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Queen's Economics Department Working Paper No. 888

Splitting Orders

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10-1993

Discussion Paper #888

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October 1993

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Draft: October, 1993

Abstract

A standard presumption of market microstructure models is that competition between risk neutral market makers inevitably leads to prices schedules that leave market makers zero expected profits conditional on the order flow. This paper shows that this result ***does not*** hold when traders can split orders between market makers. When traders can split orders, market makers set less competitive price schedules that earn them ***strictly positive*** profits and hence raise trading costs. Indeed, if noise traders have completely inelastic demands (as in Kyle 1985), market makers want to set arbitrarily uncompetitive price schedules: no equilibrium exists. Our results imply that if feasible, regulation banning order splitting on an exchange is optimal. Analogous results obtain when price schedules are set by any finite number of agents who compete using limit orders. Further, since limit orders, by their very nature, are split against incoming market orders, the analysis suggests that regulated market maker competition will provide better prices.

*The first author is grateful to the SSHRC for financial support. We wish to thank Peter Bossaerts, Rob Heinkel, Burton Hollifield, and seminar participants at Queen's University and the University of British Columbia for their insights. The usual disclaimer applies.

1 Introduction

A standard presumption in market microstructure is that competition between risk neutral market makers inevitably leads to prices schedules that leave market makers zero expected profits conditional on the order flow. This paper shows that this result *does not* hold when traders can split orders between market makers. That is, when traders can split orders, market makers set price schedules that earn them *strictly positive profits* and hence lead to greater total trading costs. The analysis also extends to a limit order book, which by its nature is split against incoming market orders: equilibrium limit order schedules set by a finite number of agents necessarily yield those agents positive expected profits.

The intuition is simple. Consider an environment where both informed and uninformed agents trade a risky asset so that the equilibrium price is necessarily increasing in order size. When traders cannot split orders among market makers, then to receive an order, a market maker must offer a better price than that offered by any other market maker. Since market makers who do not receive the order earn zero profits, it must be that the market maker who wins the trade also earns zero expected profits (provided that market makers face no constraints on the prices they set). Were a market maker to take an order on which he expects profits, then it would behoove other market makers to offer slightly more favorable prices and take the order themselves. Demand is perfectly price elastic, with all trade going to the market maker who offers the best price. Consequently, this Bertrand competition among the market makers inevitably leads to zero expected profit pricing.

In contrast, when traders can split orders among market makers, demand is not perfectly price elastic because when one market maker offers a slightly better price schedule, the other market makers still receive a portion: traders equate marginal trading costs across market makers. If one market maker sets a schedule that would earn zero expected profits when matched by another market maker, then the other market maker can ensure positive profits by setting a less competitive schedule (both market makers would then earn positive profits). This is because trade of informed agents, who only have profit motivations for trading, is more price elastic than the trade of agents who care about portfolio balance. Hence, informed traders reduce their orders by more than liquidity traders in response to a price increase. As a result, in equilibrium, when traders can split orders, market makers must set price schedules that earn strictly positive expected profits.

There is a direct analogy to duopolistic competition between firms. If firms compete in prices, then demand is perfectly price elastic. If one firm slightly undercuts the other's price, then it receives not only the incremental market order, but also any orders that would go to the other firm. The incentives to undercut the other firm's price in the Bertrand competition drive profits down to zero. If, instead, firms compete in quantities, then demand is not perfectly price elastic so that firms produce quantities that earn them strictly positive profits. Price competition among duopolists corresponds to the case where orders cannot be split among market makers and order flow is completely price elastic; quantity competition corresponds to the case where orders can be split so that order flow is less price elastic.

We consider two illustrative examples. The first features inelastic noise trader demand (as in Kyle 1985). We show that each market maker wants to set a steeper price schedule than the other, so that no equilibrium can exist. The contrast between this result and the standard zero–profit result that obtains when traders cannot split orders could not be more stark! We then consider an example with elastic liquidity trade, and show that in equilibrium, market makers set schedules that earn them strictly positive expected profits. These results extend to an environment with an arbitrary finite number of market makers.

The next section of the paper demonstrates that the results extend to almost any market where there is informed trade. Since traders can clearly split orders across exchanges if not between market makers within an exchange, our results call into question the robustness of the standard market microstructure formulation with competitive market makers. Simply put, when traders can split orders, competition does not lead to zero–expected profit pricing. An immediate implication of this finding is that if feasible, traders would value exchange rules that would eliminate splitting. These could take the form of fixed transactions costs for submitting orders.

