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Optimal Commodity Taxation for Reduction of Envy

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Abstract This paper derives optimal commodity taxes in a two-class economy, based on Chaudhuri (1986) and Diamantaras and Thomson's (1990) λ-equitability. An allocation is λ-equitable if no agent envies a proportion λ of the bundle of any other agent. We examine the properties of Pareto undominated allocations for various λ-equitability requirements. In contrast with the classic Ramsey rule and its extension, *ceteris paribus*, the goods preferred by the low skilled agent and/or of high Hicksian elasticities are taxed more heavily. As to the total tax burden, the envying agent may bear a higher tax burden, since the good which he likes should be taxed more heavily to reduce envy. Also, due to the conflict between welfare of the envying agent and his envy, there exists an economy in which the Diamantaras-Thomson allocation – an allocation which maximizes λ in the range of Pareto efficient allocations – is the Pareto efficient allocation which minimizes the welfare of the envying agent.

1 Introduction

In an economy where agents have different skill levels, there are several ethical reasons to consider redistribution. One of them is envy. Agent i envies agent j if the former thinks the latter's bundle better than his own. In this paper, we choose the reduction of envy as the social objective, and analyze the tax policy implications on the basis of that objective.

As in the real world, when sophisticated tax-transfer policies based on one's ability are infeasible, there arises a hierarchy of envy: for any allocation implementable by commodity taxation, the high skilled agent never envies the low skilled agent, whereas, unless their preferences are so diverse or their productivities are sufficiently close, the low skilled always envies the high skilled, except for some trivial allocations (Bös and Tillmann (1985)). The celebrated equity concept of no-envy faces difficulty in this situation, and some cardinal measure to evaluate intensity of envy is required when the government's objective is the reduction of envy.

Chaudhuri (1986) and Diamantaras and Thomson (1990) considered the following measure of envy. An allocation is λ-equitable if no agent envies a proportion λ of the bundle of any other agent. The value of λ measures the intensity of envy. If λ is unity, an allocation satisfies no-envy. Any feasible allocation satisfies zero-equitability. They propose to choose the Pareto efficient allocations which maximize λ (hereafter referred to as *the Diamantaras-Thomson allocations*). This paper, more broadly, studies how given λ-equitability requirements affect the qualitative properties of the tax implementable allocations, by looking at the efficient allocations among the λ-equitable allocations. Consider the maximal welfare combinations if the government is constrained by a given

λ-equitability requirement. Call this *the λ-constrained second-best frontier*. Varying λ, we can depict the set of the second-best welfare frontiers with various envy constraints. The Diamantaras-Thomson allocations are included in some frontier(s).

In the context of social choice, the λ-constrained second-best frontier corresponds to the maximal elements of the following λ-equity first and efficiency second principle. An allocation x is better than y in the sense of the λ-equity first and efficiency second principle iff 1) x is λ-equitable and y is not λ-equitable or 2) both x and y are λ-equitable or neither of them is λ-equitable, and x Pareto dominates y. This is an extension of Tadenuma's (1998) equity-first principle. The Pareto criterion, although secondary, serves for selection, to exclude trivial allocations such as no production. On the other hand, some Pareto inefficient allocations may well be included, since Pareto efficiency is secondary. But global Pareto efficiency is not necessarily appealing, if efficiency considerations are not primary. A social choice theoretic foundation is given in Nishimura (1998).

We consider a two-class economy, commonly used in the theory of optimal taxation, in which there are agents with high and low skill levels. We examine the case in which the low skilled agent envies the high skilled agent. Proposition 1 shows that the following goods are taxed more heavily (in the sense of having a larger reduction in compensated demand), other things being equal.

1. A good which is preferred by the low skilled (envying) agent and/or its Hicksian elasticities for the high skilled agent are large.
2. A good which is consumed more by those whose social marginal utility of income (the social value of giving \$ 1 to agent i) is lower (a sufficient condition for the low skill's social marginal utility of income to be higher is given in Lemma 3).

Point 1 is a natural consequence of the envy-reducing taxation problem: if the envied agent decreases the consumption of the good which the envying agent prefers, envy is reduced; similarly, if preferences are monotonic, taxing the good with a high price elasticity moderates envy. For those who are familiar with the utilitarian optimal taxation problem, this can be contrasted with the classic tax rules: the good with high Hicksian own elasticity should *not* be taxed because it accrues large efficiency loss (Ramsey (1927)); the goods which the low utility agent, whose weight in the social welfare function is higher under concavity, prefers should *not* be taxed (Diamond (1975) etc.). This rule may conflict with point 2. As a result, the low skilled agent may bear a higher tax burden, since the good which he likes tends to be taxed heavily.

We also discuss the welfare properties of the Diamantaras-Thomson allocations in the second-best taxation. Numerical examples, which are available upon request to the author, suggest that a Rawlsian type allocation, an allocation which maximizes the low skill's welfare, coincides with the allocation which maximizes λ in many cases. This property can be shown to apply rigorously under identical homotheticity (Proposition 2). However, a striking numerical example exists: there exists an economy in which the second-best Diamantaras-Thomson allocation is the Pareto efficient point which maximizes the welfare of the high skilled agent (Proposition 3).

Section 2 introduces basic notations and clarifies the class of admissible policies. Section 3 introduces the equity objective under consideration. Section 4 examines the optimal tax rule and tax burden of the allocations on the λ-constrained second-best frontier. Section 5 discusses the welfare properties of the Diamantaras-Thomson allocations. Section 6 concludes. Proof of lemmas are gathered in an appendix.

