beta_n^{-1} e^{-z/(\sqrt{2/v})}(vz)^{-1}\mathbf{1}_{\{z>1/n\}}$. The jumps in N therefore have a high frequency and are concentrated near the origin. This property motivates our approximation to the equivalent martingale measure in the next section. This approximation will also prove useful in simulating the value of complex contingent claims adapted to the process N.

3. EQUIVALENT MARTINGALE MEASURES FOR THE V. G. PROCESS

The results of Duffie (1988) and Back (1990) quoted in the introduction assert that discounted asset price processes of traded securities in an economic equilibrium are martingales under an equivalent change of measure. We suppose the existence of a constant instantaneous interest rate r.

ASSUMPTION 3.1. *A pure discount bond maturing at T with face value of \$1 is traded and has a price process given by* $e^{-r(T-t)}$.

It follows in equilibrium that there exists a measure $Q \approx P$ (Q equivalent to P; i.e., $Q(A) > 0$ if and only if $P(A) > 0$ for all $A \in \mathcal{F}$), such that

(i) $e^{-rt}S(t)$ is a Q martingale, and

(ii) the price W_t at t of a traded call option on the stock with maturity T and a strike price K is

$$W_t = E_t^Q[e^{-r(T-t)}(S(T) - K)^+], \tag{3.1}$$

where E_t^Q is the expectation operator with respect to the measure Q and conditional on $\mathcal{F}_t$.

Condition (i) is used to identify the measure Q, and then condition (ii) is employed to evaluate the option price. The measure Q is identified by determining the Radon-Nikodym derivative of Q with respect to P and the associated density process

$$\lambda(t) = E^P[dQ/dP|\mathcal{F}_t]. \tag{3.2}$$

The density process can be identified in terms of the effect of the change of measure on the characteristics of the underlying martingale process N.

Semimartingales can be characterized in terms of their characteristics (see Jacod and Shiryayev 1987, p. 76). The characteristics are defined relative to a truncation function $h(x)$, and we shall use the function $h(x) = x\mathbf{1}_{\{|x|<1\}}$. The characteristics consist of a triplet (B, C, ν), where B is a predictable real-valued process of finite variation, C is the continuous process of the quadratic characteristic of the continuous martingale component of the semimartingale, and ν is a predictable random measure that is the compensator of the random measure μ^N.

For a function $W(\omega, t, x)$ measurable with respect to the product σ-field $\tilde{\mathcal{O}}$ (resp., $\tilde{\mathcal{P}}$) of $\mathcal{O}$ (resp., $\mathcal{P}$) on $\Omega \times [0, T]$ with the Borel σ-field on $\mathbb{R}$ and of a random measure $\mu(\omega; dt, dx)$, we define the stochastic integral $W * \mu$ by

$$(W * \mu)_t = \int_{[0,t]\times\mathbb{R}} W(\omega, t, x)\mu(\omega; dt, dx)$$

if $\int_{[0,t]\times\mathbb{R}} |W(\omega, t, x)|\mu(\omega; dt, dx)$ is finite; otherwise $(W * \mu)_t$ is infinite.

A semimartingale X can be written in terms of its characteristics (B, C, ν) (Jacod and Shiryayev 1987, p. 84) as

$$X = X_0 + X^c + h * (\mu^X - \nu) + (x - h(x)) * \mu^X + B,$$

where X^c is the continuous martingale component of X. The process $h * (\mu^X - \nu)$ describes the local (near the origin) discontinuous martingale behavior, $(x - h(x)) * \mu^X$ gives the distant jump behavior, and B is a predictable finite-variation part. If the semimartingale X is special with canonical decomposition $X = X_0 + M + A$ for a martingale component M and predictable finite-variation process A, then we have that (Jacod and Shiryayev 1987, p. 85)

$$X = X_0 + X^c + x * (\mu^X - \nu) + A. \tag{3.3}$$

For the process N, which is a process of independent increments, the characteristics are identified from the Lévy-Khintchine representation of the characteristic function (Jacod and Shiryayev 1987, p. 107). If K is a positive measure that integrates $|x|^2 \wedge 1$ and satisfies $K(\{0\}) = 0$, and if for all $t \in [0, T]$ and $u \in \mathbb{R}$ we have

$$E(e^{iuX_t}) = \exp t\left[iub - \frac{1}{2}c^2u + \int (e^{iux} - 1 - iuh(x))K(dx) \right], \tag{3.4}$$

then the characteristics are $B = bt$, $C = ct$, and $\nu(\omega; dt, dx) = dt\, K(dx)$.

