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This paper proves the existence of a general equilibrium in a financial model with transaction costs. A general equilibrium is shown to exist in a model with convex trading technology, in which the agents include consumers, production firms, brokers or dealers. When the trading technology is non-convex, an individual approximate equilibrium, introduced by Heller and Starr (1976), is proved in the above model. And, moreover, under a further assumption of finite p-convexity on the commodity excess demand correspondence, a general equilibrium for a non-convex exchange economy is obtained for an economy with consumers, brokers or dealers.

Keywords: Arbitrage, general equilibrium, transaction cost, individual approximate equilibrium, finite p-convexity.

1. Introduction

A number of authors have considered financial markets with transaction costs, particularly the impact of transaction costs on optimal portfolio selection (e.g., Magill and Constantinides (1976), Kandell and Ross (1983), Taksar, Klass and Assaf (1988), Duffie and Sun (1990), Fleming *et al.* (1989), Davis and Norman (1990)); and the pricing and hedging of derivative securities using the underlying stock and bond (e.g., Leland (1985), Boyle and Vorst (1992), Bensaid, *et al.* (1992), Edirisinghe, Naik and Uppal (1993), Constantinedes and Zariphopoulou (1995)).

More recently, some authors (e.g., Jouini and Kallal (1995), Ortu (1995), Milne and Neave (1996)) have investigated economies with transaction costs and the implications for asset prices and allocations. Jouini and Kallal (1995) use arbitrage methods as introduced by Harrison and Kreps (1979) to obtain a set of equivalent Martingale measures that are deduced from an economy with transaction costs and an absence of arbitrage. Ortu (1995) uses duality and linear programming methods in finite dimensions extending Jouini and Kallal's results. Milne and Neave

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(1996) formulate a competitive economy with multiple dates, uncertainty, a single physical commodity and a set of consumers trading assets through a set of broker/intermediaries, who have explicit transaction technologies. Their formulation draws upon an older literature in General Equilibrium theory that characterises transaction costs by discriminating between bought and sold commodities or assets (see Ostroy and Starr (1990) for a survey of this literature). Assuming the existence of an equilibrium, Milne and Neave characterise pricing and asset allocations for a number of special cases of their model, emphasising the model's flexibility in encompassing many cases discussed separately in the literature (e.g., broking, personal transaction costs, fixed and variable transaction costs, inventory-type models, incomplete markets). The Milne–Neave model is consistent with the formulations of Jouini–Kallal and Ortu, in providing a general primal formulation. The latter papers exploit no arbitrage/duality methods to obtain similar, or complementary results.

This paper constructs a more general version of Milne and Neave (1996) including many physical commodities, producers/firms and general assumptions on feasible consumption, production sets, and transaction technologies. We provide conditions that guarantee the existence of an equilibrium in this economy when the transaction technology is convex. To incorporate non-convexities (or fixed costs) in transactions, the model considers a modification introduced in an earlier general equilibrium literature (see Heller and Starr (1976)) that allows us to prove the existence of an approximate equilibrium. In addition we introduce a different method for proving existence in a version of our economy with non-convex transaction technologies. It is well-known that transactions often involve a fixed cost for each transaction, and some element of marginal cost: our non-convex technology deals with that case. We suppose a condition of finite p-convexity on the commodity excess demand correspondence. This assumption allows a limited degree of non-convexity in the asset trade technology, and yet a well-defined element of convexity in commodity demand to generalise the existence of an exact equilibrium.

The rest of this paper is organized as follows. In Sec. 2, we outline the basic model and introduce the concept of no arbitrage and its equivalent condition. In Sec. 3, we will prove some preliminary results and, finally, show the existence of equilibrium for the model with convex trading technology. Section 4 is devoted to individual approximate equlibrium in the model with non-convex trading technology. In Sec. 5, a concept of finite p-convexity is introduced and the general equilibrium is proved in an exchange economy. In Sec. 6, we conclude with a discussion of special cases and possible extensions of our model. The appendix includes a proof of one preliminary result.

2. The Model and No Arbitrage

Consider an economy with uncertainty characterized by a event tree such as that depicted in Fig. 1 of Duffie (1987). This tree consists of a finite set of nodes $\mathbf{E}$

and directed arcs $\mathbf{A} \subset \mathbf{E} \times \mathbf{E}$, such that $(\mathbf{E}, \mathbf{A})$ forms a tree with a distinquished root e_0. For any node $e \in \mathbf{E}$ other than the root node e_0, e^- designates the unique predecessor of e. The number of immediate successor nodes of any $e \in \mathbf{E}$ is denoted $\#e$. A node $e \in \mathbf{E}$ is terminal if it has no successor node. The immediate successor nodes of any non-terminal node $e \in \mathbf{E}$ are labeled $e^{+1}, \ldots, e^{+K}$, where $K = \#e$. The sub-tree with root e is denoted $\mathbf{E}(\mathbf{e})$. Particularly, $\mathbf{E} = \mathbf{E}(e_0)$. Suppose there are N securities and M commodities at any node $e \in \mathbf{E}$. Assume that all assets pay dividends at each node.

There are J production firms (indexed by j) with objective function $V_j(\cdot)$, each of whom chooses a production plan and a trade plan. A production plan of firm j is an array of numbers $y_{j,m}(e)$, one for each $m \in \mathbf{M} = \{1, \ldots, M\}$, and $e \in \mathbf{E}$ with the usual sign convention for inputs (non-positive) and outputs (non-negative). Thus a contingent production plan $y_j(e) = (y_{j,1}(e), \ldots, y_{j,M}(e))$ of firm j is a point of vector space $\mathbf{R}^M$. The set of all contingent production plans that are technologically feasible for firm j will be denoted by $Y_j \subseteq \mathbf{R}^{|E| \times M}$, where $|\mathbf{E}|$ is the total number of nodes of $\mathbf{E}$.

A trade plan of firm j is a vector $\theta_j = (\theta_j(e))_{e \in \mathbf{E}} = (\theta_j^B(e), \theta_j^S(e))_{e \in \mathbf{E}} = (\theta_{j,1}^B(e), \ldots, \theta_{j,N}^B(e), \theta_{j,1}^S(e), \ldots, \theta_{j,N}^S(e))_{e \in \mathbf{E}}$ **in** $\mathbf{R}_+^{2(|\mathbf{E}| \times N)}$, where $\theta_{j,n}^B(e)(\theta_{j,n}^S(e))$ represents the accumulated purchase (sale) of asset n by firm j after trading at node e. Let $\theta_{j,0,n}^B$ and $\theta_{j,0,n}^S$ denote the initial shares of asset n bought and sold respectively by firm j just before trading begins at node e_0. And let $\theta_{j,0} = (\theta_{j,0,1}^B, \ldots, \theta_{j,0,N}^B, \theta_{j,0,1}^S, \ldots, \theta_{j,0,N}^S)$.

There are H brokers or intermediaries (indexed by h) with objective function $W_h(\cdot)$. They are intermediaries specializing in the transaction technology that transforms bought and sold assets. Let $\phi_{h,0} = (\phi_{h,0}^S, \phi_{h,0}^B) = (\phi_{h,0,1}^S, \ldots, \phi_{h,0,N}^S, \phi_{h,0,1}^B, \ldots, \phi_{h,0,N}^B)$ be the initial trading by broker h. $\phi_{h,n}^B(e)(\phi_{h,n}^S(e))$ be the accumulated number of bought (sold) asset n supplied by intermediary h after trading at e and denote $\phi_h = (\phi_h^S, \phi_h^B) = (\phi_h^S(e), \phi_h^B(e))_{e \in \mathbf{E}}$; and $z_h(e) = (z_{h,1}(e), \ldots, z_{h,M}(e))$ be the vector of contingent commodities used up in the activity of intermediation at date t and denote $(z_h(e))_{e \in \mathbf{E}}$ by z_h. For intermediary h, let $T_h(e) \subseteq \mathbf{R}_+^N \times \mathbf{R}_+^N \times \mathbf{R}_+^M$ denote its technology at node e.