2 The Model

Consider a two-agent economy with k consumption goods and labor. Each agent i ($i = 1, 2$) has a preference relation represented by the utility function $u_i(c_i, l_i)$, where $c_i \equiv (c_i^m)_{m=1}^k \in I\!R_+^k$ represents the consumption bundle and $l_i \in [0, \bar{l}]$ represents leisure. The utility function is increasing in all the elements, strictly quasi-concave and twice differentiable. Agents are also endowed with an exogenous skill level w_i, with $w_2 > w_1 > 0$. Production exhibits constant marginal costs, and firms are competitive, so that consumption taxes do not change pre-tax prices. Let $p = (p^m)_{m=1}^k$ be the marginal cost of producing private goods. The set of feasible allocations in an economy is $\{((c_1, l_1), (c_2, l_2)) \in (I\!R_+^k \times [0, \bar{l}])^2 | p \cdot (c_1 + c_2) \leq w_1(\bar{l} - l_1) + w_2(\bar{l} - l_2)\}$. A feasible allocation is often denoted by $x = (x_1, x_2)$.

The commodity taxes are used as a policy tool, with labor in efficiency units being the numeraire. Commodity tax $t \equiv (t^m)_{m=1}^k \in I\!R^k$ is proportional to transactions and is the same for both agents. The tax revenue is used for lump-sum transfer of $-T$ to each agent.[1]

Given the tax schedule, the indirect utility function is derived.

$$v_i(t, T) \equiv \max_{c_i, l_i} u_i(c_i, l_i) \quad \text{s.t. } (p + t) \cdot c_i \leq w_i(\bar{l} - l_i) - T \tag{1}$$

From the solution of (1), c_i, l_i ($i = 1, 2$) become functions of t and T.

3 The Equity Objective

We consider envy with respect to agents' consumption-leisure bundles. Agent i envies agent j in allocation x if $u_i(x_j) > u_i(x_i)$. Under the tax policy introduced in the previous section, we can observe the following hierarchy of envy. First, as $w_2 > w_1$, agent 2 *never* envies agent 1, since agent 2 can purchase agent 1's consumption bundle with fewer amount of labor.[2] On the other hand, unless their preferences are very diverse or their productivities are sufficiently close, agent 1 always envies the consumption-leisure bundle of agent 2. Typically, the low skilled agent envies the high skilled agent in any second-best Pareto efficient allocation. In fact, if we apply the classic concept of no-envy – no agent envies any other agent – to the current institutional framework, we can say the following: *in a standard economy where everyone has the identical preferences, the only no-envy and tax implementable state is 'no production'* (Bös and Tillmann (1985)).[3] If we attach a negative weight to envy, and moreover efficiency considerations are required, we are faced with reconstructing an alternative concept of no-envy.

In the social choice literature, the principal approach to evaluate a social state based on envy is to introduce a cardinal measure for the intensity of envy. Among these, we will adopt a radial contraction measure advocated by Chaudhuri (1986) and Diamantaras and Thomson (1990).[4] Given an allocation x, let $\lambda_{ij}^x \in I\!\!R_+$ be a value such that $u_i(x_i) = u_i(\lambda_{ij}^x x_j)$ when $u_i(x_i) \leq u_i(x_j)$, and $\lambda_{ij}^x \equiv 1$ when $u_i(x_i) > u_i(x_j)$.[5] When agent i envies agent j, the value of λ_{ij}^x measures the amount by which one would have to shrink j's bundle in order for i to stop envying j, and thus indicates the intensity of envy which agent i has against agent j in allocation x. Figure 1 depicts the idea in the case of one output ($k = 1$). Given an allocation x, agent i envies agent j. The straight line connecting the point $(0, 0)$ and $x_j = (c_j, l_j)$ crosses the indifference curve of agent i passing through x_i exactly once, due to continuity and monotonicity. The intersection is denoted by $(\tilde{c}, \tilde{l})$, and thus

$u_i(x_i) = u_i(\tilde{c}, \tilde{l})$. The amount of envy that agent i has against agent j at allocation x is regarded to be negatively correlated with $\lambda_{ij}^x = \dfrac{\tilde{l}}{l_j} = \dfrac{\tilde{c}}{c_j}$, the ratio between the hypothetical leisure-consumption bundle and agent j's actual bundle.

Figure 1 around here.

Let $\lambda^x \equiv \min_{i,j} \lambda_{ij}^x$. An allocation x is λ-equitable iff $\lambda^x \geq \lambda$, or equivalently, $u_i(x_i) \geq u_i(\lambda x_j)$ $\forall i,\ j$. With this envy measure, we can depict how utility possibilities are altered under the considerations of λ-equitability requirements. Figure 2 depicts the utility possibilities with various λ-equitability requirements, $\lambda_2 > \lambda_1 > 0$.

Figure 2 around here.

The outer frontier is the second-best Pareto frontier (the second-best welfare frontier with a zero-equitability requirement): no allocation is excluded by the zero-equitability requirement. Increasing λ to λ_1, some points are judged inequitable. But the others are still feasible, i.e., satisfy λ_1-equitability. As a result, the λ_1-constrained second-best frontier contracts from the second-best Pareto frontier.[6] Similarly, as λ increases, the constrained second-best frontier shrinks further, and there may be a maximal λ which is compatible with second-best Pareto efficiency.[7] This is the main concern of Diamantaras-Thomson. The allocations that maximize λ among the Pareto efficient allocations are called *the Diamantaras-Thomson allocations*.[8] The welfare properties of them are discussed in Section 5.

The maximal welfare combinations when the government is constrained by a given λ-equitability can be derived by solving the following restricted Pareto optimization problem,

$$
\begin{aligned}
& \max_{t,T} v_1(t,T) \\
\text{s.t.} \quad & v_2(t,T) \geq \bar{v}_2 \\
& v_i(t,T) \geq u_i(\lambda(x_j(t,T))) \quad (i = 1,2,\ i \neq j) \\
& \sum_{i=1,2} [t \cdot c_i(t,T) + T] \geq 0
\end{aligned}
\tag{2}
$$

with varying $\bar{v}_2 \in I\!R$. Denote the pairs of the maximal utility level of agent 1 (where it exists) and the corresponding utility level $\bar{v}_2$ *the λ-constrained second-best frontier*. Varying λ, we can depict the set of the welfare frontiers with various envy constraints.