Madan and Seneta (1990) obtained the Lévy-Khintchine representation for N, and this yields $b = c = 0$ and

$$K(dx) = \frac{\exp(-|x|\sqrt{2}/\sigma\sqrt{\nu})}{\nu|x|}\, dx \qquad \text{for } x \neq 0. \tag{3.5}$$

One observes from (3.5) that $(|x|^2 \wedge |x|) * \nu$ is of integrable variation, and this implies (Jacod and Shiryayev 1987, p. 82) that $N(t)$ is a special semimartingale. Since N is a martingale with no continuous component and $N(0) = 0$, we have from (3.5) that

$$N(t) = x * (\mu^N - \nu),$$

with $\nu = dt\, K(dx)$. Note however that $x * \nu$ is zero, so N is $x * \mu^N$, as it should be.

A change of measure from P to a measure Q absolutely continuous with respect to P results by Girsanov's theorem (Jacod and Shiryayev 1987, p. 159) in N being a Q semimartingale with altered characteristics. In fact, for a pure jump process like N, there exists a $\tilde{\mathcal{P}}$-measurable nonnegative function $Y(\omega, t, x)$ satisfying $|h(x)(Y - 1)| * \nu_t < \infty$ Q-a.s. such that the characteristics of N relative to Q are

$$B' = h(x)(Y - 1) * \nu, \qquad C' = 0, \quad \nu' = Y \cdot \nu,$$

where $Y \cdot \nu(\omega; dt, dx) = Y(\omega, t, x)\nu(\omega; dt, dx)$.

The function Y serves to alter the jump compensator and may be thought of as recharging all the jump magnitudes in a predictable fashion. Equivalence of Q to P requires that Y be strictly positive, so we may write

$$Y(\omega, t, x) = e^{\psi(\omega, t, x)}. \tag{3.6}$$

An explicit computation of the density of Q with respect to P is provided by Jacod and Shiryayev (1987, p. 179), provided all P martingales have the representation property relative to N. This is implied by Jacod and Shiryayev (1987, Theorem 4.34, p. 176), as N is a process of independent increments. For the specific case considered here with $\nu(\omega; \{t\} \times \mathbb{R}) = 0$, the density is obtained as the Doléans-Dade exponential of the martingale $M = \{M(t), t \in [0, T]\}$ defined by

$$M = (Y - 1) * (\mu^N - \nu). \tag{3.7}$$

The Doléans-Dade exponential of a semimartingale X, denoted $\mathscr{E}(X)$, is defined as the semimartingale solution Z to the equation

$$Z = 1 + Z_- \cdot X,$$

or, in differential notation,

$$dZ = Z_- \, dX$$

with $Z_0 = 1$. For a process with no continuous martingale component we have that (Jacod and Shiryayev 1987, p. 59)

$$\mathscr{E}(X)_t = \exp(X_t - X_0)X \prod_{s \le t} (1 + \Delta X_s)e^{-\Delta X_s}. \tag{3.8}$$

Applying (3.8) to the process M, written as $(Y - 1) * \mu^N - (Y - 1) * \nu$, we note that

$$\begin{aligned} \mathscr{E}(M)_t &= \exp\left(-(Y - 1) * \nu_t + \sum_s \Delta M_s \right) X \prod_{s \le t} (1 + \Delta M_s)e^{-\Delta M_s} \\ &= \exp\left(-\int_{[0,t]\times\mathbb{R}} (e^{\psi(\omega,s,x)} - 1) \, ds \, K(dx) + \sum_{s \le t} \psi(\omega, s, \Delta N_s) \right). \end{aligned} \tag{3.9}$$