There are I consumers, indexed by i, with endowment $\omega_{i,t}$, initial trading $\psi_{i,0}$, and utility function $U_i(\cdot)$ and consumption set $X_i = \mathbf{R}_+^{|\mathbf{E}| \times M}$. The consumer i chooses a consumption plan $x_i = (x_{i,1}(e), \ldots, x_{i,M}(e))_{e \in \mathbf{E}} \in X_i$ and a portfolio plan $\psi_i = (\psi_i^B, \psi_i^S) \in \mathbf{R}_+^{2(|\mathbf{E}| \times N)}$ which can be explained analogously to θ_j.

Now we turn to assets. At each node $e \in \mathbf{E}$, asset $n (n = 1, \ldots, N)$ has a buying price $B^n(e)$ and selling price $S^n(e)$ and dividend $d_n(e)$ denominated in the first commodity (numeraire). Suppose that at each node e, the asset n pays its dividend d_n (denominated in the numeraire commodity) and is then available for trade at prices $B_n(e)$ and $S_n(e)$.

Let $d = \{d(e) = (d_1(e), \ldots, d_N(e)) : e \in \mathbf{E}\}, B = \{B(e) = (B^1(e), \ldots, B^N(e)) : e \in \mathbf{E}\}$ and $S = \{S(e) = (S^1(e), \ldots, S^N(e)) : e \in \mathbf{E}\}$. A dividend process d^θ

generated by a generic strategy $\theta = (\theta^B, \theta^S)$ is defined by

$$d^\theta(e) = (\theta^B(e^-) - \theta^S(e^-))d(e) + \triangle\theta^S(e)S(e) - \triangle\theta^B(e)B(e), e \in \mathbf{E},$$

where $\theta(e_0^-) = (\theta^B(e_0^-), \theta^S(e_0^-))$ is the initial shares bought and sold by an agent just before the trading takes place at node e_0. Here $\triangle\theta^B(e) = \theta^B(e) - \theta^B(e^-)$; and $\triangle\theta^S(e) = \theta^S(e) - \theta^S(e^-)$. And define $\triangle\theta(e) = \theta(e) - \theta(e^-)$.

Let $p = (p)_{e\in\mathbf{E}} = (p_1(e), \ldots, p_M(e))_{e\in\mathbf{E}} (\in \triangle_0)$ denote the spot price of commodities, where $\triangle_0$ is the unit simplex of $\mathbf{R}^{M\times|\mathbf{E}|}$.

Now the problem of firm j can be specified as

$$\sup_{(\theta_j, y_j)\in\Gamma_j^1(p)} V_j(d^{\theta_j} + py_j), \qquad (*)$$

where $\Gamma_j^1(p)$ denote the set of feasible production-trade plans (θ_j, y_j) of firm j given price p, which satisfies

(2.1) y_j is in Y_j;
(2.2) $d^{\theta_j}(e) + p(e)y_j(e) \geq 0$, $\forall e \in \mathbf{E}$.

The maximization problem of broker h can be stated as

$$\sup_{(\phi_h, z_h)\in\Gamma_h^2(\gamma_h)} W_h(-d^{\phi_h} - pz_h), \qquad (**)$$

where $\Gamma_h^2(\gamma_h)(\gamma_h = ((\psi_i), (\theta_j), (\phi_{h'\neq h}), p)$ is the space of feasible trade-production plans $(\phi_h, z_h) = (\phi_h^S, \phi_h^B, z_h)$ given γ_h, which satisfies, at each node,

(2.3) $(\triangle\phi_h^B(e), \triangle\phi_h^S(e), z_h(e)) \in T_h(e)$ and $z_h(e) \geq 0$, $\forall e \in \mathbf{E}$;
(2.4) $-d^{\phi_h}(e) - p(e)z_h(e) \geq 0$, $\forall e \in \mathbf{E}$;
(2.5) $\sum_h \triangle\phi_h(e) \geq \sum_i \triangle\psi_i(e) + \sum_j \triangle\theta_j(e)$, $\forall e \in \mathbf{E}$.

Comments: 1. The condition (2.5) requires that all consumers and all production firms buy and sell securities through brokers.

2. The productive firms and intermediary firms are treated similarly to consumers. Because of transaction costs, it is well-known that the Fisher separation theorem fails. Therefore we assume that each firm has an objective function (utility function) derived in some fashion from the preference of the owners. For example, we can either assume one-owner firms or draw on DeMarzo (1988) and argue that the objective is derived from a more complicated composition of owner preferences.

3. The intermediary formulation allows the agent to trade on its own account; or by interpreting the transaction technology more narrowly, it can be restricted to a pure broker with direct pass through of assets bought and sold (see Milne and Neave (1996) for further discussion).

The problem of consumer i is as follows:

$$\sup_{(\psi_i, x_i)\in\Gamma_i^3(\tau)} U_i(x_i), \qquad (***)$$

where $\tau = ((\phi_h, z_h), (\theta_j, y_j), p)$ and $\Gamma_i^3(\tau)$ is the set of feasible portfolio-consumption plans (ψ_i, x_i) given τ, which satisfies

(2.6) x_i is in X_i;
(2.7) $p(e)x_i(e) \leq p(e)\omega_i(e) + d^{\psi_i}(e) + \sum_j \alpha_{i,j}\bar{y}_j(e) + \sum_h \beta_{i,h}\bar{z}_h(e)$, $\forall e \in \mathbf{E}$.

Here $\bar{y}_j(e) = d^{\theta_j}(e) + p(e)y_j(e), \bar{z}_h(e) = -d^{\phi_h}(e) - p(e)z_h(e); \alpha_{i,j} \geq 0(\sum_i \alpha_{i,j} = 1)$ is consumer i's initial share of the net cash flow of firm j; and $\beta_{i,h} \geq 0(\sum_i \beta_{i,h} = 1)$ is consumer i's initial share of net cash flow of broker h.

Now we can define the abstract economy: $\mathcal{E} = (X_1 \times \mathbf{R}^{2(N\times|\mathbf{E}|)}, \ldots, X_I \times \mathbf{R}^{2(N\times|\mathbf{E}|)}, Y_1 \times \mathbf{R}^{2(N\times|\mathbf{E}|)}, \ldots, Y_J \times \mathbf{R}^{2(N\times|\mathbf{E}|)}, T_1', \ldots, T_H', \triangle_0, U_1(x_1), \ldots, U_I(x_I), V_1(d^{\theta_1} + py_1), \ldots, V_J(d^{\theta_J} + py_J), W_1(-d^{\phi_1} - pz_1), \ldots, W_H(-d^{\phi_H} - pz_H), \sum_{e\in\mathbf{E}} \pi(e)p(e)w(e),\ \Gamma_1^3(\tau), \ldots, \Gamma_I^3(\tau),\ \Gamma_1^1(p), \ldots, \Gamma_J^1(p),\ \Gamma_1^2(\gamma_1), \ldots, \Gamma_H^2(\gamma_H), \triangle_0)$, where $\pi(e)(e \in \mathbf{E})$ will be defined in Lemma 2.1 below, $w(e) = \sum_i x_i(e) + \sum_h z_h(e) - \sum_j y_j(e) - \sum_i \omega_i(e)$ and

$$T_h' = \{(\phi_h^S, \phi_h^B, z_h) : (\triangle\phi_h^B(e), \triangle\phi_h^S(e), z_h(e)) \in T_h(e), \forall e \in \mathbf{E}\}.$$

A point $e^* = ((\psi_i^*, x_i^*), (\theta_j^*, y_j^*), (\phi_h^*, z_h^*), p^*)$ is called an equilibrium solution of economy $\mathcal{E}$ given the market system (B, S, d) if e^* solves problems $(*)$, $(**)$ and $(***)$ and

$$\sum_i x_i^* + \sum_h z_h^* = \sum_j y_j^* + \sum_i \omega_i,$$

$$\sum_h \triangle\phi_h^* = \sum_i \triangle\psi_i^* + \sum_j \triangle\theta_j^*.$$

The following assumptions are made in the remainder of this paper.
For consumer i:

(A.1) $U_i(\cdot)$ is a continuous, concave and strictly increasing function.

For firm j:

(A.2) Y_j is a closed convex subset of $\mathbf{R}^{M\times|\mathbf{E}|}$ containing $-R_+^{M\times|\mathbf{E}|}$;
(A.3) $Y_j \cap \mathbf{R}_+^{M\times|\mathbf{E}|} = \{0\}$;
(A.4) $(\sum_j Y_j) \cap (-\sum_j Y_j) = \{0\}$;
(A.5) $V_j(\cdot)$ is a continuous, concave and strictly increasing function.