A tax policy implication of the solution is that, from the allocation there is no direction of tax changes which is Pareto improving and maintains λ-equitability. Some second-best Pareto inefficient allocations are included in the solutions, since our standpoint is that global Pareto efficiency is not necessarily appealing when there are additional equity requirements. However, the Pareto criterion is used as a secondary criterion, to avoid allocations such as no production. The approach to require λ-equitability as a prerequisite and give efficiency consideration second is an extension of Tadenuma's (1998) *Equity-first Principle*. The formalization and discussions are given in Nishimura (1998).

4 Optimal Tax Rules

Set up the Kuhn-Tucker problem corresponding to (2). The Lagrange expression is:

$$\begin{aligned} \mathcal{L} \equiv\ & v_1(t,T) + \zeta_2(v_2(t,T) - \bar{v}_2) + \\ & \mu_1(v_1(t,T) - u_1(\lambda \cdot (c_2(t,T), l_2(t,T)))) + \mu_2(v_2(t,T) - u_2(\lambda \cdot (c_1(t,T), l_1(t,T)))) + \\ & \gamma(\sum_{i=1,2}[t \cdot c_i(t,T) + T]) \end{aligned} \tag{3}$$

Let α_i be the Lagrange multiplier of (1) (marginal utility of income). We make use of the following envelope property

$$\frac{\partial v_i}{\partial T} = -\alpha_i,\ \frac{\partial v_i}{\partial t^m} = -\alpha_i x_i^m\ (i = 1, 2,\ m = 1, ..., k).$$

The first order conditions are as follows.

$$\partial \mathcal{L}/\partial t^m = 0 \iff$$

$$\sum_{i,j=1,2,i\neq j}[-(\zeta_i + \mu_i)\alpha_i x_i^m - \mu_i \frac{\partial u_i}{\partial(\lambda x_j)} \cdot \frac{\lambda \partial x_j}{\partial t^m} + \gamma(t \cdot \frac{\partial c_i}{\partial t^m} + x_i^m)] = 0\ (m = 1, ..., k), \tag{4}$$

$$\partial \mathcal{L}/\partial T = 0 \iff$$

$$\sum_{i,j=1,2,i\neq j}[-(\zeta_i + \mu_i)\alpha_i - \mu_i \frac{\partial u_i}{\partial(\lambda x_j)} \cdot \frac{\lambda \partial x_j}{\partial T} + \gamma(t \cdot \frac{\partial c_i}{\partial T} + 1)] = 0 \tag{5}$$

where $\zeta_1 = 1$ and $\partial u_i/\partial(\lambda x_j) = \partial u_i/\partial(c,l)|_{(c,l)=\lambda x_j}$.

4.1 Optimal Tax Formula

The first order condition for good m is manipulated to yield the standard formula using the Slutsky equation:[9]

Proposition 1 *At any allocation along the λ-constrained second-best frontier, there exist nonnegative numbers μ_i, ζ_i (i=1,2), γ (some of them positive) such that*

$$\sum_{i=1,2} t \cdot \frac{\partial \tilde{c}_i}{\partial t^m} = \sum_{i,j=1,2,i \neq j} \frac{\mu_i}{\gamma} \frac{\partial u_i}{\partial(\lambda x_j)} \cdot \frac{\lambda \partial \tilde{x}_j}{\partial t^m} + 2Cov(b_i,\ c_i^m) \tag{6}$$

where $\tilde{c}_i^m = \tilde{c}_i^m(t,u)$ is a Hicksian demand function, Cov denotes covariance, and

$$b_i \equiv \left(\frac{\zeta_i + \mu_i}{\gamma}\right) \alpha_i - \left((t,0) - \lambda \frac{\mu_j}{\gamma} \frac{\partial u_j}{\partial(\lambda x_i)}\right) \cdot \frac{\partial x_i}{\partial T},\ i \neq j, \tag{7}$$

where $(t,0) \in I\!R^{k+1}$ is a vector to adjust for vectorial calculation.[10]

The interpretation of the first order condition depends on which constraints are binding. Below we will impose reasonable restrictions on the sign of the Lagrange multipliers.

First, by our definition in Section 3, $\lambda \leq 1$. The restriction implies $\mu_2 = 0$ by the complementary slackness condition.

Next, we will consider the case with $\mu_1 > 0$. A sufficient condition is the following:

Lemma 1 *At any allocation on the λ-constrained second-best frontier, if there are Pareto improving directions of infinitesimal tax changes, $\mu_1 > 0$.*[11]

The qualitative feature of the tax formula along the λ-constrained second-best frontier is divided in two cases: those in which the envy constraint does bind (i.e., $\mu_1 > 0$) and those in which it does not (i.e., $\mu_1 = 0$). The latter corresponds to the second-best Pareto efficient tax formula, which has already been studied in the optimal taxation literature (e.g. Diamond (1975), Mirrlees (1975)). Here, we examine the case with $\mu_1 > 0$. As discussed above, there is a positive reason to consider some Pareto inefficient allocations, if the efficiency requirement is not primary.

Our final restriction concerns the government budget balance. The increment in the uniform reimbursement by $-\Delta T$ may increase envy: if the increment in welfare $\alpha_i \equiv \partial u_i/\partial x_i \cdot \partial x_i/\partial(-T)$ is sufficiently lower than that in $\partial u_i/\partial(\lambda x_j) \cdot \lambda \partial x_j/\partial(-T)$, such a payment may be retained, so that γ, the shadow price of government revenue, may be zero. However, we can verify that the sign of the Lagrange multiplier is positive in the class of preferences commonly used in the quantitative analysis.