Integrability of $e^{\psi} - 1$ with respect to $K(dx)$ requires that $\psi(0) = 0$, and for ψ smooth we may construct a first-order Taylor series approximation of ψ at zero and write

$$\psi(\omega, t, x) = \alpha(\omega, t)x + e(\omega, t, x),$$

where e is an error or remainder term. Since most of the jump magnitudes are concentrated near the origin for the process N, a reasonable approximation to the change of measure density for an incomplete markets equilibrium may be specified by employing the function

$$\psi(\omega, t, x) = \alpha(\omega, t)x. \tag{3.10}$$

Substituting (3.10) into (3.9) and performing the integration with respect to $K(dx)$ (which can be checked by power series expansion as in Madan and Seneta 1990), we get that the change of measure density process $\lambda = \{\lambda(t), t \in [0, T]\}$ has the form

$$\lambda(t) = \exp\left(\sum_{s \le t} \alpha(\omega, s) \, \Delta N_s - \int_0^t \phi\left(\frac{\alpha(\omega, s)}{i} \right) ds \right). \tag{3.11}$$

The coefficients α can be identified by employing condition (i), which requires the discounted stock price process to be a Q martingale. This requires (Elliott 1982, p. 161) that $e^{-rt}\lambda(t)S(t)$ be a P martingale. Writing this expression, we have

$$e^{-rt}\lambda(t)S(t) = S(0)\exp\left(\sum_{s\leq t}(\alpha(\omega, s) + \sigma)\,\Delta N_s + \left(\mu - r - \phi\left(\frac{\sigma}{i}\right)\right)t - \int_0^t \phi\left(\frac{\alpha(\omega, s)}{i}\right) ds\right).$$

From the form of (3.11) and the fact that $\boldsymbol{\lambda}$ is a martingale, it follows that

$$\gamma(t) = \exp\left(\sum_{s\leq t}(\alpha(\omega, s) + \sigma)\,\Delta N_s - \int_0^t \phi\left(\frac{\alpha(\omega, s) + \sigma}{i}\right) ds\right)$$

is a martingale. Observe now that

$$e^{-rt}\lambda(t)S(t) = S(0)\gamma(t)\exp\left(\left(\mu - r - \phi\left(\frac{\sigma}{i}\right)\right)t - \int_0^t \phi\left(\frac{\alpha(\omega, s)}{i}\right) ds + \int_0^t \phi\left(\frac{\alpha(\omega, s) + \sigma}{i}\right) ds\right). \tag{3.12}$$

Hence, (3.11) and (3.12) are martingales only if

$$\mu - r = \phi(\alpha(\omega, s)/i) + \phi(\sigma/i) - \phi((\alpha(\omega, s) + \sigma)/i),$$

in which case $\alpha(\omega, s)$ must be the constant function α satisfying

$$\mu - r = \phi(\alpha/i) + \phi(\sigma/i) - \phi((\alpha + \sigma)/i), \tag{3.13}$$

and from (3.11) the change of measure density is

$$\lambda(t) = e^{\alpha N(t) - \phi(\alpha/i)t}. \tag{3.14}$$

The change of measure is now identified using (3.13) and (3.14). The option price can be determined by integration, which is taken up in Section 5. The density (3.14) can also be given an exact interpretation for a Naik and Lee (1990) equilibrium, which we discuss next.

4. THE CHANGE OF MEASURE IN A PARTICULAR GENERAL EQUILIBRIUM

This section describes a general equilibrium in which the change of measure has the form (3.14) and the coefficient α is related to the coefficient of relative risk aversion for a representative agent. The model is set up as in Naik and Lee (1990), where there is one fully-equity-financed firm with a unit of stock outstanding engaged in the costless production of a single perishable consumption good. Let the probability space be as described in Section 2, except that the time span is now $[0, \infty]$. Suppose the dividend process $D = \{D(t), t \in [0, \infty]\}$ is exogenously given as

$$D(t) = D(0)\exp(\mu t - \phi(\sigma/i)t + \sigma N(t)). \tag{4.1}$$

This is precisely the process supposed for the stock price earlier.