For broker h:

(A.6) For each e, $T_h(e)$ is a closed and convex set with $0 \in T_h(e)$.
(A.7) For any e and given $x = (x_1, \ldots, x_{2N}, z_1, \ldots, z_M) \in T_h(e)$, If $y = \sum_{n=1}^{2N} x_n \to \infty$, then $|(z_1, \ldots, z_M)| = \sum_{m=1}^{M} z_m \to \infty$.
(A.8) For each e, if $(\psi, z) \in T_h(e)$ and $z' \geq z$, then $(\psi, z') \in T_h(e)$ (free disposal).

(A.9) $W_h(\cdot)$ is a continuous, concave and strictly increasing function:

(A.10) The initial holdings $\psi_{i,0}, \theta_{j,0}, \phi_{h,0}$ of securities of all consumers, all firms and all brokers are taken as given and satisfy

$$\sum_i \psi_{i,0} + \sum_j \theta_{j,0} = \sum_h \phi_{h,0} \,.$$

The assumptions (A.1)–(A.6) are standard. (A.7) says that transactions must consume resources. (A.8) and (A.9) are standard. And (A.10) says that the initial net trading is zero.

Now we conclude this section by introducing the concept of no-arbitrage.

Let an agent have initial shares $\theta_0 = (\theta_0^B, \theta_0^S)$ just before the trading takes place at node e_0. Then his or her initial capital will be $d_0 = d(e_0)(\theta_0^B - \theta_0^S) + S(e_0)\theta_0^B - B(e_0)\theta_0^S$ if he or she sells the holding and buys back the short-sale. Given a price-dividend triple (B, S, d) for N securities, a trading strategy θ is an arbitrage if $-d_0 \geq 0$ and $d^\theta(e) \geq 0$, $\forall e \in \mathbf{E}$, and moreover, $-d_0 > 0$ or there exists at least one node $e \in \mathbf{E}$ such that $d^\theta(e) > 0$.

(A.11) The price-dividend triple (B, S, d) admits no-arbitrage.

The following equivalent condition of no arbitrage is similar to Proposition 2C of Duffie (1996) and will play an important role in the proof of market clearing later.

Lemma 2.1. *There is no arbitrage if and only if there is a strictly increasing linear function $F : \mathbf{R}^{|\mathbf{E}|+1} \to \mathbf{R}^1$ such that $F((-d_0, d^\theta)) \leq 0$ for any trading strategy $\theta \in \Theta$, where Θ denotes the space of trading strategies and is a closed and convex set.*

Proof. There is no arbitrage if and only if the cones $\mathbf{R}_+^{|\mathbf{E}|+1}$ and $M^0 = \{(-d_0, d^\theta) : \theta \in \Theta\}$ intersect precisely at zero. If there is no arbitrage, the theorem "Linear Separation of Cones" in Appendix B of Duffie (1996) implies the existence of a nonzero linear functional F such that $F(x) < F(y)$ for each x in M^0 and each nonzero y in $\mathbf{R}_+^{|\mathbf{E}|+1}$. Since M^0 is a cone, this implies $F(x) \leq 0$ for each x in M^0. And, moreover, $0 \in M^0$; thus $F(y) > F(0) = 0$ for each nonzero $y \in \mathbf{R}_+^{|\mathbf{E}|+1}$. That is, F is strictly increasing. The converse is immediate. □

In comparison with Proposition 2C in Duffie (1996), the next result shows the difference between the model without transaction costs and the model with transaction costs caused by the bid-ask spread at some node.

Lemma 2.2. *Suppose there is no arbitrage and $B^n(\tilde{e}) > S^n(\tilde{e})$ for security n at a certain node $\tilde{e}$. Then $F((-d_0, d^\theta)) \not\equiv 0$ over Θ.*

Proof. Suppose not. Then $F((-d_0, d^\theta)) = 0$ for each $\theta \in \Theta$. Since $F(\cdot)$ is a strictly increasing linear functional on $\mathbf{R}_+^{|\mathbf{E}|+1}$, this implies that there exists a

vector $\pi = (\pi^0, (\pi^e)_{e\in\mathbf{E}}) \in \mathrm{int}(\mathbf{R}_+^{|\mathbf{E}|+1})$ such that $F(x) = \pi^0 x_0 + \sum_{e\in\mathbf{E}} \pi^e x_e$ for each $x = (x_0, (x_e)_{e\in\mathbf{E}}) \in \mathbf{R}^{|\mathbf{E}|+1}$. Without loss of generality, we assume $N = n = 1$. For a fixed positive integer M, set

$$\theta^B(e) = 0, \forall e \in \mathbf{E} - \{\widetilde{e}\}, \theta^B(\widetilde{e}) = M\,;$$
$$\theta^S(e) = 0, \forall e \in \mathbf{E}\,.$$

Then $d^\theta(e) = d(e)(\theta_0^B - \theta_0^S), \forall e \notin \mathbf{E}(\widetilde{e}), d^\theta(\widetilde{e}) = d(\widetilde{e})(\theta_0^B - \theta_0^S) - MB^1(\widetilde{e}), d^\theta(e) = d(e)(\theta_0^B - \theta_0^S + M), \forall e \in \mathbf{E}(\widetilde{e}) - \{\widetilde{e}\}$.

Hence

$$M\pi(\widetilde{e})B^1(\widetilde{e}) = -\pi^0 d_0 + \sum_{e\in\mathbf{E}} \pi(e)d(e)(\theta_0^B - \theta_0^S) + M \sum_{e\in\mathbf{E}(\widetilde{e})-\{\widetilde{e}\}} \pi(e)d(e)\,.$$

Likewise, by setting

$$\theta^S(e) = 0, \forall e \in \mathbf{E} - \{\widetilde{e}\}, \theta^S(\widetilde{e}) = M\,;$$
$$\theta^B(e) = 0, \forall e \in \mathbf{E}\,.$$

$$M\pi(\widetilde{e})S^1(\widetilde{e}) = \pi^0 d_0 - \sum_{e\in\mathbf{E}} \pi(e)d(e)(\theta_0^B - \theta_0^S) + M \sum_{e\in\mathbf{E}(\widetilde{e})-\{\widetilde{e}\}} \pi(e)d(e)\,,$$

and therefore,

$$M\pi(\widetilde{e})(B^1(\widetilde{e}) - S^1(\widetilde{e})) = 2\left[\sum_{e\in\mathbf{E}} \pi(e)d(e)(\theta_0^B - \theta_0^S) - \pi^0 d_0\right].$$

Thus $B^1(\widetilde{e}) = S^1(\widetilde{e})$ since M is arbitrary, which provides a contradiction and proves the conclusion of the lemma. □

Therefore, from Proposition 2C of Duffie (1996), in a security market without friction and with no arbitrage, the initial value of any trading strategy is uniquely determined by the inner product of its future cash flows with a martingale measure. But, by the above two lemmas, this conclusion does not hold in the security market with transaction costs: the martingale measure is not unique (see Jouini and Kallal (1995) for similar observations).

3. Proof of Existence of Equilibrium

We will adopt the technique of proof used in Arrow–Debreu (1954). First of all, we will show that the set of attainable plans for economy $\mathcal{E}$ is bounded, and replace the original economy $\mathcal{E}$ by a bounded one. Secondly, we will show the continuity of the constrained correspondences.

For broker h, define

$Z_h = \{z_h$: there exists $(\phi_h^S, \phi_h^B) \geq 0$ such that $(\phi_h^S, \phi_h^B, z_h) \in T_h'\}$;

$\hat{Z}_h = \{z_h \in Z_h$: there exist $z_{h'} \in Z_{h'}$ for each $h' \neq h, x_i \in X_i$ for each i and $y_j \in Y_j$ for each j such that $w = \sum_i x_i + \sum_h z_h - \sum_j y_j - \sum_i \omega_i \leq 0\}$;

$\Phi_h = \{\phi_h = (\phi_h^S, \phi_h^B)$: there exists $z_h \in \hat{Z}_h$ such that $(\phi_h, z_h) \in T_h'\}$;

$\hat{\Phi}_h = \{\phi_h = (\phi_h^S, \phi_h^B) \in \Phi_h$: there exist $\phi_{h'} \in \Phi_{h'}$ for each $h' \neq h, \triangle\theta_j \geq 0$ for each j and $\triangle\psi_i \geq 0$ for each i such that $\sum_i \triangle\psi_i + \sum_j \triangle\theta_j \leq \sum_h \triangle\phi_h\}$.