Throughout the analysis we assume that market makers only observe those orders that they process. They can, of course, invert back to determine the equilibrium order flow with each market maker. Thus, even were market makers to observe total order flow, the positive–profit equilibrium would be unaffected. However, we also show that if market makers can observe total order flow, the standard zero–profit market maker schedule can obtain as an equilibrium outcome: each market maker sets a price schedule that depends not only on his own order flow, but also the order flow of all other market makers. One way to interpret this is that when order splitting is allowed, the standard model has multiple equilibria, one with positive market maker profits and one with zero profits. However, standard equilibrium refinements (e.g. cheap talk) select the equilibrium with positive profits.

In the last section of the paper we consider an environment in which a finite number of agents compete by setting limit order schedules. It is natural to consider order splitting in this context because the nature of limit orders is such that incoming market orders are split up against the limit book. The results are analogous to those obtained when market makers set average price schedules. When liquidity demand is inelastic, no equilibrium exists. For elastic liquidity demand, limit order schedules earn strictly positive expected profits. For a zero–expected profit limit book to emerge in equilibrium, an arbitrarily large number of agents must compete to submit limit orders. This is in sharp contrast to the equilibrium obtained when market makers submit average price schedules and order splitting is prohibited. There, market makers expect zero profits for *any* number of market makers greater than one.

The results in this paper are closely related to Glosten's [1993] concept of immunity. We consider a price schedule to be immune from competition *if and only if* no competing schedule can enter, take trade with positive probability, and earn non–negative profits. In a related paper (Bernhardt and Hughson 1993), we show that the class of immune price schedules is quite large when traders can split orders, and includes market maker schedules that earn substantial expected profits in excess of those earned by informed traders. However, an immune schedule can *not* be an equilibrium outcome. Given that a competing schedule would not be set, a market maker would wish to set an even less competitive schedule than the immune schedule .... but that would then draw entry from a competing schedule.

2 Example 1: Inelastic Demand

Let the current value of a claim to an asset be one. There is an innovation δ to the claim's value that is uniformly distributed on the interval [-1,1], that is private information to a risk neutral insider. The insider therefore knows the asset's true value, $V = 1 + \delta$. There are also noise traders whose inelastic trading demands, Q, are exogenously given, and are drawn from a uniform [-1,1] distribution.

Traders can submit their orders to either of two risk neutral, uninformed market makers, A and B. Orders are handled individually. The probability that a trader who comes to the market is informed is one–half. The model is thus the standard inelastic noise trader environment (e.g. Kyle 1985). The only differences are that orders are not aggregated across agents, and we consider uniform rather than normal distributions.

The timing is as follows:

1. The identity of the trader (informed or noise) is determined.
2. The insider perfectly observes the innovation to the asset value, δ.
3. Market makers simultaneously set price schedules, detailing for each trade quantity a price.
4. The trader chooses how much to trade with each market maker.
5. Payoffs are realized.

2.1 No Splits

When traders cannot split orders between market makers, in equilibrium, an informed trader chooses an order quantity, $q_I(\delta, p_a(\cdot), p_b(\cdot))$, that maximizes expected profits given his information, and both types of traders choose the market maker who will execute their orders most cheaply. Given correct beliefs about the schedule set by the other market maker and the trading strategies of the two trader types, each market maker's price schedule maximizes his expected profits.

In the equilibrium, as is standard when agents cannot split their trade, Bertrand competition demands that the two market makers set identical price schedules, price schedules that break even in expectation conditional on the order flow.

It is easy to show that the price schedule that leaves a market maker zero expected profits conditional on the order size q is given by $p(q) = 1 + \frac{q}{2}$. Facing this schedule, an insider observing δ trades to maximize $q(V - p(q)) = q(\delta - \frac{q}{2})$ which has solution, $q^* = \delta$. Hence, expected market maker profits from order q are $q[.5(\frac{q}{2} - 0) - .5(q - \frac{q}{2})] = 0$, and the expected value of the asset given order q is $1 + .5(0) + .5(q) = 1 + \frac{q}{2} = p(q)$.

2.2 Splits

To facilitate the analysis when agents can split orders we make the simplifying restriction that market makers set linear schedules that satisfy $P(0) = 1$. The restriction to linear schedules when splitting is possible permits the solution for equilibrium schedules: since pricing is no longer pinned down by a zero expected profit condition, the solution to the more general equilibrium problem would be extremely difficult to obtain. Since the schedules that earn zero expected profits conditional on the order flow are *still* linear (with a slope twice as great as when agents cannot split orders), it follows that if the zero–profit price schedules do not survive as equilibrium schedules when pricing strategies are restricted to be linear, then they cannot survive extensions to more general classes of pricing strategies. We will show here that when traders can split orders that no equilibrium exists when we restrict schedules to be linear.