The economy has trading in the stock of the single firm with price process $S = \{S(t), t \in [0, \infty]\}$ as well as trading in a zero net supply European call option maturing at T with exercise price K. The price process of the option is $W = \{W(t), t \in [0, T]\}$. There is also a risk-free bond traded with maturity T and face value \$1 with price process $B = \{B(t), t \in [0, T]\}$.

The economy has a representative agent who chooses a consumption plan $c = \{c(t), t \in [0, \infty]\}$ so as to maximize expected utility

$$E\int_0^\infty e^{-\delta t}U(c(t))\,dt \tag{4.2}$$

subject to a budget constraint. The pure rate of time preference is δ, and we suppose that the utility function is one of constant relative risk aversion; that is,

$$U(c) = (1/\eta)c^\eta, \tag{4.3}$$

where $1 - \eta$ is the coefficient of relative risk aversion. The budget constraint requires that all consumption be financed through the gains associated with an investment strategy in the financial assets.

A competitive equilibrium for this economy is a set of security price processes such that expected utility of the representative agent is maximized subject to the budget constraint with the representative agent holding the total stock of the firm and consuming all the dividends.

For such an economy, following Naik and Lee, one must have

$$S(t)D(t)^{\eta-1} = \int_t^\infty E_t^P[e^{-\delta(s-t)}D(s)^\eta\,ds, \tag{4.4}$$

and on computing out the expectation (see Appendix A), we obtain

$$S(t) = \frac{D(t)}{\delta - \eta\mu + \eta\phi(\sigma/i) - \phi(\eta\sigma/i)}. \tag{4.5}$$

Hence, $S(t)$ follows a process like $D(t)$, which is, by construction, consistent with (2.3).

The interest rate for the economy is obtained from the equation

$$B(t)D(t)^{\eta-1} = e^{-\delta(T-t)}E_t^P D(T)^{\eta-1}, \tag{4.6}$$

which on computation (see Appendix A) yields

$$B(t) = e^{-(T-t)[\delta+(1-\eta)\mu+(\eta-1)\phi(\sigma/i)-\phi((\eta-1)\sigma/i)]}. \tag{4.7}$$

Hence, there is a constant risk-free interest rate of

$$r = \delta + (1-\eta)\mu + (\eta - 1)\phi(\sigma/i) - \phi((\eta - 1)\sigma/i). \tag{4.8}$$

The change of measure process $\lambda(t)$ may be obtained from the conditional expectation process of normalized marginal utilities at T (Back 1990), and this gives

$$\lambda(t) = E_t^P\left[\frac{e^{-rT}D(T)^{\eta-1}}{E^P[e^{-rT}D(T)^{\eta-1}]}\right], \tag{4.9}$$

which yields (see Appendix A)

$$\lambda(t) = e^{(\eta-1)\sigma N(t)-\phi((\eta-1)\sigma/i)t} \tag{4.10}$$

which is of the form (3.14) with $\alpha = (\eta - 1)\sigma$. The coefficient of relative risk aversion is $1 - \eta$ or $-\alpha/\sigma$.

5. THE EXPLICIT CHANGE OF MEASURE

The change of measure is determined explicitly on solving (3.13) for the value of α in (3.14). Defining $\theta = e^{v(\mu-r)}$ and $\gamma = v\sigma^2/2$, we show in Appendix B that the solution to (3.13) for α/σ is

$$\frac{\alpha}{\sigma} = \begin{cases} (\theta(1-\gamma)-1)^{-1} + \left\{\dfrac{\theta(1-\gamma)}{[\theta(1-\gamma)-1]^2} + \dfrac{1}{\gamma}\right\}^{1/2} & \text{if } \theta(1-\gamma) < 1, \\ -\dfrac{1}{2} & \text{if } \theta(1-\gamma) = 1, \\ (\theta(1-\gamma)-1)^{-1} - \left\{\dfrac{\theta(1-\gamma)}{[\theta(1-\gamma)-1]^2} + \dfrac{1}{\gamma}\right\}^{1/2} & \text{if } \theta(1-\gamma) > 1. \end{cases} \tag{5.1}$$

One observes directly from (3.13) and (5.1) that if $\mu = r$ then $\theta = 1$, and in this case $\alpha = 0$ and one has the risk-neutral case with no change of measure involved.