For consumer i, define

$\hat{X}_i = \{x_i \in X_i$: there exist $x_{i'} \in X_{i'}$ for each $i' \neq i, z_h \in Z_h$ for each h and $y_j \in Y_j$ for each j such that $w = \sum_i x_i + \sum_h z_h - \sum_j y_j - \sum_i \omega_i \leq 0\}$;

$\hat{\Psi}_i = \{\psi_i = (\psi_i^B, \psi_i^S)$: there exist $\triangle\psi_{i'} \geq 0$ for each $i' \neq i, \phi_h \in \Phi_h$ for each $h, \triangle\theta_j \geq 0$ for each j such that $\sum_i \triangle\psi_i + \sum_j \triangle\theta_j \leq \sum_h \triangle\phi_h\}$.

Likewise, we can define $\hat{Y}_j$ and $\hat{\Theta}_j$ for firm j.

By use of the technique of 3.3.1 of Arrow–Debreu (1954) and by Lemma A1, we can show the following result.

Lemma 3.1. *The sets $\hat{X}_i, \hat{Y}_j$ and $\hat{Z}_h$ are all compact and convex.*

In exactly the same method as Lemma 3.1 (and A1 in Appendix), we can show the following boundedness result of trade plan for broker h.

Lemma 3.2. *The set Φ_h is a compact and convex subset of $\mathbf{R}^{2(|\mathbf{E}|\times N)}$. And so is $\hat{\Phi}_h$, $\hat{\Psi}_i$ and $\hat{\Theta}_j$.*

Thus there exist cubes $C^1(\subseteq \mathbf{R}^{|\mathbf{E}|\times M})$ and $C^2(\subseteq \mathbf{R}_+^{2(|\mathbf{E}|\times N)})$ so that C^1 contains in its interior all $\hat{X}_i$, all $\hat{Y}_j$ and all $\hat{Z}_h$; C^2 contains all $\hat{\Psi}_i$, all $\hat{\Theta}_j$ and all $\hat{\Psi}_h$. Define $\tilde{X}_i = C^1 \cap X_i, \tilde{\Psi}_i = C^2, \tilde{Y}_j = C^1 \cap Y_j, \tilde{\Theta}_j = C^2, \tilde{Z}_h = C^1 \cap Z_h$ and $\tilde{\Phi}_h = C^2$. And let $\tilde{\Gamma}_j^1(p), \tilde{\Gamma}_h^2(\gamma_h)$ and $\tilde{\Gamma}_i^3(\tau)$ be the resultant modification of $\Gamma_j^1(p), \Gamma_h^2(\gamma_h)$ and $\Gamma_i^3(\tau)$ respectively.

We now turn to the proof of continuity of $\tilde{\Gamma}_j^1(p), \tilde{\Gamma}_h^2(\gamma_h)$ and $\tilde{\Gamma}_i^3(\tau)$. We only investigate the continuity of $\tilde{\Gamma}_h^2$, the continuity of the others can be shown similarly.

Lemma 3.3. *Given p, all ψ_i, all θ_j and all $\phi_{h'}(h' \neq h)$, and there exists $(\phi_h, z_h) \in T_h'$ such that $\sum_i \triangle\psi_i + \sum_j \triangle\theta_j \ll \sum_h \triangle\phi_h$ and $0 \ll -d^{\phi^h} - pz_h$. Then $\tilde{\Gamma}_h^2(\gamma_h)$ is continuous.*

Proof. Without loss of generality, we show the continuity of $\tilde{\Gamma}_1^2(\gamma_1)$. Let $\gamma_1^k = (\psi_1^k, \ldots, \psi_I^k, \theta_1^k, \ldots, \theta_J^k, \phi_2^k, \ldots, \phi_H^k, p^k) \to \gamma_1 = (\psi_1, \ldots, \psi_I, \theta_1, \ldots, \theta_J, \phi_2, \ldots, \phi_H, p)$. Consider a point $(\phi_1, z_1) \in \tilde{\Gamma}_1^2(\gamma_1)$, then

$$0 \leq -d^{\phi_1} - pz_1, \ \sum_i \triangle\psi_i + \sum_j \triangle\theta_j \leq \sum_h \triangle\phi_h .$$

If $0 \ll -d^{\phi_1} - pz_1$, $\sum_i \triangle\psi_i + \sum_j \triangle\theta_j \ll \sum_h \triangle\phi_h$, then for k sufficiently large, $0 \ll -d^{\phi_1} - p^k z_1$, and $\sum_i \triangle\psi_i^k + \sum_j \triangle\theta_j^k \ll \sum_{h\neq 1} \triangle\phi_h^k + \triangle\phi_1$. By taking $(\phi_1^k, z_1^k) = (\phi_1, z_1)$, we prove the conclusion of the lemma.

If the above case does not hold, then there exist $\mathbf{E}_0 \subseteq \mathbf{E}$ and $\mathbf{E}_0', \widetilde{\mathbf{E}}_0' \subseteq \widetilde{\mathbf{E}} = \mathbf{E} \times \{1, \ldots, N\}$ (there is at least one nonempty set among $\mathbf{E}_0$ and $\mathbf{E}_0'$ and $\widetilde{\mathbf{E}}_0'$) such that

$$-d^{\phi_1}(e) - p(e)z_1(e) = 0\,,\ e \in \mathbf{E}_0\,;$$
$$-d^{\phi_1}(e) - p(e)z_1(e) > 0\,,\ e \in \mathbf{E} - \mathbf{E}_0\,;$$

$$\sum_i \triangle\psi^B_{i,q}(e) + \sum_j \triangle\theta^B_{j,q}(e) = \sum_h \triangle\phi^S_{h,q}(e),\ (e,q) \in \mathbf{E}_0'\,,$$
$$\sum_i \triangle\psi^B_{i,q}(e) + \sum_j \triangle\theta^B_{j,q}(e) < \sum_h \triangle\phi^S_{h,q}(e),\ (e,q) \in \widetilde{\mathbf{E}} - \mathbf{E}_0'\,,$$

and

$$\sum_i \triangle\psi^S_{i,q}(e) + \sum_j \triangle\theta^S_{j,q}(e) = \sum_h \triangle\phi^B_{h,q}(e),\ (e,q) \in \widetilde{\mathbf{E}}_0'\,,$$
$$\sum_i \triangle\psi^S_{i,q}(e) + \sum_j \triangle\theta^S_{j,q}(e) < \sum_h \triangle\phi^B_{h,q}(e),\ (e,q) \in \widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}_0'\,.$$

By assumption, we can choose $(\phi_1', z_1') \in T_1'$ such that

$$0 \ll -d^{\phi_1'} - pz_1'\,,$$

and

$$\sum_i \triangle\psi_i + \sum_j \triangle\theta_j \ll \sum_{h \neq 1} \triangle\phi_h + \triangle\phi_1'\,.$$

Clearly

$$\triangle\phi'^S_{1,q}(e) > \triangle\phi^S_{1,q}(e),\ (e,q) \in \mathbf{E}_0'\,,$$

and,

$$\triangle\phi'^B_{1,q}(e) > \triangle\phi^B_{1,q}(e),\ (e,q) \in \widetilde{\mathbf{E}}_0'\,.$$