Given this strategy space, in the (Perfect–Bayesian) equilibrium when traders can split orders between market makers, the informed trader maximizes expected profits with his choice of order quantities, $q_I^A(\delta, p_a(\cdot), p_b(\cdot))$, $q_I^B(\delta, p_a(\cdot), p_b(\cdot))$ in markets A and B respectively; the noise trader maximizes expected profits with his choice of order quantities, $q_L^A(Q, p_a(\cdot), p_b(\cdot))$, $q_L^B(Q, p_a(\cdot), p_b(\cdot))$, where $q_L^A(Q, p_a(\cdot), p_b(\cdot)) + q_L^B(Q, p_a(\cdot), p_b(\cdot)) = Q$; and the two market makers set linear price schedules, $p_a(q) =$

$1+aq$, $p_b(q) = 1+bq$, that maximize each market maker's expected profits given correct beliefs about the schedule set by the other market maker and correct inferences about how each type of trader responds to the market maker schedules.

The zero profit schedule when traders can split orders is twice as steep as the zero profit schedule when they cannot:

$$p_a(q) = p_b(q) = 1 + q.$$

If market makers set these schedules, then traders split orders in half so that the effective price schedule for the total order is the zero–profit schedule set when traders cannot split. However, we will show that this cannot be an equilibrium schedule. Indeed, for any given schedule set by market maker B, $p_b(q) = 1+bq$, market maker A will want to set a schedule with steeper slope, $a > b$. Since market maker B has similar incentives, an equilibrium with linear pricing cannot exist.

Suppose that market maker A has beliefs that market maker B will set schedule $p_b(q) = 1 + bq$. Given these beliefs we solve for market maker A's optimal schedule, $p_a(q) = 1 + aq$.

The insider observing δ trades q_I^A with market maker A, where q_I^A solves

$$max_{q_I^A} \;\; q_I^A(\delta - aq_I^A).$$

Solving, the informed agent buys claims

$$q_I^A = \frac{\delta}{2a}$$

to the risky asset. Henceforth, we focus on buy orders. The problem is symmetric for sel orders. Conditional on trading, the insider's expected profits from trading with market maker A (and hence market maker A's losses) are

$$\int_0^1 \frac{\delta}{2a}(\delta - a\frac{\delta}{2a})d\delta = \int_0^1 \frac{\delta^2}{4a}d\delta = \frac{1}{12a}.$$

A noise trader with demand Q, minimizes his trading costs by buying q_L^A of claims to the risky asset from market maker A and $Q - q_L^A$ from B. His optimization problem is

$$min_{q_L^A} \; q_L^A a q_L^A + (Q - q_L^A) b (Q - q_L^A),$$

which has solution

$$q_L^A = \frac{bQ}{a+b}, \; q_L^B = \frac{aQ}{a+b}.$$

Market maker A's optimization problem is then given by

$$max_a \; \frac{-1}{12a} + \int_0^1 \left[\frac{Qb}{a+b}\right]^2 adQ.$$

The first term is market maker A's expected loss to the informed traders; the second term is his expected profit from noise traders. Differentiating with respect to a yields first order conditions

$$\frac{1}{12a^2} + \frac{b^2}{3}\left[\frac{1}{(a+b)^2} - \frac{2a}{(a+b)^3}\right] = 0.$$

Simplifying, and re–arranging yields

$$4a^2b^2(a-b) = (a+b)^3.$$

Since both a and b are positive, this implies that $a > b$. Since each market maker always wants to set a steeper price schedule than its competition, no equilibrium exists. When traders can split orders, market makers want to soak noise traders for an infinite amount of money. Section 4 shows that this result extends to any microstructure model with inelastic liquidity trade (e.g. Kyle 1985). Note also that the result immediately extends to any finite number $N > 2$ of competing market makers.

3 Example 2: Elastic Demand

We now change the environment of example 1 to allow for price–elastic liquidity demand. Agents trade consumption at date 1 for claims to an asset with random consumption payoff at date 2. Liquidity traders have preferences

$$c_1 + \beta c_2,$$

where c_i is consumption on date $i, i = 1, 2$. β can take on one of two values, $\bar{\beta} \geq 2 > 1 > \underline{\beta}$, where the probability that a liquidity trader has discount $\bar{\beta}$ is one–half. Traders with discount $\bar{\beta}$ want to buy claims to the risky asset; traders with discount $\underline{\beta}$ want to sell. The current value of the asset is equal to one. Suppose that the innovation δ to the claim's value is again uniformly distributed on $[-1, 1]$.

The informed trader now has preferences

$$c_1 + c_2,$$

and has a sufficiently large endowment that he can trade any desired equilibrium quantity.

Agents can trade with either or both of two risk neutral market makers. Market makers also have preferences

$$c_1 + c_2.$$

Market makers have sufficiently large endowments that, in equilibrium, they are not endowment constrained. The role of a market maker is to set a price at which he will fill any order that he receives.

A single trader comes to the market each period. The trader is equally likely to be an informed or liquidity trader.