It follows from (5.1) that if $\theta(1 - \gamma) > 1$ then $\alpha < 0$. On the other hand, if $\theta(1 - \gamma) < 1$ then $\alpha > 0$ can be shown to be equivalent to $\gamma(1 - \theta) < (1 - \theta)$. Thus, for $\theta > 1$, which corresponds to $\mu > r$, $\alpha > 0$ only if $\gamma > 1$. Since we have $\gamma < 1$, α is always negative for $\theta > 1$. If $\theta < 1$ or $\mu < r$, then $\alpha > 0$ with $\gamma < 1$. Hence, for $\mu > r$, $\alpha < 0$, and as $-\alpha/\sigma$ is the coefficient of relative risk aversion for the equilibrium of Section 4, the implied utility function is always concave.

In the region $\theta > 1$ and $0 \le \gamma \le 1$ we have $\alpha/\sigma = 0$ at the boundary $\theta = 1$. It is also zero at the boundary $\gamma = 1$ and rises without bound at the boundary $\gamma = 0$. On the arc $\gamma = 1 - 1/\theta$ we have $-\alpha/\sigma = 1/2$, so the contour plot of $-\alpha/\sigma$ as a function of θ and γ in this region has concave level curves written with γ as a function of θ. As $\mu \to \infty$ for a given v and σ^2, θ rises without bound and $-\alpha/\sigma$ approaches $\sqrt{1/\gamma}$ or $(1/\sigma)\sqrt{2/v}$.

6. THE OPTION PRICING FORMULA

The option price is determined simply by evaluating the expectation of condition (ii) given by (3.1). It is sufficient to evaluate the option price at time 0, with maturity being at t. We first write μ in terms of α in the description (2.3) of the stochastic process for $S(t)$.

Then μ is expressed in terms of α using (3.13) and the precise form of ϕ given in (2.2). This yields

$$S_t = S_0 \exp\left\{\sigma N(t) + \left[r + \left(\frac{1}{v} \ln \frac{1 - v(\alpha + \sigma)^2/2}{1 - v\alpha^2/2}\right)\right]t\right\}. \tag{6.1}$$

In addition, we have the precise change of measure density

$$\lambda(t) = e^{\alpha N(t) - (t/v)\ln(1/(1 - v\alpha^2/2))}. \tag{6.2}$$

The option price can now be written as

$$W = E^P[e^{-rt}\lambda(t)(S(t) - K)^+]. \tag{6.3}$$

We now have to integrate with respect to the density of $N(t)$. This is done by first conditioning on the gamma variate $G(t) = G$, for, conditional on this, $N(t)$ is normal with mean 0 and variance G. The conditional option value $W(G)$ is therefore obtained by standard methods on integrating with respect to the normal density.

Substituting (6.1) and (6.2) into (6.3), we find that the variable to be integrated is

$$\left[S_0 \exp\left\{\sigma N + \frac{t}{v} \ln\left[\frac{1 - v(\alpha + \sigma)^2/2}{1 - v\alpha^2/2}\right]\right\} - Ke^{-rt}\right]^+ e^{\alpha N + (t/v)\ln(1 - v\alpha^2/2)}, \tag{6.4}$$

with domain of integration

$$N > -d = -\frac{1}{\sigma}\left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{v}\ln\left[\frac{1 - v(\alpha + \sigma)^2/2}{1 - v\alpha^2/2}\right]\right)t\right],$$

and N is distributed normal with mean 0 and variance G. By standard methods the integration yields

$$\begin{aligned} W(G) = {} & S_0\left(\frac{1 - v(\alpha + \sigma)^2}{2}\right)^{t/v} e^{(\alpha+\sigma)^2 G/2}\Phi\left(\frac{d}{\sqrt{G}} + (\alpha + \sigma)\sqrt{G}\right) \\ & - Ke^{-rt}\left(\frac{1 - v\alpha^2}{2}\right)^{t/v} e^{\alpha^2 G/2}\Phi\left(\frac{d}{\sqrt{G}} + \alpha\sqrt{G}\right), \end{aligned} \tag{6.5}$$

where Φ is the standard normal distribution function.