Let

$$\lambda^1_k = \min\left\{1, \frac{\triangle\phi'^S_{1,q}(e) - (\sum_i \triangle\psi^{k,B}_{i,q}(e) + \sum_j \triangle\theta^{k,B}_{j,q}(e) - \sum_{h\neq 1} \triangle\phi^{k,S}_{h,q}(e))}{\triangle\phi'^S_{1,q}(e) - \triangle\phi^S_{1,q}(e)} : (e,q) \in \mathbf{E}_0'\right\};$$

$$\bar{\lambda}^1_k = \min\left\{1, \frac{\triangle\phi'^B_{1,q}(e) - (\sum_i \triangle\psi^{k,S}_{i,q}(e) + \sum_j \triangle\theta^{k,S}_{j,q}(e) - \sum_{h\neq 1} \triangle\phi^{k,B}_{h,q}(e))}{\triangle\phi'^B_{1,q}(e) - \triangle\phi^B_{1,q}(e)} : (e,q) \in \widetilde{\mathbf{E}}_0'\right\};$$

and

$$\lambda_k^2 = \min\left\{1, \frac{-d^{\phi'_1}(e) - p^k(e)z_1'(e)}{-d^{\phi'_1}(e) - p^k(e)z_1'(e) - (-d^{\phi_1}(e) - p^k(e)z_1(e))} : e \in \mathbf{E}_0\right\}.$$

Let $\lambda_k = \min(\lambda_k^1, \bar{\lambda}_k^1, \lambda_k^2)$ and $(\phi_1^k, z_1^k) = \lambda_k(\phi_1, z_1) + (1 - \lambda_k)(\phi'_1, z_1')$.
It is easy to verify that $\lambda_k^i \to 1 (i = 1, 2)$ and

$$-d^{\phi_1^k}(e) - p^k(e)z_1^k(e) \geq 0, \ e \in \mathbf{E}_0\,; \tag{3.1}$$

$$\sum_i \triangle\psi_{i,q}^{k,B}(e) + \sum_j \triangle\theta_{j,q}^{k,B}(e) \leq \sum_h \triangle\phi_{h,q}^{k,S}(e), \ (e,q) \in \mathbf{E}_0'\,;$$

and

$$\sum_i \triangle\psi_{i,q}^{k,S}(e) + \sum_j \triangle\theta_{j,q}^{k,S}(e) \leq \sum_h \triangle\phi_{h,q}^{k,B}(e), \ (e,q) \in \widetilde{\mathbf{E}}_0'\,;$$

for k sufficiently large.

On the other hand, since $\alpha = \min_{e \in \mathbf{E} - \mathbf{E}_0}\{-d^{\phi_1}(e) - p(e)z_1(e)\} > 0$ and

$$\lim_{k \to \infty}(-d^{\phi_1^k}(e) - p^k(e)z_1^k(e)) = -d^{\phi_1}(e) - p(e)z_1(e)\,.$$

Hence, for k sufficiently large,

$$-d^{\phi_1^k}(e) - p^k(e)z_1^k(e) \geq 0 : e \in \mathbf{E} - \mathbf{E}_0\,,$$

which, combining with (3.1), implies that for k sufficiently large,

$$-d^{\phi_1^k} - p^k z_1^k \geq 0\,.$$

Likewise, we can show that for k sufficiently large,

$$\sum_i \triangle\psi_i^k + \sum_j \triangle\theta_j^k \leq \sum_h \triangle\phi_h^k\,.$$

Consequently, $(\phi_1^k, z_1^k) \in \tilde{\Gamma}_1^2(\gamma_1^k)$ and converges to (ϕ_1, z_1), proving the continuity of $\tilde{\Gamma}_1^2(\cdot)$. □

For firms and consumers, we have the following similar results.

Lemma 3.4. *For firm j, given any price $p \in \triangle_0$, assume there exist $\theta_j' \geq 0$ and $y_j' \in Y_j$ such that $0 \ll d^{\theta_j'} + py_j'$. Then $\tilde{\Gamma}_j^1(\cdot)$ is continuous.*

Lemma 3.5. *For consumer i, given any price $p \in \Delta_0$, assume there exist $\psi_i' \geq 0$ and $x_i' \in X_i$ such that*

$$px_i' \ll p\omega_i + d^{\psi_i'} + \max\left\{0, \sum_j \alpha_{i,j}\bar{y}_j + \sum \beta_{i,h}\bar{z}_h\right\}.$$

Then $\tilde{\Gamma}_i^3(\cdot)$ is continuous.

Remark 3.1. The conditions in Lemmas 3.4 and 3.5 will be satisfied if there exists a portfolio θ such that $d^\theta(e) > 0, \forall e \in \mathbf{E}$, which is the Assumption 2.1 of Ortu (1995) called the "Internality Condition". This condition can be guaranteed if the agent has a positive initial long position and has no initial short position and adopt a buy-hold strategy, that is, $\theta^B(e) = \theta_0^B > 0$ and $\theta^S(e) = \theta_0^S = 0$, $\forall e \in \mathbf{E}$. Observe that these conditions are extensions of well-known standard assumptions that ensure continuity of budget correspondences. Weaker conditions could be found, but these will suffice for our purpose.

Let

$$\mu_i = \mu_i(\tau) = \left\{(\psi_i, x_i) : U_i(x_i) = \sup_{(\bar{\psi}_i, \bar{x}_i) \in \bar{\Gamma}_i^3(\tau)} U_i(\bar{x}_i)\right\};$$

$$\upsilon_j = \upsilon_j(p) = \left\{(\theta_j, y_j) : V_j(d^{\theta_j} + py_j) = \sup_{(\bar{\theta}_j, \bar{y}_j) \in \bar{\Gamma}_j^1(p)} V_j(d^{\bar{\theta}_j} + p\bar{y}_j)\right\};$$

$$\tau_h = \tau_h(\gamma_h) = \left\{(\phi_h, z_h) : W_h(-d^{\phi_h} - pz_h) = \sup_{(\bar{\phi}_h, \bar{z}_h) \in \bar{\Gamma}_h^2(\gamma_h)} W_h(-d^{\bar{\phi}_h} - p\bar{z}_h)\right\};$$

$$\bar{p} = \bar{p}(w) = \left\{p : \sum_{e \in \mathbf{E}} \pi(e)p(e)w(e) = \sup_{p' \in \Delta_0} \sum_{e \in \mathbf{E}} \pi(e)p'(e)w(e)\right\},$$

and

$$\Psi = \prod_{i=1}^{I} \mu_i \times \prod_{j=1}^{J} \upsilon_j \times \prod_{h=1}^{H} \tau_h \times \bar{p}.$$

By Berge's Maximum Theorem and standard methods, we can prove that the correspondences $\mu_i, \upsilon_j, \tau_h$ and $\bar{p}$ are upper hemi-continuous and convex valued. This implies Ψ is also upper hemi-continuous and convex valued.

The correspondence Ψ has been shown to satisfy the hypotheses of the Kakutani fixed point theorem, and therefore to have a fixed point, say $e^* = ((\psi_i^*, x_i^*), (\theta_j^*, y_j^*), (\phi_h^*, z_h^*), p^*)$. Especially, this fixed point satisfies

$$\sum_i \Delta\psi_i^*(e) + \sum_j \Delta\theta_j^*(e) \leq \sum_h \Delta\phi_h^*(e), \ \forall e \in \mathbf{E}, \tag{3.2}$$

$$\sum_{e \in \mathbf{E}} \pi(e)p^*(e)w^*(e) \geq \sum_{e \in \mathbf{E}} \pi(e)p(e)w^*(e), \ \forall p \in \Delta_0. \tag{3.3}$$

By assumption (A.10) and Lemma 2.1,

$$\begin{aligned}
\sum_{e\in\mathbf{E}} \pi(e)p^*(e)w^*(e) &\le \sum_{e\in\mathbf{E}} \pi(e)d^{\sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^*}(e) \\
&= -\pi_0 \left[d(e_0) \left(\sum_i \psi_{i,0} + \sum_j \theta_{j,0} - \sum_h \phi_{h,0} \right) \right. \\
&\quad + S(e_0) \left(\sum_i \psi_{i,0}^B + \sum_j \theta_{j,0}^B - \sum_h \phi_{h,0}^S \right) \\
&\quad \left. -B(e_0) \left(\sum_i \psi_{i,0}^S + \sum_j \theta_{j,0}^S - \sum_h \phi_{h,0}^B \right) \right] \\
&\quad + \sum_{e\in\mathbf{E}} \pi(e)d^{\sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^*}(e) \\
&= F\left(-d_0, d^{\sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^*} \right) \le 0\,,
\end{aligned}$$

where

$$\begin{aligned}
d_0 = d(e_0) &\left(\sum_i \psi_{i,0} + \sum_j \theta_{j,0} - \sum_h \phi_{h,0} \right) \\
&+ S(e_0) \left(\sum_i \psi_{i,0}^B + \sum_j \theta_{j,0}^B - \sum_h \phi_{h,0}^S \right) - B(e_0) \left(\sum_i \psi_{i,0}^S + \sum_j \theta_{j,0}^S - \sum_h \phi_{h,0}^B \right).
\end{aligned}$$

Hence, from (3.3), $w^*(e) \le 0, \forall e \in \mathbf{E}$.