Because a single trader's order determines the total order flow, without loss of generality we can focus on positive purchases of claims to date–two consumption: the possible traders are liquidity traders with discount factor $\bar{\beta}$ and informed traders who observe a positive innovation.

We assume that a liquidity trader's endowment of date–one consumption, Q is drawn from a distribution with density,

$$f(Q) = \frac{1}{\sqrt{(2Q+1)}}$$

on the support $[0, \frac{3}{2}]$. This density 'serendipitously' implies that the equilibrium price schedule when traders cannot split orders is linear. Liquidity traders must consume non–negative quantities in each period: short–sales are prohibited. All random variables are independently distributed.

3.1 No Splits

When traders cannot split orders between market makers, in equilibrium, the informed trader chooses an order quantity, $q_I(\delta, p_a(\cdot), p_b(\cdot))$, that maximizes expected profits given his information; the liquidity trader chooses an order quantity, $q_L(\beta, Q, p_a(\cdot), p_b(\cdot))$, that maximizes expected discounted profits given his discount factor and endowment; and both trader types choose the market maker who will execute their orders most cheaply. Given correct beliefs about the schedule set by the other market maker and the trading strategies of the two trader types, each market maker's price schedule maximizes his expected profits.

Proposition 1 *When traders cannot split orders, the equilibrium market maker schedule is:*

$$p(q) = 1 + \frac{q}{2}.$$

Associated insider profits are $\frac{1}{6}$ and the market makers earn zero expected profits conditional on the order flow.

Proof: See appendix. ∎

In the proof, we first conjecture that the equilibrium ask schedules set by the market makers are linear, and solve for the resulting orders of the informed and liquidity traders. We then verify that, conditional on the order flow, the market maker who takes the order expects zero profits, thus validating our initial conjecture that the equilibrium features linear ask schedules.

3.2 Splits

To facilitate the analysis when agents can split orders we again make the simpifying restriction that market makers set linear ask schedules that satisfy $P(0) = 1$. Again, since pricing is no longer pinned down by a zero expected profit condition, this makes possible the solution for equilibrium schedules (albeit with restricted strategies). We will show that when traders can split orders, market makers always set price schedules from which they expect strictly positive profits. Since the restricted equilibrium schedule generates positive expected market–maker profits, we thus prove that the zero–profit schedule is not an equilibrium schedule when strategies are unrestricted.

Given this strategy space, in the (Perfect–Bayesian) equilibrium when traders can split orders between the market makers, the informed trader maximizes expected profits with his choice of order quantities, $q_I^A(\delta, p_a(\cdot), p_b(\cdot))$, $q_I^B(\delta, p_a(\cdot), p_b(\cdot))$; given his discount factor and his endowment, the liquidity trader maximizes expected discounted profits with his choice of order quantities, $q_L^A(\beta, Q, p_a(\cdot), p_b(\cdot))$, $q_L^B(\beta, Q, p_a(\cdot), p_b(\cdot))$

given his endowment Q; and the two market makers set linear price schedules, $p_a(q) = 1+aq$, $p_b(q) = 1+bq$, that maximize each market maker's expected profits given correct beliefs about the schedule set by the other market maker and correct inferences about how each type of trader responds to the market maker schedules.

In equilibrium, market makers must set identical price schedules, else the one who sets a strictly more competitive schedule for some order flow does better to set a less competitive schedule, thereby increasing profits. The zero–profit schedules are now twice as steep as that obtained when agents cannot split orders:

$$p_a(q) = p_b(q) = 1 + q.$$

To see this, note that traders split their orders in half so that the effective price schedule is the zero–profit schedule set when traders cannot split. However, this cannot be an equilibrium schedule.

The construction of the equilibrium schedule is as follows. First market maker A conjectures a schedule $p_b(q) = 1+bq$. Given that conjecture he then chooses his schedule $p_a(q) = 1 + aq$ to maximize expected profits. In the appendix we show that

Proposition 2 *The slope, a, of market maker A's linear price schedule is given by the solution to:*

$$max_a \ \frac{-1}{12a} + \int_0^{\hat{Q}} \left(\frac{-1 + a\sqrt{\frac{1}{a^2} + \frac{4Qb}{a(a+b)}}}{2a} \right)^2 af(Q)dQ + \int_{\hat{Q}}^{\frac{3}{2}} \left(\frac{\bar{\beta} - 1}{2a} \right)^2 af(Q)dQ,$$

where $\hat{Q} = \frac{(\bar{\beta}^2-1)(a+b)}{4ab}$, *and* $f(Q) = \frac{1}{\sqrt{1+2Q}}$.

Proof: See appendix. ■

The first term represents the expected losses to informed trade; the second term is expected profit from constrained liquidity trade; and the third term represents expected profit from unconstrained liquidity traders. Differentiating with respect to a and setting $a = b$ in the first order conditions yields the symmetric equilibrium price schedule. The results are graphed in figure 1.