Lemma 2 *At any allocation along the λ-constrained second-best frontier, $\gamma > 0$ if preferences are homothetic or quasi-linear with respect to the same good.*[12]

Below we will give interpretations of the tax formula when $\mu_1 > 0$, $\mu_2 = 0$ and $\gamma > 0$.[13]

4.2 Interpretations of the Tax Formula

The left hand side of (6) is the so-called *index of discouragement*. When the taxes are infinitesimal or Hicksian derivatives are constant, it can be approximated by the welfare-preserving change of the demand for good m from no commodity taxes to the current tax rate (Diamond and Mirrlees (1971, p.262)).[14] The smaller this value is, the more demand is *discouraged*, which implies that the good and its complements tend to be taxed more.[15]

The right hand side is decomposed to two parts. The first term is the inner product of the marginal utility of the low skilled agent and substitution effect of the compensated demand, which reflects the reduction of envy through discouragement of consumption by the high skilled agent due to taxation. The term arises even if lump-sum taxes are feasible, so that it can be called the first-best term, in the terminology of Seade (1980). The second term, on the other hand, is analogous to the so-called covariance rule in the second-best utilitarian optimal taxation problem (see, for example, Diamond (1975)). Hence the term can be called the second-best term. The b_i's correspond to the social marginal utility of income. It is the social value of giving \$ 1 to agent i. From (5), average $\bar{b} = 1$.[16]

Interpretation of the First-best Term: Following the standard in the optimal taxation literature, we interpret the effect of each term '*ceteris paribus*'. Begin with the first term of the R.H.S. of (6). Consider a) the product of the own substitution effect of the envied agent (here, the high skilled) and its marginal utility to the envying agent (here, the low skilled), and b) the product of the cross substitution effect of a complement of the taxed good and its marginal utility to the envying agent. These terms are negative. The smaller (larger in absolute value) these values are, the more demand for the good is discouraged. We interpret high values of $\partial u_1/\partial c^m|_{(c,l)=\lambda x_2}$ as 'agent 1 prefers good m'.[17] Then, we can derive an implication of the first-best term of (6) as follows:

> *if good m or its Hicksian complement in terms of the high skilled agent's preference is preferred by the low skilled agent, and/or their Hicksian derivatives with respect to t^m of the high skilled agent are small, ceteris paribus, demand is more discouraged.*

It is justified for the reduction of envy to tax more heavily (in the sense of having a larger reduction in compensated demand) the goods that the low skilled agent prefers and whose Hicksian elasticities are high. This observation, although a natural consequence of the envy-reducing taxation problem, contrasts with the traditional Ramsey (1927) inverse elasticity rule and Diamond's (1975) extension to the many person economy. The former says that, under the linear tax schedule which maximizes the welfare of a representative individual, the tax rate is proportional to the reciprocal of the commodity's Hicksian own elasticity. The latter says that, under a linear tax schedule which maximizes a Bergson-Samuelson social welfare function, the goods which the low utility agent, whose weight in the social welfare function is higher under concavity, prefers should not be taxed.

A sufficient condition for the first-best term to be zero is worth mentioning. Suppose that *agents have the identical homothetic preferences*. Then, the λ-contraction of the path the envied agent's compensated demand (the shift of the consumption-leisure bundle along the indifference curve) with the price change moves along some indifference curve of the envying agent. Therefore, $\frac{\partial u_1}{\partial(\lambda x_2)} \cdot \frac{\lambda \partial \tilde{x}_2}{\partial t^m}$ is zero for all m. The case of identical homotheticity will be discussed further in Section 5.

Interpretation of the Second-best Term: The interpretation of the second term of the R.H.S. of (6) depends on whether $b_1 \gtrless b_2$, i.e., whether agent 1 or agent 2 should use an additional dollar for envy-constrained Pareto efficiency. The covariance rule suggests that, if $b_i \leq b_j$ $(i \neq j)$, *ceteris paribus*, demand for a good which agent i (resp. j) consumes more is more discouraged (resp. less discouraged). As in the case of utilitarianism (see Mirrlees (1975)), it might not always be true that the social marginal utility of income of the low skilled agent is higher. However, there is a case in which we can secure $b_1 > b_2$ (equivalently, $b_2 < 1$). Ignore the welfare constraint of agent 2. That is, consider the envy-constrained Pareto efficient allocation which maximizes the utility of agent 1. If demands are normal, we can show that $b_2 < 1$.

Lemma 3 *Suppose that the fiscal budget is balanced at the optimum. If demands are normal, then $b_1 > b_2$ at any envy-constrained Pareto efficient allocation which maximizes the utility of the low skilled agent.*

Notice that the above statement holds for any λ, including 0. Mirrlees' (1975) conjecture that there might exist an economy at which the order of 'gross social weight', a weight given by a social welfare function, is contrary to 'net social weight', b_i in our context, does not arise at the endpoint of utility possibility frontier.

Thus, in the case of the maximization of the low skilled agent's welfare, the covariance component tends to make the good which the low skill agent prefers less discouraged. Even in this case, the first term tends to make such a good more discouraged. The offsetting effects of these terms are central in determining the tax burden of the agents, which will be discussed in the next subsection.

Tax Burden: Corresponding to Diamond (1975, p.339), we can derive a formula of tax burden as follows.

$$2Cov(b_i, t \cdot c_i + T) = -\lambda \frac{\mu_1}{\gamma} \frac{\partial u_1}{\partial(\lambda x_2)} \cdot \left(t \cdot \frac{\partial \tilde{x}_2}{\partial t} \right) + \sum_{i=1,2} t \cdot \frac{\partial \tilde{c}_i}{\partial t} \cdot t. \tag{8}$$

where $\frac{\partial \tilde{x}_i}{\partial t}$ is a $k \times (k+1)$ matrix whose (m,j) element is $\frac{\partial \tilde{c}_i^j}{\partial t^m}$ for $j \leq k$ and $\frac{\partial \tilde{l}_i}{\partial t^m}$ for $j = k+1$. In other words, $t \cdot \frac{\partial \tilde{x}_i}{\partial t} \in I\!R^{k+1}$ is a list of discouragement indices for all goods of agent i. $\frac{\partial \tilde{c}_i}{\partial t}$ is a $k \times k$ matrix omitting the last column of $\frac{\partial \tilde{x}_i}{\partial t}$.