Let $\triangle y_J^*(e) = -w^*(e) \ge 0, \triangle\theta_J'^*(e) = \sum_h \triangle\phi_h^*(e) - \sum_i \triangle\psi_i^*(e) - \sum_j \triangle\theta_j^*(e) \ge 0$, $e \in \mathbf{E}$. And set

$$\bar{y}_J^* = y_J^* - \triangle y_J^*,\ \triangle\bar{\theta}_J^* = \triangle\theta_J^* + \triangle\theta_J'^*\,.$$

Clearly,

$$\bar{y}_J^* \in Y_J,\ \sum_i x_i^* = \sum_{j\ne J} y_j^* + \bar{y}_J^* + \sum_h z_h^*\,,$$

and

$$\sum_i \triangle\psi_i^* + \sum_{j\ne J} \triangle\theta_j^* + \triangle\bar{\theta}_J^* = \sum_h \triangle\phi_h^*\,.$$

Moreover

$$\begin{aligned}
p^*(\triangle y_J^*) &= -p^* w^* = -d^{\sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^*} \\
d^{\theta'^*} &= -d^{\sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^*}\,,
\end{aligned}$$

this implies that

$$d^{\bar{\theta}_J^*} + p * \bar{y}_J^* = d^{\theta_J^*} + p^* y_J^* .$$

Finally, in exactly the same method as Arrow and Debreu (1954), it is not difficult to show $\bar{e}^* = ((x_i^*, \psi_i^*), (y_{j\neq J}^*, \theta_{j\neq J}^*), (\bar{y}_J^*, \bar{\theta}_J^*), (\phi_h^*, z_h^*), p^*)$ is an equilibrium point of the original Economy $\mathcal{E}$.

4. Non-Convex Production Economy

This section is devoted to a economy in which the trading technology of each broker is not necessarily convex so that we allow for fixed costs in trading assets. By using the technique of Heller and Starr (1976), we will show the existence of an individual approximate equilibrium defined by Heller and Starr (1976). An approximate equilibrium is generally defined as a price p^* and two allocations, a^* and $a^{*\prime}$. One, a^*, is the allocation desired by households, firms and brokers at this price, which may not clear the market. The other, $a^{*\prime}$, is an allocation obeying the market clearance condition although it need not represent agents' optimizing behaviour. The equilibrium is approximate of a modulus C if some suitably chosen norm of the difference between these two allocations is no larger than C. The desired allocation represents an approximate equilibrium in the sense that the failure to clear the market at this price is bounded by C. And, furthermore, the bound of the approximation improves as the number of the agents in the economy increases.

We will continue to make all the assumptions in Sec. 2 except the convexity of the broker's technology. We further assume the following:

(A.12) $B_\omega = Y \cap (X + Z - \omega)$ is bounded, where $X = \sum_{i\in I} X_i, \omega = \sum_{i\in I} \omega_i$, $Y = \sum_{j\in J} Y_j$ and $Z = \sum_{h\in H} Z_h$.

Since the assumptions of Theorem 1 of Hurwicz and Reiter (1973) can be easily verified through Assumptions A.4 and A.12, we can show the boundedness of $\hat{Z}_h, \hat{X}_i, \hat{Y}_j$; and, hence, $\hat{\Phi}_h, \hat{\Psi}_i$ and $\hat{\Theta}_j$ are all bounded.

In order to show that the equilibrium of the bounded economy is the equilibrium of the original economy, an additional assumption is required.

(A.13) There is a positive number L_0 such that $|z_h| \leq L_0, \forall z_h \in Z_h$.

That is, the quantity of commodities used in transaction of assets is limited. This is reasonable since a quantity larger than the total supply of the world is not feasible. So the feasible plan of the broker should satisfy the additional assumption (A.13). The cubes C^1 and C^2 used in defining the bounded economy can be chosen to be large enough to contain the feasible plan of any broker.

As in Heller and Starr (1976), in order to prove the continuity of $\tilde{\Gamma}_h^2(\gamma_h)$, we give the definition of local interior.

Definition 4.1. $\tilde{\Gamma}_h^2$ is said to be locally interior if for each $(\phi, z) \neq 0, (\phi, z) \in \tilde{\Gamma}_h^2(\gamma_h)$ there is (ϕ^*, z^*) so that

(i) $(\phi^*, z^*) \in \tilde{\Gamma}_h^2(\gamma_h)$.
(ii) $0 \ll -d^{\phi^*} - pz^*$.
(iii) There exists a continuous function $f : [0,1] \to \tilde{\Gamma}_h^2(\gamma_h)$ so that $f(0) = (\phi^*, z^*), f(1) = (\phi, z)$ and for all $\sigma(\in [0,1)), f(\sigma)$ satisfies the strict inequility in (ii).

(A.14) $\tilde{\Gamma}_h^2(\gamma_h)$ is locally interior.

Now we are in a position to show the existence of an individual approximate equilibrium of the economy with non-convexities. But we omit its proof since it can be obtained in the same method as Heller and Starr (1976). In the proof, we use the correspondence Ψ (defined in Sec. 3) instead of $\gamma(p)$ defined in Heller and Starr (1976). The boundedness of $R(\Psi)$ defined in Heller and Starr (1976) can be clearly guaranteed by the assumption (A.13).

Theorem 4.1. *Under the assumptions (A.1)-(A.14) omitting the convexity of the brokers' technologies, there exists an individual approximate equilibrium of modulus C which only depends on M, N, L_0 and $R(p^*)$, where p^* is an approximate equilibrium price. That is, there exist two vectors $a^* = (\phi_1^*, z_1^*, \ldots, \phi_H^*, z_H^*, \psi_1^*, x_1^*, \ldots, \psi_I^*, x_I^*, \theta_1^*, y_1^*, \ldots, \theta_J^*, y_J^*)$ and $a^{*\prime} = (\phi_1^{*\prime}, z_1^{*\prime}, \ldots, \phi_H^{*\prime}, z_H^{*\prime}, \psi_1^{*\prime}, x_1^{*\prime}, \ldots, \psi_I^{*\prime}, x_I^{*\prime}, \theta_1^{*\prime}, y_1^{*\prime}, \ldots, \theta_J^{*\prime}, y_J^{*\prime})$ such that*

(i) *$a^{*\prime}$ satisfies market clearance with respect to p^*.*
(ii) *a^* solves problems $(*)$, $(**)$ and $(***)$ with respect to p^*.*
(iii) $(\phi_i^*, x_i^*) = (\phi_i^{*\prime}, x_i^{*\prime}), (\theta_j^*, y_j^*) = (\theta_j^{*\prime}, y_j^{*\prime}), i \in I, j \in J$.
(iv) $(\sum_h |(\psi_h^*, z_h^*) - (\psi_h^{*\prime}, z_h^{*\prime})|^2)^{1/2} \leq C$.

5. An Exchange Economy with Finite p-Convexity

In this section, the existence of a general equilibrium of an exchange economy is investigated, which only includes consumers, brokers or dealers. For simplicity we omit productive firms. In this model, the trading technology of each broker is not necessarily convex. The model includes some cases of fixed costs, e.g., a model with set-up cost, as special cases. By introducing the more restrictive concept of finite p-convexity we are able to prove the existence of an exact equilibrium, even when there are some fixed costs in transacting.

A fixed cost is represented by an initial fixed amount of input before there is any output; after that, we can allow outputs and inputs to increase. For example, setting up an office with a computer etc., will require inputs independent of how much work is done in the office. Clearly, this gives a non-convex production set.

We will retain all the assumptions in Sec. 4 except (A.12), (A.13) and (A.14). It is not difficult to show the boundedness of sets $\hat{X}_i, \hat{Z}_h, \hat{\Psi}_i$ and $\hat{\Phi}_h$. Moreover, we will introduce another assumption called finite p-convexity. Finally, the existence of general equilibrium is proved. To this end, we give the following definition of finite p-convexity.