The V. G. option price requires that we integrate G in (6.5) with respect to the gamma density with a mean t and variance vt, namely,

$$f(G) = \frac{G^{t/v - 1}e^{-G/v}}{v^{t/v}\Gamma(t/v)}. \tag{6.6}$$

Defining $G/v = g$, we may write the V. G. option price as

$$W = \int_0^\infty W(vg)\frac{g^{t/v - 1}e^{-g}}{\Gamma(t/v)}\, dg. \tag{6.7}$$

The integration in (6.7) must be performed numerically, and $W(vg)$ must be obtained from (6.5).

This numerical integration really only needs to be performed for small values of t/v. For large values of t/v, observe from (2.2) that the log characteristic function of $N(t)/\sqrt{t}$ may be written as

$$\phi(u) = \frac{t}{v}\ln\left(\frac{1}{1 + u^2/2(t/v)}\right),$$

and this tends to $-u^2/2$ as t/v tends to infinity. Hence, the distribution of $N(t)/\sqrt{t}$ may be approximated by a standard normal variate for large values of t/v. This asymptotic distribution is independent of the parameter v. However, the change of measure involves v, reflecting the fact that though one has normality for the large t's one does not have Brownian motion.

Using this approximation for the density of $N(t)$, one may compute the option price for large t by evaluating the integral of (6.4) with respect to the normal density with mean 0 and variance t. This yields

$$W = S_0 e^{(\alpha+\sigma)^2 t/2}(1 - v(\alpha + \sigma)^2/2)^{t/v}\Phi(d_1) - Ke^{-rt+\alpha^2 t/2}(1 - v\alpha^2/2)^{t/v}\Phi(d_2),$$

where

$$d_1 = \frac{\ln(S_0/K)}{\sigma\sqrt{t}} + \left[\frac{r + (1/v)\ln[(1 - v(\alpha + \sigma)^2/2)/(1 - v\alpha^2/2)]}{\sigma} + (\alpha + \sigma)\right]\sqrt{t},$$

$$d_2 = d_1 - \sigma\sqrt{t}.$$

A special case of some interest is the risk-neutral case of $\mu = r$ when $\alpha = 0$. Substitution into (6.8) shows that the option price is given precisely by the Black-Scholes formula applied to an initial price of $S_0 e^{\sigma^2 t/2}(1 - v\sigma^2/2)^{t/v}$, with exercise price K, interest rate r, maturity t, and variance rate σ^2. In this case the altered stochastic process has the effect of applying the multiple $(1 - v\sigma^2/2)^{t/v}e^{\sigma^2 t/2}$ to the initial price. As $v \to 0$, the process approaches Brownian motion, and this multiple approaches unity from below; therefore, the V. G. option price approaches the Black-Scholes price from below.

In many practical applications the closed-form approximation (6.8) would suffice. Our experimentation with the numerical integration and its comparison with (6.7) reveals that even for values of t/v as small as 2, for options worth at least a cent, the formulas (6.7) and (6.8) agree to three decimal places. For our comparisons with Black-Scholes reported in the next section, the use of (6.8) was sufficient.

7. PROPERTIES OF THE V. G. OPTION PRICE

We report here on comparisons of V. G. option prices with their Black-Scholes values. The basic finding is that, relative to the V. G., the Black-Scholes formula underprices options, with the percentage bias with respect to the Black-Scholes price rising as the stock gets out of the money. The percentage bias also rises with the time to maturity, the expected rate of return on the stock, and the level of kurtosis, while it falls with increases in the standard deviation. The presentation here is restricted to sensitivity with respect to the two new parameters influencing the V. G. option price, and these are the expected rate of return on the stock μ and the level of kurtosis, the percentage increase in kurtosis being $100v$. Madan and Seneta (1990) reported on the effects of changing v in the risk-neutral valuation case; we report here on the simultaneous effects of changing μ and v.