The amount of tax burden depends on whether $b_1 \gtrless b_2$ and the sign of the right hand side. As the Slutsky matrix (omitting the components of leisure) is negative semidefinite, the second term of the R.H.S. is negative. If $\mu_1 = 0$, i.e., if the allocation is second-best Pareto efficient, the argument given in Diamond (1975) and Mirrlees (1975) applies: the social marginal utility of income and the tax burden are negatively correlated. In particular, if the social objective is the maximization of agent 1's welfare, agent 2 bears higher tax burden (Lemma 3). However, in the current context the first-best term appears. According to the above interpretation of the tax formula, the good which the envying agent prefers tends to be taxed in order to reduce envy. Thus, the first term would be positive. The tax burden depends on whether the former surpasses the latter, which may or may not be the case. Therefore, contrary to the Paretian optimal taxation, we can say the following:

> *even though the social objective is the maximization of the utility of the low skilled agent, when the government is constrained by envy reduction, the low skilled agent may bear higher tax burden.*

It is easy to find an example where the envying agent bears higher tax burden at *an* allocation belonging to the λ-constrained second-best frontier. In fact, there is a more striking example where the envying agent bears higher tax burden at the Diamantaras-Thomson allocation, an allocation which maximizes λ^x defined in Section 3, among the Pareto efficient allocations. This will be discussed in the next section.

5 Welfare Properties of the Diamantaras-Thomson Allocations

So far we have discussed the qualitative features of the tax rules which are common to any allocation along the λ-constrained second-best frontier. The allocations of particular interest are the second-best Pareto efficient allocations which maximize λ^x advocated by Diamantaras and Thomson (1990).[18] In this section we examine the welfare properties of the allocation.

Consider first the case of identical homotheticity, commonly used in quantitative optimal taxation studies. Then the following can be shown:

Proposition 2 *Suppose utility of the agents is identical and homothetic. That is,*

$$u_i(x_i) = g(h(x_i)) \ (i = 1, 2), \ g \text{ is an increasing function}, h \text{ is homogeneous of degree 1}. \tag{9}$$

Then, along the second-best utility possibility frontier, λ^x is maximized at the second-best Pareto efficient point which maximizes the utility of the low skilled agent.

Proof: Note that

$$u_1(x_1) = g(h(\lambda^x x_2)) = g(\lambda^x h(x_2)). \tag{10}$$

Along the utility possibility frontier, $h(x_2)$ falls with increasing the utility of agent 1. Thus, λ^x is higher at the allocation which is more favorable to agent 1. *Q.E.D.*

With Proposition 2 and our earlier observation that the first-best term is zero in the optimal tax formula, we can characterize the λ-constrained second-best frontier under identical homotheticity. This is given in Figure 3.

Figure 3 around here.

Let $\lambda_2 > \lambda_1 > 0$. As the requirement of equitability increases, the frontier shrinks. By Proposition 2, the frontier is contracted from the north-western area. Moreover, there is no allocation which belongs to the λ-constrained second-best frontier but does not belong to the original second-best Pareto frontier: as is discussed in Section 4, the first-best term vanishes under identical homotheticity, hence the tax formula does not deviate from the usual second-best Pareto efficient formula.[19]

The assumption of identical homotheticity is stronger than that of identical preferences. Even though preferences are identical, as agents are consuming at different income levels, the paths of envy and welfare are not necessarily proportional. In fact, the income elasticity of goods and leisure does matter. See Nishimura (1999).

In general, on the other hand, since the good which the envying agent prefers tends to be taxed when we consider envy-constrained Pareto optimization, the welfare possibility of agent 1 shrinks. Then the conclusion of the above proposition will not hold when the preference does not satisfy identical homotheticity. We can show that in some economy there exists a severe trade-off between welfare of the envying agent and envy measure.

Proposition 3 *There exists an economy in which the second-best Diamantaras-Thomson allocation is the Pareto efficient point which minimizes the welfare of the low skilled agent, the tax burden of the low skilled agent is maximal, and moreover corresponding λ^x is less than unity.*

Proof: Consider the following one-good economy.

$$u_i = c_i^{a_i} l_i, \ \bar{l} = 1, \ a_1 = 1, \ a_2 = 0.5, \ w_1 = 1, \ w_2 = 1.2 \tag{11}$$

Here, a_i is a parameter of propensity to consume leisure for agent i. We assume that agent 1 is more industrious. Labor in efficiency units is taxed with the proportional income tax t^w. Agent i's tax payment is $t^w w_i(\bar{l} - l_i) + T$. The values of u_1, u_2, λ^x_{12} and $R_1 \equiv t^w w_1(\bar{l} - l_1) + T$ for various t^w (with T being an endogenous variable, taking account of the budget balance) are given in Table 1.

Table 1 around here.

The second-best Pareto efficient allocation is $t^w \in [-0.2, 0.2]$, and the welfare of the high skilled agent increases with t^w. On the other hand, λ^x_{12} keeps increasing in the range of less than unity (agent 1 envies agent 2). Also, R_1 keeps increasing. *Q.E.D.*

The source of this result can be seen in the observations through numerical examples in Nishimura (1999). He examined the change in welfare of agents and λ^x through Cobb-Douglas preferences with one good or two goods. He found that high tax rates reduce envy in most of the case, which is true in this case as well. However, it decreases the welfare of the low skilled agent in this example. This is an application of the interpretation of the tax formula in Section 4.2. High income tax is equivalent to high commodity tax in the labor-corn economy, which implies the discouragement of what the industrious agent prefers.

In fact, numerical examples, which are available upon request to the author, suggest that a conflict between lowering envy and the welfare of the low skilled agent arises only if the difference in the skills is not large (and moreover the low skilled agent is more industrious), and a Rawlsian type allocation, an allocation which maximizes the low skill's welfare, coincides with the allocation which maximizes λ^x in most of the cases.[20] Nevertheless, Proposition 3 tells us explicitly that there may be a severe conflict between envy reduction and compensation to the low skilled agent.

6 Conclusion

This paper points out an aspect of the taxation based on envy: to determine the tax rate, the preference of the envying agent and the price elasticity of the envied agent play important roles. We discussed that the goods preferred by the low skilled agent and/or of high Hicksian elasticities are taxed heavily, and discussed that the lower skilled agent may well bear higher tax. We also showed that there is an economy at which the Diamantaras-Thomson allocation does not necessarily coincide with a Rawlsian type allocation, the maximization of the welfare of the low skilled agent. These observations show a possible conflict between envy reduction and compensation to the low skilled agent under second-best taxation.