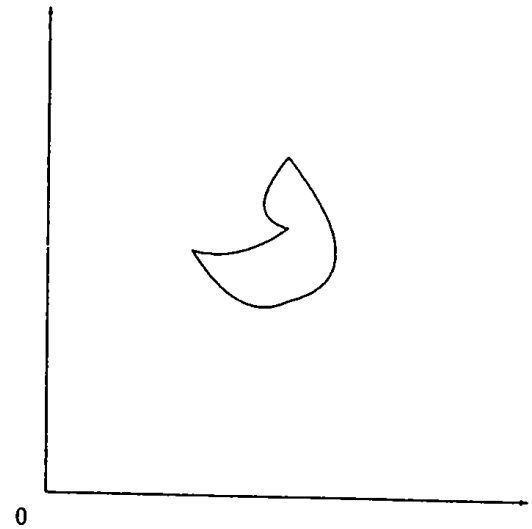

Fig. 1. Finite p-convex set.

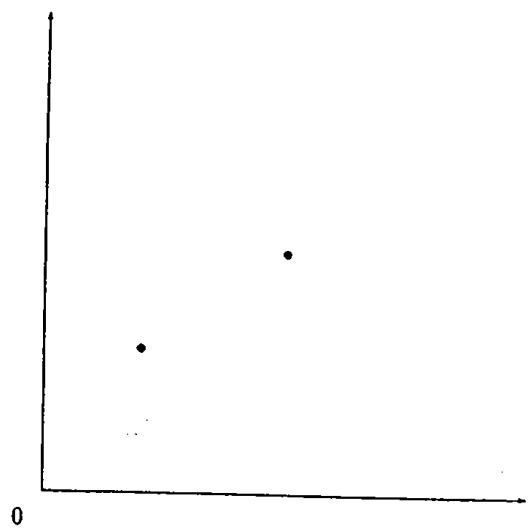

Fig. 2. Finite p-convex set.

Definition 5.1. Let X be a subset of R^n and $\triangle^{(n-1)}$ be the simplex of R^n. Then X is called finitely p-convex if for any x_1, $x_2 \in X$ and $p_1, \ldots, p_m \in \text{int}(\triangle^{(n-1)})$ there is $\bar{x} \in X$ such that $p_i\bar{x} \leq p_i(\frac{x_1+x_2}{2})$ for all $i = 1, 2, \ldots, m$ (see Figs. 1 and 2).

Let $(\phi, z) = (\phi_1, z_1, \ldots, \phi_h, z_h)$ and define the feasible set $\tilde{\Gamma}_i(p, \psi, z)$ (given price p and broker's plan (ϕ, z)) of consumer i analogously to $\tilde{\Gamma}_i^3(\tau)$; and the feasible set $\tilde{\Gamma}_h(\gamma_h)$ of broker h analogously to $\tilde{\Gamma}_h^2(\gamma_h)$ in Sec. 3. Define the demand function $\mu_i(p, \phi, z)$ of consumer i as $\mu_i(\tau)$ and the demand function $\tau_h(\gamma_h)$ of broker h as $\tau_h(\gamma_h)$ in Sec. 3.

Let

$$\xi(p) = \left\{ \sum_h (-\triangle\phi_h, z_h) + \sum_i (\triangle\psi_i, x_i) - \sum_I (0, \omega_i) | (\phi_h, z_h) \in \tau_h(\gamma_h), \right.$$

$$\left. \forall h \in H, (\psi_i, x_i) \in \mu_i(p, \phi, z), \forall i \in I \right\}.$$

As shown in Heller and Starr (1976), it can be shown that the sets $\tilde{\Gamma}_h(\gamma_h)$ and $\tilde{\Gamma}_i(p, \phi, z)$ are all continuous. Thus, the correspondences $\mu_i(p, \phi, z)$ and $\tau_h(\gamma_h)$ are upper hemi-continuous and also compact valued. Now it is not difficult to show that the correspondence $\xi(p)$ is upper hemi-continuous. And, moreover, the projection $\xi_0(p)$ of $\xi(p)$ onto the commodity space is also upper hemi-continuous.

Before the proof of the main result of this section, we introduce two lemmas.

Lemma 5.1. *Let $P \subseteq R^l$ be a compact set and let $\phi : P \to R^m$ be an upper hemi-continuous correspondence. If $\forall p \in P$,*

$$\Phi(p) = \{z \in R^m : z\mu > 0, \ \forall \mu \in \phi(p)\} \neq \emptyset,$$

then there exists a continuous function, $W : P \to R^m$, such that $W(p) \in \Phi(p)$, $\forall p \in P$ (cf. McCabe (1981)).

Lemma 5.2. *Suppose that X and Y are two non-empty compact spaces and that $f : X \times Y \to R$ is a real-valued function such that*

(i) *$x \to f(x, y)$ is lower hemi-continuous on X for each $y \in Y$; $y \to f(x, y)$ is upper hemi-continuous for each $x \in X$.*
(ii) *X is finitely f-convex: i.e., for any $x_1, x_2 \in X$ and $y_1, \ldots, y_n \in Y$ there is $\bar{x} \in X$ such that $f(\bar{x}, y_i) \leq \frac{1}{2}[f(x_1, y_i) + f(x_2, y_i)]$ for all $i = 1, \ldots, n$;*
(iii) *Y is finitely f-concave: i.e., for any $y_1, y_2 \in Y$ and $x_1, \ldots, x_m \in X$ there exist $\bar{y} \in Y$ such that $f(x_j, \bar{y}) \geq \frac{1}{2}[f(x_j, y_1) + f(x_j, y_2)]$ for all $j = 1, \ldots, m$.*

Then

$$\min_X \max_Y f(x, y) = \max_Y \min_X f(x, y)$$

(cf. Granas and Fon-Che Liu (1987)).

We now turn to the main result of this section.

Theorem 5.1. *Suppose that $\xi_0(p)$ is finitely p-convex and all assumptions omitting producers in Sec. 2 hold. Then there exists a general equilibrium $e^* = ((\psi_i^*, x_i^*)_{i \in I}, (\phi_h^*, z_h^*)_{h \in H}, p^*)$ in the non-convex exchange economy, i.e., e^* satisfies the following condition:*

(i) (ψ_i^*, x_i^*) *solves problem* $(***)$ *for each* $i \in I$;
(ii) (ϕ_h^*, z_h^*) *solves problem* $(**)$ *for each* $h \in H$;
(iii) e^* *satisfies market clearance, i.e.,*

$$\sum_h z_h^* + \sum_i x_i^* - \sum_i \omega_i = 0\,;$$

and

$$\sum_h \triangle\psi_h^* = \sum_i \triangle\phi_i^*\,.$$

Proof. We first truncate by a natural number n the set Z_h (defined in Sec. 3) and prove the existence of general equilibrium in the truncated economy $\mathcal{E}^n$. And then by taking limits, the existence of equilibrium can be obtained as in Geanakoplos and Polemarchakis (1986).

Furthermore, the cubes C^1 and C^2 are also chosen large enough to include the truncated feasible sets of all brokers.

Note that the consumption sets of all consumers are $R_+^{|\mathbf{E}|\times M}$. Hence, by the definition of $\xi(p)$, to prove the existence of general equilibrium it suffices to show that there exists $p_0 \in \triangle^{(|\mathbf{E}|\times M-1)}$ such that $\xi_0(p_0) \cap (-R_+^{|\mathbf{E}|\times M}) \neq \emptyset$.

It is equivalent to that there exist $z_0 \in \pi\xi_0(p_0)$ such that $\max_{p\in\triangle^{(|\mathbf{E}|\times M-1)}} pz^0 \leq 0$, where $\pi = (\pi(e))_{e\in\mathbf{E}}$ as defined in Sec. 2 and

$$\pi\xi_0(p_0) = \{(\pi(e)z_1(e), \ldots, \pi(e)z_M(e))_{e\in\mathbf{E}} | z = (z_1(e), \ldots, z_M(e))_{e\in\mathbf{E}} \in \xi_0(p_0)\}\,.$$

And it is easy to show that $\pi\xi_0(p)$ is upper hemi-continuous and finitely p-convex.