Observe that the schedules all have slopes greater than one and hence are strictly steeper than the zero–expected profit schedule. Indeed, it is necessarily the case that market makers set sufficiently steep schedules that some liquidity traders do not face binding endowment constraints. That is, the price schedules are sufficiently steep that liquidity traders respond to steeper price schedules by reducing their purchases. An immediate consequence is that as $\bar{\beta}$ goes to infinity, i.e. as liquidity trade becomes perfectly inelastic, the price schedules set by the market makers become arbitrarily steep, and their profits at the expense of liquidity traders become arbitrarily great.

4 The General Argument

Consider the properties of a canonical insider trading model. Agents trade consumption today for claims to an asset with random consumption payoff tomorrow. Some agents have private information about the value of the claim to the risky asset; other liquidity agents want to trade to rebalance their portfolios. All trade is through one of two risk neutral, uninformed market makers who set continuous, twice differentiable, monotone increasing, price schedules detailing the prices at which they are willing to handle any given order flow. Trade by informed and uninformed agents can be characterized by their associated first order conditions. The trades of informed agents are more price elastic than those who also trade to rebalance their portfolios, because informed traders have no reasons to trade other than profit.

First suppose that an agent cannot split his order between the two market makers. Then equilibrium price schedules must provide market makers zero expected profits conditional on the order flow. This zero–profit price schedule is continuous and monotonically increasing in order flow, and not 'too' concave for $q > 0$ and not 'too' convex for $q < 0$ (the meaning of 'too' is detailed below). Clearly, it is an equilibrium for both market makers to set a schedule that breaks even conditional on the order flow: given that one sets that schedule, the other is indifferent between matching and not.

Were one market maker to set a less competitive schedule for a range of order flows (continuity demands that the market maker set a less competitive schedule for a range), i.e. were one market maker to set a schedule on which he expects strictly positive profits for a range of order flows, then the other would have an incentive to offer slightly better prices on that range and take those profitable orders himself. Since the effect on informed and liquidity trade of offering a slightly better schedule can be made arbitrarily small, it is strictly profitable for the other market maker to offer those slightly better prices. This competition inevitably demands that the two market makers set identical price schedules, price schedules that break even in expectation conditional on the order flow.

Let $P(q)$ be the equilibrium price schedule set when agents cannot split orders. When agents *can* split trades between two market makers, then the zero–profit schedule that each market maker sets satisfies $P_S(\frac{q}{2}) = P(q)$. Traders split orders evenly since the price schedules are monotonically increasing, so that the price for the total order flow of q remains $P(q)$. Since the effective price schedule is unchanged, then so must the total order choices of the traders. But schedule $P_S(\cdot)$ is not an equilibrium price schedule when agents can split orders: in equilibrium, market makers must set price schedules that earn them strictly positive expected profits.

Since orders are handled independently, it is without loss of generality to focus on buy orders, $q > 0$. Let market maker B set price schedule

$$P_B(q) = P_S(q)$$

and let A set schedule

$$P_A(q) = P_S(q) + kq, \ k > 0.$$

An insider with information that a claim to the asset has value V trades to maximize

$$max_{q_I^A, q_I^B} \ q_I^A(V - P_A(q_I^A)) + q_I^B(V - P_B(q_I^B)),$$

where q_I^i is the insider's trade with market maker $i, i = A, B$. The first order condition determining the insider's trade with market maker B,

$$V - P_S(q_I^B) - q_I^B \frac{dP_S(q_I^B)}{dq_I^B} = 0,$$

is not a function of the price schedule set by market maker A, $P_A(q_I^A)$. The first order condition for the insider's trade with market maker A is given by

$$V - P_S(q_I^A) - q_I^A \frac{dP_S(q_I^A)}{dq_I^A} = 2q_I^A k.$$

The second order conditions for a maximum are given by

$$2\frac{dP_S(q)}{dq} + \frac{d^2P_S(q)}{dq^2} > 0.$$

Note that the second order conditions place a limit on the concavity of the ask price schedule.