The extension of the basic equations to the many person economy is straightforward, but the interpretation is not easy. Even if we omit counting the psychic compensation that the higher skilled agent obtains from the lower skilled agent's misfortune (namely, greater than one λ_{ij}^{x}), we still have to look at $n(n-1)/2$ pairs of envy (envy of the lower skilled against the higher skilled). They might conflict: reducing envy of the lowest skilled agent against the highest skilled agent may raise that of the middle skilled against the highest. It is also an important question whether we adopt Chaudhuri's (1986) criterion or Diamantaras-Thomson's (1990) criterion: that is, whether we admit a skewed distribution of envy at the cost of minimization of the sum of envies or paying attention to the most envying agent.[21] In the case of identically homothetic preferences, the latter approach boils down to minimizing envy of the lowest skilled agent against the highest skilled.

Our objective is different from a welfaristic approach. A value judgment is welfaristic if the evaluation of social states is based solely on the welfare (utility) of the individuals. As envy concerns the bundle of the opponent, λ-equitability is not welfaristic. It has been long time since the problematic features of welfaristic approach as an evaluation of social states and public policies were pointed out (Sen (1979)). Yet such an approach is dominant in the optimal taxation. No-envy and its alternatives have been one of the most celebrated concepts in the social choice literature, but, needless to say, there are other criteria to be considered. Further discussions on non-welfaristic value judgments, taking account of tax implementability, are left open.

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Appendix

Proof of Lemma 1

We make use of Theorem 3.1 of Guesnerie (1996) and first order conditions (4) and (5). Suppose to the contrary that $\mu_1 = 0$. Then, as $\mu_2 = 0$, there is no budget balance and Pareto improving direction of infinitesimal tax changes, by the Guesnerie's theorem. Q.E.D.

Proof of Lemma 2

It is sufficient to show that $-\alpha_1 - \frac{\partial u_1}{\partial(\lambda x_2)} \cdot \frac{\lambda \partial x_2}{\partial T} < 0$. If so, the claim follows from (5) since $\sum_i \left(t \cdot \frac{\partial c_i}{\partial T} + 1 \right) > 0$ (which is equivalent to $\sum_i \left(p \cdot \frac{\partial c_i}{\partial T} + w_i \frac{\partial l_i}{\partial T} \right) < 0$) by normality of demand.

1. Preferences are homothetic.

As λ-equitability is ordinal, it is enough if we show when the preferences of the agents are homogeneous of the degree 1; that is, $\forall a > 0 \forall x_i,\ u_i(ax_i) = au_i(x_i)$ $(i = 1, 2)$. The $u_1(\cdot)$ and $u_2(\cdot)$ may well be different functions. By monotonicity of preferences, $u_i(x_i) \geq 0$. In this case, corresponding to the decrease in income, agents decrease the demand of all goods in the same proportion. If the lump-sum tax increases by ΔT, the ratio of demand reduction is $\left(1 - \dfrac{\Delta T}{w_i \bar{l} - T}\right)$, taking account of 'full income' $w_i \bar{l}$.

Consider the infinitesimal change in lump-sum tax ΔT from the optima $v_1(t, T) = u_1(\lambda x_2(t, T))$. $v_1(t, T)$ and $u_1(\lambda x_2(t, T))$ becomes, respectively,

$$v_1(t, T + \Delta T) = \left(1 - \frac{\Delta T}{w_1 \bar{l} - T}\right) v_1(t, T)$$
$$u_1(\lambda x_2(t, T + \Delta T)) = \left(1 - \frac{\Delta T}{w_2 \bar{l} - T}\right) u_1(\lambda x_2(t, T)).$$

After some rearrangement letting $\Delta T \to 0$,

$$\frac{\partial v_1(t, T)}{\partial T} = -\frac{v_1(t, T)}{w_1 \bar{l} - T}$$
$$\frac{\partial u_1(\lambda x_2(t, T))}{\partial T} = -\frac{u_1(\lambda x_2(t, T))}{w_2 \bar{l} - T}.$$

As $v_1(t, T) = u_1(\lambda x_2(t, T))$ and $w_1 < w_2$, $\dfrac{\partial v_1(t, T)}{\partial T} = -\alpha_1 < \dfrac{\partial u_1(\lambda x_2(t, T))}{\partial T} = \dfrac{\partial u_1}{\partial(\lambda x_2)} \cdot \dfrac{\lambda \partial x_2}{\partial T}$. As $\lambda \leq 1$, we are home.

2. Preferences are quasi-linear.

Suppose the preference of the agents are quasi-linear with the same good being transferable. For example, let good 1 be transferable.

$$u_i(x) = f_i(c^2, ..., c^k, l) + c^1 \ (i = 1, 2).$$

As before, $f_1(\cdot)$ and $f_2(\cdot)$ may differ.

Then, if negative holding of the transferable good (good 1 in the above example) is allowed, increment in lump-sum tax ΔT is reflected to the increment in the transferable good (hence welfare $v_1(t,T)$) by $-\Delta T$. On the other hand, increment in $u_1(\lambda x_2(t,T))$ is $-\lambda \Delta T$. Since $\lambda \leq 1$, again we showed that $-\alpha_1 < \frac{\partial u_1}{\partial(\lambda x_2)} \cdot \frac{\lambda \partial x_2}{\partial T}$. *Q.E.D.*

Proof of Lemma 3

By the total differentiation of the budget constraint with respect to T,

$$(p+t) \cdot \frac{\partial c_i}{\partial T} = -w_i \frac{\partial l_i}{\partial T} - 1. \tag{12}$$

By the use of (12), we can say the following when $\zeta_2 = 0$. Noting also that $\mu_2 = 0$ by the complementary slackness condition,

$$b_2 \gtrless 1 \iff \lambda \frac{\mu_1}{\gamma} \frac{\partial u_1}{\partial(\lambda x_2)} \cdot \frac{\partial x_2}{\partial T} \gtrless -p \cdot \frac{\partial c_2}{\partial T} - w_2 \frac{\partial l_2}{\partial T}.$$

If demands are normal, $\lambda \frac{\mu_i}{\gamma} \frac{\partial u_1}{\partial(\lambda x_2)} \cdot \frac{\partial x_2}{\partial T}$ is negative, and $-p \cdot \frac{\partial c_2}{\partial T} - w_2 \frac{\partial l_2}{\partial T}$ is positive. So $b_2 < 1$. *Q.E.D.*

Footnotes

1. The extension considering public good provision from tax revenue is straightforward. We can obtain an optimal public good provision rule which is a variant of Diamond (1975), eq. (19). Details are available upon request to the author.