We take σ to be 0.25, the time to maturity to be 0.25, and the interest rate to be 10%. Note that the same value of σ is relevant in using the Black-Scholes or V. G. option pricing formulas. This is because the relevant estimate in both cases is the unit period variance rate for log returns. The exercise price is fixed at 100, and to allow for the analysis of the effects of being out of the money, at the money, and in the money, we consider three values for the stock price: 90, 100, and 110, respectively. The values for v are the same as those in Madan and Seneta (1990), 0.25, 0.5, 0.75, and 1.0, allowing for a doubling of kurtosis at the extreme. Three values for μ are investigated: 0.2, 0.3, and 0.4.

Table 7.1 presents the results in a 3×4 grid, with μ varying across rows and v varying across columns. Each cell of the table contains seven numbers, namely the risk aversion coefficient at the top of the cell and the V. G. option prices for the cases out of the money, at the money, and in the money, listed vertically immediately below with the percentage biases from Black-Scholes listed alongside. The biases are a percentage of the Black-Scholes value.

The implied relative risk aversion coefficient estimates rise with μ and fall with kurtosis. The biases rise with μ and kurtosis, but fall as the stock gets in the money.

TABLE 7.1
Comparison of V. G. Option Values with Black-Scholes

	Percentage Change in Kurtosis							
μ	25		50		75		100	
.2	3.04		2.92		2.82		2.73	
	1.36	3.0%	1.40	6.06%	1.43	8.33%	1.45	9.84%
	5.10	2.0%	5.18	3.6%	5.26	5.2%	5.32	6.4%
	11.86	1.37%	11.99	2.48%	12.11	3.5%	12.21	4.36%
.3	4.29		3.95		3.71		3.51	
	1.46	10.61%	1.57	18.94%	1.65	25.0%	1.72	30.30%
	5.35	7.0%	5.60	12.0%	5.79	15.8%	5.95	19.0%
	12.24	4.6%	12.62	7.86%	12.91	10.34%	13.15	12.39%
.4	5.31		4.71		4.31		4.02	
	1.64	24.24%	1.84	39.39%	1.99	50.75%	2.11	59.85%
	5.77	15.4%	6.24	24.80%	6.58	31.60%	6.84	36.80%
	12.85	9.83%	13.54	15.73%	14.04	20.0%	14.41	23.16%

8. CONCLUSION

An approximation to an incomplete markets equilibrium price was developed in the case of the underlying uncertainty following the pure jump V. G. stochastic process introduced by Madan and Seneta (1990). The approximation was developed by taking a first-order

Taylor series approximation to the change in the jump compensator induced by the measure change.

The density process was also identified as the relevant change of measure for an exact equilibrium of a Lucas (1978) type of economy following the methods of Naik and Lee (1990). The coefficient of the change of measure was then interpreted as the coefficient of relative risk aversion scaled by the asset return standard deviation.

The equilibrium martingale condition for discounted asset prices was used to determine the measure change, and the option price was then evaluated by integration. A closed-form expression was obtained for the longer maturities.

The V. G. option values were observed to be typically higher than Black-Scholes values, with the percentage underpricing by Black-Scholes being higher for out of the money options, with long maturity, high mean rates, low variance rates, and high kurtosis.

APPENDIX A.

The Derivation of the Relationship of $S(t)$ to $D(t)$ in (4.5)

From (4.4) we have that

$$S(t)D(t)^{\eta-1} = E_t^P \int_t^\infty e^{-\delta(s-t)} D(s)^\eta = \int_t^\infty e^{-\delta(s-t)} E_t^P D(s)^\eta ds. \tag{A.1}$$

Now by construction

$$D(s) = D(t)e^{\mu(s-t)-\phi(\sigma/i)(s-t)+\sigma(N(s)-N(t))},$$

and, therefore,

$$D(s)^\eta = D(t)^\eta e^{\eta\mu(s-t)-\eta\phi(\sigma/i)(s-t)+\eta\sigma(N(s)-N(t))}.$$

Employing the log characteristic function of the V. G. process, we have that

$$E_t^P[D(s)^\eta] = D(t)^\eta e^{\eta\mu(s-t)-\eta\phi(\sigma/i)(s-t)+\phi(\eta\sigma/i)(s-t)}.$$

Substituting into (A.1) and integrating, we obtain (4.5).