Suppose that the liquidity trader's reduced–form objective can be written as[1]:

$$max_{q_L^A, q_L^B} \ W(q_L^A + q_L^B \mid \phi) - q_L^A P_A(q_L^A) - q_L^B P_B(q_L^B),$$

[1]Note that liquidity traders are not endowment constrained so that the model of the previous section is not captured within this formulation.

where $W(\cdot)$ is a strictly concave monotonic function for $q_L^A + q_L^B < \phi$. That is, ϕ represents the optimal portfolio mix; the further away the liquidity trader's holdings are from ϕ, the more he values trading a marginal unit to get closer to ϕ. The associated first order conditions are given by

$$\frac{dW(q_L^A + q_L^B \mid \phi)}{dq_L^B} - P_S(q_L^B) - q_L^B \frac{dP_S(q_L^B)}{dq_L^B} = 0,$$

and

$$\frac{dW(q_L^A + q_L^B \mid \phi)}{dq_L^A} - P_S(q_L^A) - q_L^A \frac{dP_S(q_L^A)}{dq_L^A} = 2q_L^A k.$$

Consider now liquidity and informed traders who would submit the same order were the two market makers to set the same schedules. The effects of an increase in the slope of market maker A's price schedule on their orders are given by:

$$\frac{dq_I^A}{dk} = \frac{-2q_I^A}{2P_S' + q_I^A P_S''} < 0,$$

$$\frac{dq_I^B}{dk} = 0,$$

$$\frac{dq_L^A}{dk} = \frac{+2q_L^A(-W'' + 2P_S' + P''q_L^A)}{4W''P_S' - 4(P_S')^2 + 2P_S''W''q_L^A - 4P_S'P_S''q_L^A - (P''q_L^A)^2} < 0,$$

$$\frac{dq_L^B}{dk} = \frac{2q_L^B W''}{4W''P_S' - 4(P_S')^2 + 2P_S''W''q_L^B - 4P_S'P_S''q_L^B - (P''q_L^B)^2} > 0,$$

where we drop the arguments of the functions and the primes refer to the associated derivatives.

Since $W(\cdot)$ is strictly concave, at $q_L^A = q_I^A$, $\frac{dq_I^A}{dk} < \frac{dq_L^A}{dk}$. Intuitively, a liquidity trader has a reason other than profit maximization to trade, so that when A sets a less competitive price schedule, a liquidity trader who would trade what an insider would trade were $k = 0$, reduces his order with A by less than the insider. To see this, differentiate $\frac{dq_L^A}{dk}$ with respect to $W''(q_L^A + q_L^B)$ (i.e. make the liquidity trader's valuation function locally less concave). Solving, we see that

$$sign[\frac{d^2 q_L^A}{dkdW''}] = sign[-(2P_S' + P_S'' q)^2] < 0.$$

The more concave is $W(\cdot)$, the less price elastic is liquidity trade with market maker A, and hence the less by which liquidity trade with A falls when A increases price.

We now show that the profits that A expects from informed and liquidity traders who would trade the same arbitrary quantity $q > 0$ were A to set ask price schedule $P_S(\cdot)$ are increased by a marginal price increase. Since informed trade is monotone in V and liquidity trade in ϕ, one can invert from q and determine the associated V and ϕ. Were A to set schedule $P_S(\cdot)$ then his expected profits conditional on order flow q would be 0:

$$P_S(q) = E\{V \mid q\} = Pr(informed \mid q)V^{-1}(q) + Pr(liquidity \mid q)E_0\{V\},$$

where $V^{-1}(q)$ is the asset value associated with an inside trade of q, and $E_0\{V\}$ is the ex ante expected value of a claim to the risky asset. When market maker A sets a steeper price schedule than the zero–profit schedule, the effect on the price at which informed agent $V^{-1}(q)$ trades is given by

$$P_S' \frac{dq_I^A}{dk} + q = P_S' \frac{-2q}{2P_S' + qP_S''} + q,$$

which is zero if $P_S(\cdot)$ is linear and positive if $P_S(\cdot)$ is strictly convex. Since liquidity traders reduce their trades with market maker A by less than informed traders, it must be that a marginal increase in the price schedule set by A leads liquidity trader $\phi^{-1}(q)$ to trade at a higher price than before. Consequently, a marginal increase in the price schedule leads to a reduction in trade quantities from informed trader $V^{-1}(q)$ relative to liquidity trader $\phi^{-1}(q)$ *and* the price at which each trades is greater. Therefore, conditional on trading with one of these two trader types, market maker A must expect strictly positive profits. Since q was arbitrary, it must be that market maker A expects strictly positive profits when he sets a schedule that is steeper than the zero–expected profit schedule. Hence, the zero–profit schedule cannot be an equilibrium schedule. Finally, it is immediate that the above arguments follow when $P_S(\cdot)$ is concave for $q > 0$, provided that $P_S(\cdot)$ is not 'too' concave.