2. Let (c_1^*, l_1^*) be agent 1's optimal choice. As $(p+t) \cdot c_1^* \leq w_1(\bar{l} - l_1^*) - T \leq w_2(\bar{l} - l_1^*) - T$, the bundle (c_1^*, l_1^*) is affordable for agent 2. Therefore, agent 2's optimal choice under his budget constraint gives at least as high utility level as $u_2(c_1^*, l_1^*)$.

3. Notice that no-envy is different from so-called *self-selection* in the optimal taxation literature. An allocation satisfies self-selection iff no agent envies consumption-*efficient unit* labor of any other agent, which is a necessary condition for tax implementability. Typically, when the productivity of agents is different, there are no-envy allocations which violate self-selection, and most of the self-selective states invite envy of the lower skilled, as discussed above.

4. Feldman and Kirman (1974) first advocate reducing envy based on a cardinal measure; their measure is the difference in utilities (see Bös and Tillmann (1985) for tax policy implication of their concept). However, their concept is not invariant with respect to monotonic transformations of the utility functions. The property does not suit the classic assumption of the new welfare economics: social choice should be ordinally invariant. On other alternative concepts advocated by the mid 1980's (see Thomson and Varian (1985)), Tillmann (1984) and Bös and Tillmann (1985) show that most of them are not successful in addressing the Bös-Tillmann impossibility theorem discussed above. On the concepts in economies with different consumptive talents (Arnsperger (1994), Section 6), we can easily see that the application of these criteria into the current environment also results in the Bös-Tillmann impossibility.

5. Here, we modify the definition of Diamantaras and Thomson (1990). In the current environment, it may happen that the λ-expansion of the leisure of the opponent meets the upper bound $\bar{l}$, so that a value of λ_{ij}^x such that $u_i(x_i) = u_i(\lambda_{ij}^x x_j)$ may not exist when $u_i(x_i) > u_i(x_j)$.

6. There is also some contraction in the interior of the utility possibility set, which we omit discussing here. Our only concern is the frontiers.

7. However, such an allocation might not exist: in fact, as is shown in Guesnerie (1996), the set of the second-best Pareto optima does not have normal topological properties such as closedness and connectedness. As a result, the solution may not be single valued, and in some case it might be empty.

8. This is in general different from Chaudhuri's (1986) concept. However, they are the same in a two-class economy with a hierarchy of envy. The difference in a many-person economy will be discussed in conclusion.

9. $\frac{\partial c_i^m}{\partial t^j} = \frac{\partial \tilde{c}_i^m}{\partial t^j} + \frac{\partial c_i^m}{\partial T} \cdot c_i^j, \quad \frac{\partial l_i}{\partial t^m} = \frac{\partial \tilde{l}_i}{\partial t^m} + \frac{\partial l_i}{\partial T} \cdot c_i^m \quad (j, m = 1, ..., k, \ i = 1, 2).$

10. Remember that we let leisure be the untaxed good.

11. Notice that the lemma is a *sufficient* condition: for second-best Pareto efficient allocation, there is a corresponding vector of Lagrange multipliers, *which may be different from $\delta_1, \delta_2, \gamma$ in equations (4) and (5)*. Thus, there is a possibility of $\mu_1 > 0$ at some second-best Pareto efficient allocation.

12. Also, when there are public goods, if the demand of some of the public goods is separable with private goods and the sub-utility of the public goods is homogeneous of degree s, $s \in \mathbb{R}_+$, we can show that $\gamma > 0$. This type of utility function is also common in the quantitative analysis.

13. As in the typical optimal taxation problems, (3) is not well-behaved. As is standard, the existence of such a solution is simply supposed.

14. When the above suppositions are not valid, the term is the change in the compensated demand for good m induced by increasing all tax rates by the same proportion (Mirrlees (1976, p.331)).

We will adopt the interpretation of the reduction in compensated demand, which is common in the literature.

15. This interpretation is taken by, for example, Atkinson and Stiglitz (1980).

16. The reader can readily show an alternative formula of b_i.

$$b_i - 1 = \frac{\partial \mathcal{L}/\partial M_i}{\gamma}, \text{ thus } b_i \gtrless \bar{b} \iff \frac{\partial \mathcal{L}}{\partial M_i} \gtrless 0.$$

where M_i is the transfer to agent i, γ is the shadow price of government revenue. Then b_i net of cost of the unit increment of pecuniary transfer measures the social benefit of lump-sum transfer to agent i.

The sign of the covariance component can be a guide to the desirability of *anonymous quota* or *rationing*, evaluated at the tax equilibrium. The following can be shown by a straightforward application of Guesnerie's (1996) theorem.

Proposition: Given the allocation determined by the program (2), rationing (negative anonymous in-kind transfer) of good m is desirable with respect to the social objective iff $Cov(b_i, c_i^m) > 0$.

The proof technique is an application of Guesnerie's Theorem 4.2 and 4.4. The result is comparable to Guesnerie's Corollary 4.3: if a good is more consumed by socially desirable class at the optima of tax equilibrium, the consumption is unduly, and rationing is, if possible, socially desirable. The theorem can be extended to the general non-Paretian maximization problem. The proof and its thorough interpretation are available upon request to the author.

17. If readers care that the definition depends on cardinal representation, normalize the marginal utility by that of some good, say labor, in the form of the marginal rate of substitution between good m and the numeraire. The author acknowledges professor Yongsheng Xu for the comment.