We will prove the conclusion of this theorem by a contradiction. To this end, let, for each $k \geq |\mathbf{E}| \times M$,

$$\triangle_k^{(|\mathbf{E}|\times M-1)} = \left\{p = (p_1(e), \ldots, p_M(e)) \in \triangle^{(|\mathbf{E}|\times M-1)} | p_i(e) \geq \frac{1}{k}, i = 1, \ldots, |\mathbf{E}| \times M\right\}.$$

For each $p \in \triangle_k^{(|\mathbf{E}|\times M-1)}$ and each $z \in \pi\xi_0(p)$, suppose that there exists a $p' \in \triangle_k^{(|\mathbf{E}|\times M-1)}$ such that $p'z > 0$. Hence, $\max_{\triangle_k^{(|\mathbf{E}|\times M-1)}} p'z > 0$. By the continuity of the function $\max_{\triangle_k^{(|\mathbf{E}|\times M-1)}} p'z$, we have

$$\min_{\pi\xi_0(p)} \max_{\triangle_k^{(|\mathbf{E}|\times M-1)}} p'z > 0\,.$$

In Lemma 5.2, by taking $X = \pi\xi_0(p), Y = \triangle_k^{(|\mathbf{E}|\times M-1)}$ and $f(z, p') = p'z$, it is easy to verify that all conditions in this lemma are satisfied. And, particularly, the condition finite f-convexity of X corresponds to the finite p-convexity of $\pi\xi_0(p)$.

Therefore,

$$\max_{\triangle_k^{(|\mathbf{E}|\times M-1)}} \min_{\pi\xi_0(p)} p'z > 0\,,$$

which implies that there exist $p'_k \in \triangle_k^{(|\mathbf{E}|\times M-1)}$, such that $\min_{\pi\xi_0(p)} p'_k z > 0$ and moreover, $p'_k z > 0, \forall z \in \pi\xi_0(p)$. This is equivalent to that

$$\Phi(p) = \left\{p' \in \triangle_k^{(|\mathbf{E}|\times M-1)} | p'z > 0, \forall z \in \pi\xi_0(p)\right\} \neq \emptyset$$

for each $p \in \triangle_k^{(|\mathbf{E}|\times M-1)}$.

Thus, by Lemma 5.1, there exists a continuous function $W(p) : \triangle_k^{(|\mathbf{E}|\times M-1)} \longrightarrow \triangle_k^{(|\mathbf{E}|\times M-1)}$ such that $W(p) \in \Phi(p), \forall p \in \triangle_k^{(|\mathbf{E}|\times M-1)}$. Then, by the Brouwer fixed point theorem, there is a $p_k^0 \in \triangle_k^{(|\mathbf{E}|\times M-1)}$ such that $p_k^0 = W(p_k^0)$. This means that $p_k^0 z > 0, \forall z \in \pi\xi_0(p_k^0)$, which contradicts the Walras' Law which has been established in Sec. 3. Therefore, for each $p_k \in \triangle_k^{(|\mathbf{E}|\times M-1)}$, there exists $z_k \in \pi\xi_0(p_k)$ such that $\max_{\triangle_k^{(|\mathbf{E}|\times M-1)}} pz_k \leq 0$.

Since $\{\pi\xi_0(p_k) | p_k \in \triangle_k^{(|\mathbf{E}|\times M-1)}, k = 1, 2, \ldots)\}$ are compact subsets of a compact set, there is a convergent subsequence of (p_k, z_k) with limit (p^0, z^0).

Note that the set $\xi_0(p)$ is empty for each $p \in \partial\triangle^{(|\mathbf{E}|\times M-1)}$ since $\pi\xi$ is upper hemi-continuous (as proved in Lemmas 3.3 and 3.5) and the utility function of consumer 1 is strictly increasing. It is not difficult to show that $p^0 \in \text{int}\triangle^{(|\mathbf{E}|\times M-1)}$, $z^0 \in \pi\xi_0(p^0)$ and $\max_{\triangle^{(|\mathbf{E}|\times M-1)}} pz^0 \leq 0$, proving the existence of equilibrium of the truncated economy.

In exactly the same method as that of Geanakoplos and Polemarchakis (1986), it can be shown that there exists a equilibrium in the original economy. □

We will finish this section by an example. We will show that the commodity excess demand correspondence of a model with set-up costs satisfies finite p-convexity.

An example. Consider a one-period model in which there are two commodities, N securities and only one broker. We still retain the assumption of convexity for consumers. But the broker needs a set-up cost before trading. The trading technology of the broker is described as follows:

$$T = \{(\theta^B, \theta^S, x_1, x_2) \in R_+^{N+1} : F(\theta^B, \theta^S, x_1, x_2) \leq 0, x_2 \geq k > 0\} \cup \{(0,0,0,0)\},$$

where k is positive constant, that is, the broker needs k units of commodity 2 to set up his or her trading. Here the F is a convex function and strictly increasing with respect to each component. Including $(0,0,0,0)$ in the trading technology T means no trading.

Given a strictly positive commodity price, it suffices to show the commodity excess demand correspondence of broker is finitely p-convex since the commodity excess demand correspondences of all consumers are convex. Clearly, the commodity excess demand correspondence of the broker is a convex subset in T' (where T' is the projection of set $T - \{(0,0,0,0)\}$ onto the commodity space) or the union E of a convex subset of T' and $\{(0,0)\}$. The set E is finitely p-convex since, in the

Definition 5.1, for any $x_1, x_2 \in E$ and $p_1, \ldots, p_m \in \operatorname{int}(\Delta^1)$, $p_i(0,0) \leq p_i(\frac{x_1+x_2}{2})$ for all $i = 1, \ldots, m$.

Consequently, there exists an equilibrium in this model.

6. Conclusion

This paper has attempted to attain three objectives:

1. Prove the existence of equilibrium of an asset economy with transaction costs. The model is sufficiently general to cover most cases (finite states, time horizon) in the literature.
2. The method of proof proves some new results (see Ortu (1995)) extending arbitrage pricing dual results to cover transaction costs and different buying and selling prices.
3. In addition, two proofs are provided of existence of an equilibrium with non-convex transaction technologies. These proofs are important for addressing economies with fixed costs in transacting.

Two final comments: in Milne–Neave (1996), it is shown that the basic model can be adapted easily to accomodate a number of variations common in the literature. For example, by considering $I = J = \emptyset$, and brokers are considered as ordinary consumers with a "transaction technology" representing short-sales constraints on trading, the proofs can be interpreted as proving the existence of an equilibrium with trading constraints. A special case of this formulation is an exchange economy with incomplete asset markets. We have discussed an approximate equilibrium with non-convexities. It is possible to modify our model to allow a continuum of agents and obtain an exact equilibrium via the use of the Liapunov Convexity Theorem.

Finally observe that we can incorporate the recent model of Prechac (1996) by assuming that brokers have a linear technology. He assumes that this is a simple markup, but to avoid underpricing, assumes that banking or clearance house has a monopoly with an exogenous, fixed markup.

Appendix

Lemma A1. If the assumption (A.7) holds, then the set Z_h is a closed convex set.

Proof. The convexity of Z_h is obvious. It remains to show its closedness. Suppose $z_h^k \in Z_h$ and $z_h^k \to z_h$. For each k, there exists $\phi_h^{(k)}$ such that $(\Delta\phi_h^{(k)}, z_h^k) \in T_h(e)$, and, in particular, by (A.8), $(\Delta\phi_h^{(k)}, z_h') \in T_h(e)$, where $z_h' = (\max_k z_{h,1}^k, \ldots, \max_k z_{h,M}^k)$. If $\{\Delta\phi_h^{(k)}\}$ is unbounded, we may suppose $\Delta(\phi^{(k)})_{h,1}^B \to \infty$ without loss of generality.

But, by assumption (A.7),

$$\lim_{k\to\infty} |z|_h^k = \infty\,,$$

which provides a contradiction and proves the boundedness of $\{\triangle\phi_h^{(k)}\}$. Hence, we can choose a subsequence $\{\triangle\phi_h^{(k_n)}\}$ from $\{\triangle\phi_h^{(k)}\}$ such that

$$\lim_{n\to\infty} \triangle\phi_h^{(k_n)} = \triangle\phi_h\,;$$

this implies, by closedness of $T_h(e)$, the closedness of Z_h. □

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