Note that market maker B too must earn positive unconditional expected profits when A sets a steeper price schedule because informed trade with B is unaffected by the price schedule set by A, but B receives more liquidity trade when k rises (the liquidity trader equates marginal trading costs across market makers, substituting some order share toward market maker B). □

The fact that the zero–profit schedule cannot be an equilibrium schedule when traders can split orders is easiest to discern in any model where liquidity trade is completely price inelastic (e.g. Kyle 1985). If one market maker offers schedule $P_S(\cdot)$, then losses to informed traders are reduced, and total profits from liquidity trade is increased since their total order quantities are unaffected. Indeed, when liquidity trade is completely inelastic, one can show that market makers want to set arbitrarily steep price schedules in this environment — no equilibrium exists. This is because setting a slightly steeper schedule than the other market maker reduces losses to informed traders by more than profits from liquidity traders fall. Intuitively, aggregate market maker losses to insiders are reduced, and aggregate market maker profits from liquidity trade rise. The cost to setting a slightly less competitive schedule is that liquidity trade is shifted slightly to the other market maker, but this cost is less than the gain from the reduction in losses in informed trade which accrue only to the market maker that sets the steeper price schedule.

More formally, a liquidity trader with inelastic demand Q minimizes his trading costs by buying q_L^A of claims to the risky asset from market maker A and $Q - q_L^A$ from B. His optimization problem is

$$min_{q_L^A} \; q_L^A(P(q_L^A) + kq_L^A) + (Q - q_L^A)P(Q - q_L^A).$$

The associated first order condition is

$$P(q_L^A) + 2kq_L^A + q_L^A P'(q_L^A) = P(q_L^B) + q_L^B P'(q_L^B).$$

Differenting with respect to k at $k = 0$ (so $q_L^A = \frac{Q}{2}$), yields

$$\frac{dq_L^A}{dk} = \frac{-q_L^A}{2P'(q_L^A) + q_L^A P''(q_L^A)}.$$

Consider the effect on market maker A's profits from some liquidity trader of an increase in k:

$$d\frac{(P(q_L^A) + kq_L^A)q_L^A}{dk} = (q_L^A)^2 + (q_L^A P'(q_L^A) + P(q_L^A))\frac{-q_L^A}{2P'(q_L^A) + q_L^A P''(q_L^A)}.$$

If $P(\cdot)$ is linear, market maker A's profits from liquidity trade remain unchanged, and if $P(\cdot)$ is convex, then they increase. Since losses to informed trade fall when the market maker sets a less competitive schedule, and profits from liquidity trade remain at worst unchanged, it must be that each market maker always wants to set a less competitive schedule than the other, so that no equilibrium exists.

One should note that the results follow under weaker restrictions. Further, the differentiability and continuity assumptions are unimportant because a monotone increasing discontinuous or non–differentiable schedule can be approximated arbitrarily well by a twice continuously differentiable one, with arbitrarily small effect on order selection. Also, the perturbations that we use in the proofs are global perturbations — the price increase for an order of size q is kq. We need only have considered local price increases on any open interval. Thus, the argument extends immediately unless price schedules are everywhere too concave.

5 Robustness

Here we briefly consider the robustness of our results to the informational assumptions. Throughout this analysis we have assumed that the only order flow that schedule setters observe is that which they process, although in equilibrium, they can correctly infer the total order flow. This assumption is easy to motivate if the schedule setters are exchanges.

But suppose, instead, that market makers do observe the total order flow. Clearly the equilibrium that we describe continues to exist: in our formulation market makers correctly infer the total order flow from that in their market, so that observing the total order flow directly leaves the equilibrium unaffected. However, another 'equilibrium' also exists, one in which each market maker sets the zero–expected profit price conditional on the total order flow in both markets: $p(q_1, q_2) = p(q_1 + q_2)$. Given that one market maker sets this schedule, the other market maker cannot possibly earn positive profits, and hence is indifferent between setting that schedule and submitting any other schedule that earns him non–negative expected profits. One way to interpret this observation is that when order splitting is feasible, the standard model has multiple equilibria, one with positive market maker profits, and one with zero profits. Alternatively, observe that standard equilibrium refinements eliminate the zero–expected profit schedule as an equilibrium schedule. For instance, cheap talk between market makers ensures that they select the most profitable equilibrium outcome. Trivially, the zero–expected profit outcome is the least profitable outcome for market makers!

Next consider adding additional market makers or exchanges to the economy with elastic liquidity trade. As with oligopolistic competition between firms, the effect is to make any given market maker's liquidity demand more price elastic, so that the payoff to offering schedules with 'better' prices is increased.

Were some of those additional exchanges to have regulation that prohibits splitting of orders within the exchange, then multiple equilibria arise. It follows immediately that on the regulated exchange where orders cannot be split that competition drives prices down so that the schedule setters expect zero expected profits. It also follows that one equilibrium outcome is for the regulated market to set the zero–profit schedule which takes all transactions, leaving no trade on the unregulated market. Further, this equilibrium is robust to cheap talk because the market makers on the regulated exchange cannot do better than zero expected profits in any equilibrium. Other equilibria, however, can also exist, in which regulated exchanges offer more competitive schedules (i.e. with 'better' prices) than the unregulated exchanges, schedules that earn zero profits given that there will be trade on the unregulated exchanges, and where the aggregate price schedule is more competitive than it would be were there no regulation. Consequently, the regulation is socially beneficial.