Derivation of the Risk-Free Rate of (4.8)

The discount bond price is given by

$$\begin{aligned} B(t)D(t)^{\eta-1} &= e^{-\delta(T-t)} E_t^P D(T)^{\eta-1} \\ &= e^{-\delta(T-t)} D(t)^{\eta-1} e^{[(\eta-1)\mu-(\eta-1)\phi((\sigma/i)+\phi(\eta-1)\sigma/i)](T-t)}. \end{aligned}$$

The result follows on taking the negative of the log of $B(t)$ divided by $(T - t)$.

Derivation of the Density Process of (4.10)

The t conditional expectation of marginal utility at T is

$$E_t^P[e^{-rT}D(T)^{\eta-1}] = e^{-rT}D(t)^{\eta-1}e^{[(\eta-1)\mu-(\eta-1)\phi(\sigma/i)+\phi((\eta-1)\sigma/i)](T-t)}.$$

The expression $D(t)^{(\eta-1)}$ can be written as

$$D(t)^{(\eta-1)} = C(t)e^{(\eta-1)\sigma N(t)}$$

for some function $C(t)$. On normalizing we obtain the martingale

$$\lambda(t) = e^{(\eta-1)\sigma N(t)-\phi((\eta-1)\sigma/i)t}$$

of (4.10).

APPENDIX B.

The Solution for α

On substitution for ϕ from (2.2), (3.13) implies that

$$e^{\nu(\mu-r)} = \frac{1-\nu(\alpha+\sigma)^2/2}{(1-\nu\alpha^2/2)(1-\nu\sigma^2/2)}. \tag{B.1}$$

Moreover, the positivity of $\phi(\alpha/i)$ and $\phi((\alpha+\sigma)/i)$ implies that $\nu\alpha^2/2$ and $\nu(\alpha+\sigma)^2/2$ must both be less than unity.

Define

$$\theta = e^{\nu(\mu-r)} \tag{B.2}$$

and

$$\gamma = \nu\sigma^2/2, \tag{B.3}$$

and then rewrite (B.1) as

$$\theta(1-\gamma) = \frac{1-\nu(\alpha+\sigma)^2/2}{1-\nu\alpha^2/2}. \tag{B.4}$$

Since an interesting entity is α/σ, we can write (B.4) as a quadratic in α/σ to obtain

$$\theta(1-\gamma) = \frac{1-\gamma(\alpha/\sigma+1)^2}{1-\gamma(\alpha/\sigma)^2}, \tag{B.5}$$

which yields the quadratic

$$\gamma(\theta(1-\gamma)-1)(\alpha/\sigma)^2 - 2\gamma(\alpha/\sigma) + (1-\gamma)(1-\theta) = 0. \tag{B.6}$$

Solving (B.3) for α/σ, we get for $\theta(1-\gamma) = 1$,

$$\alpha/\sigma = (1-\gamma)(1-\theta)/2\gamma = -1/2 \qquad (\text{since } \theta = 1/(1-\gamma)).$$

The solution of (B.6) for $\theta(1 - \gamma) \neq 1$ is

$$\frac{\alpha}{\sigma} = \frac{1}{\theta(1 - \gamma) - 1} \pm \left(\frac{\theta(1 - \gamma)}{(\theta(1 - \gamma) - 1)^2} + \frac{1}{\gamma}\right)^{1/2}.$$

The radicand is always positive for $\theta > 0$ and $0 < \gamma < 1$.

If $\theta(1 - \gamma) > 1$, then the choice of the sign for the radical must be negative because $(\alpha/\sigma)^2$ must be below $1/\gamma$ for the denominator of (B.5) to be positive. For the same reason the sign of the radical must be positive if $\theta(1 - \gamma) < 1$.

With these sign choices we obtain (5.1).

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