18. As noted in footnote 7, existence cannot be secured, as in the typical optimal taxation problems. As to the tax rule, consider the following problem. Let λ^* and $\bar{v}_2^*$ be the values of λ^x and v_2 corresponding to the equilibrium, respectively. Then solve the problem (2) letting $\lambda = \lambda^*$ and $\bar{v}_2 = \bar{v}_2^*$. The solution coincides with the original problem. As the resultant allocation is second-best Pareto efficient, μ_1 might be zero. See footnote 11 after Lemma 1. It is uncertain whether $b_1 \gtrless b_2$.

19. Formally, the claim is derived as follows. Suppose the λ-equitability becomes binding at some second-best Pareto efficient allocation, say x. If we seek a λ-constrained second-best allocation, say y, we must either give agent 1 at least as good as $u_1(x_1)$ or give agent 2 at least as good as $u_2(x_2)$. If we require the former, y is a second-best Pareto efficient allocation at which λ^y is higher than λ^x, by Proposition 2. If we require the latter, $u_1(y_1) < u_1(x_1)$, so that, by an application of the logic we have used in deriving Proposition 2, it does not satisfy λ-equitability or it is Pareto inferior to x, which violates the envy-constrained second-best optimality.

20. In the preference combinations in (11), fixing $w_1 = 1$, if $1 \leq w_2 \leq 1.1$, there exists a Pareto efficient allocation where $\lambda^x = 1$, and if $w_2 \geq 1.6$, there is no conflict between enhancing agent 1's utility and reducing envy. Fixing a_2, as a_1 becomes closer to a_2, the critical level of skill differences where the envy and the low skill's welfare conflict decreases.

21. Formally, Chaudhuri's measure is the minimization of the sum of $1/\lambda^x_{ij}$'s for all pairs of i, j which are greater than 1:

$$\min \sum_{\{(i,j)|\lambda^x_{ij}<1\}} \left(\frac{1}{\lambda^x_{ij}} - 1 \right).$$

In λ-equitability, $\lambda^x \equiv \min_{i,j} \lambda^x_{ij}$ matters. On the other hand, in this equity criterion the sum of envies matters.

Table 1: Proposition 3

t^w	u_1	u_2	R_1	λ^x_{12}
-0.3	0.2516	0.4104	-0.0120	0.9296
-0.2	0.2522	0.4147	-0.0086	0.9389
-0.1	0.2517	0.4185	-0.0046	0.9483
0	0.25	0.4216	0	0.9578
0.1	0.2466	0.4237	0.0053	0.9675
0.2	0.2412	0.4243	0.0116	0.9772
0.3	0.2331	0.4229	0.0188	0.9870

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Fig. 1 λ_{ij}^{x}

Fig. 2 the λ-constrained second-best frontier

Fig. 3 the λ-constrained second-best frontier under identical homotheticity

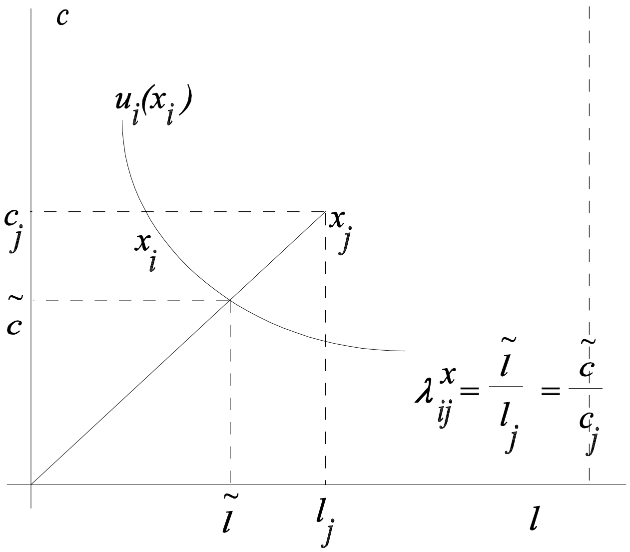

Figure 1: λ^x_{ij}

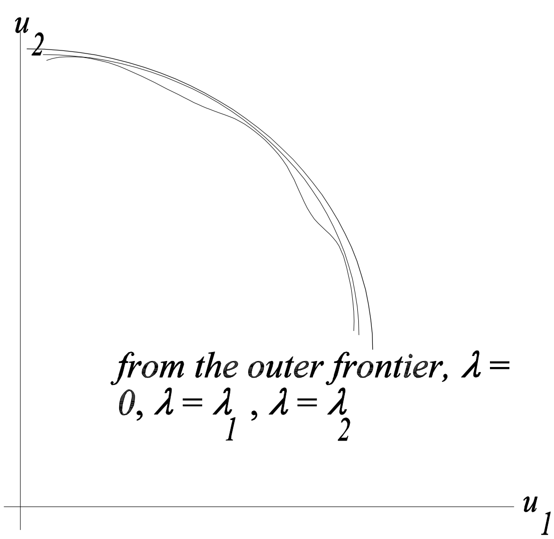

Figure 2: the λ-constrained second-best frontier

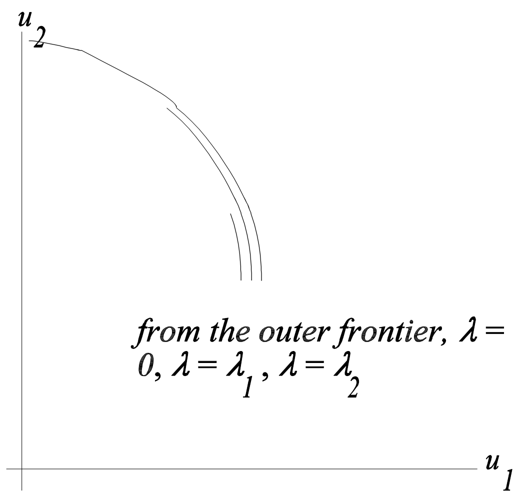

Figure 3: the λ-constrained second-best frontier under identical homotheticity