Putting the potential costs and benefits of competition from additional exchanges into perspective, without regulation, the addition of more exchanges always leads to more competitive prices. However, if one market is regulated so that splits within that market are prohibited and market makers earn zero expected profits, then the introduction of an additional exchange between which traders can split *only* be counter productive.

6 Limit Orders

We now return to the general environment of section 4, except that we now assume that price schedules are set by agents who compete using limit orders. While we assume that there are only two such competing agents, the argument generalizes to any finite number.

All trade is through one of two risk neutral, uninformed agents who set continuous, twice differentiable, monotone increasing, limit order schedules detailing the prices at which they are willing to handle any given order flow. Trade by informed and uninformed agents can be characterized by their associated first order conditions. The trades of informed agents are more price elastic than those who also trade to rebalance their portfolios, because informed traders have no reasons to trade other than profit.

We first consider the case where liquidity demand is inelastic. There, we show that not only is the zero–profit limit order schedule not an equilibrium, but also that no equilibrium exists. As when market makers set average price schedules and liquidity demand is inelastic, non–existence follows because given that one agent submits a set of limit orders, the optimal response of the other is to submit limit orders that offer less competitive prices.

Next, we show that when liquidity demand is elastic, the zero profit limit order schedule cannot be an equilibrium outcome. Again, an agent submitting limit orders can (strictly) increase his expected profits by offering less competitive prices. The zero profit limit order schedule can be improved upon because liquidity trade is less price elastic than informed trade.

Perhaps more important, this analysis shows that the zero–profit condition is not appropriate for pinning down the limit order schedule unless the number of agents who submit limit orders is arbitrarily large. When the number of agents who submit limit orders is small, limit orders can expect significant positive profits in equilibrium. Further, it is unclear what regulation prohibiting order splitting would mean when price schedules are determined by limit orders. By their nature, limit order schedules induce order splitting. In contrast, while it is ambiguous whether the unregulated market maker schedule is more competitive than the limit order schedule (when there are a finite number of agents who submit limit orders), what *is* clear is that the market maker schedule when order splitting is prohibited dominates the limit order schedule, since the regulated market maker schedule earns zero expected profits.

6.1 Inelastic demand

Risk neutral agents trade claims to an asset with *ex ante* per–share expected value given by $E_0\{V\}$. The current value of the asset, V, is private information to an informed trader. Liquidity traders have inelastic demand. A single trader comes to the market and with probability π, $0 < \pi < 1$, he is an informed trader.

Since orders are handled independently, it is without loss of generality to focus on buy orders, $q > 0$. In the analysis it is easier to focus on the limit order schedule set by each schedule setter rather than work with the aggregate schedule. Consider any limit order schedule set by agent B, $P_B^L(\cdot)$. That is, $P_B^L(q)$ is the price of a limit order set by B when there are measure q lower–priced limit orders posted by B.

We now show that limit order schedule setter A earns greater profits from setting a steeper schedule than that set by B so that no equilibrium can exist: each schedule setter wants to set a steeper schedule than the other. The proof mimics that when market makers set average price schedules.

In particular, we show that A earns greater profits from setting limit order schedule

$$P_A^L(q) = P_B^L(q) + kq, \; k > 0.$$

An insider who with information V trades with the limit books A and B until the price of the marginal unit exceeds the value of his information:

$$P_j^L(q_I^A(V)) = V, \;\; j = A, B.$$

Conditional on trading q_I^A with an informed trader, the setter of schedule A receives losses:

$$q_I^A(V)V - \int_0^{q_I^A(V)} (P_B^L(q) + kq)dq.$$

The effect of increasing k on informed trade is given by

$$\frac{dq_I^A}{dk} = \frac{-q_I^A(V)}{P_B^{L'} + k},$$

so that the effect on market maker A's losses to informed trade from an increase in k is given by:

$$\frac{d(losses)}{dk} = V\frac{-q_I^A(V)}{P_B^{L'} + k} - (P_B^L(q_I^A(V)) + kq_I^A(V))\frac{-q_I^A(V)}{P_S' + k} + \frac{q_I^A(V)^2}{2} - k\frac{q_I^A(V)^2}{2(P_B^{L'} + k)}.$$

Since

$$P_S(q_I^A(V)) + kq_I^A(V) = V,$$

at $k = 0$, the change in losses to insiders simplifies to:

$$\frac{-q_I^A(V)^2}{2}.$$

Hence setting a steeper limit order schedule reduces market maker A's losses to informed trade.

We next calculate the effect on A's profits from trade with a liquidity trader. A liquidity trader who must trade q_L equates marginal costs across the two limit order schedules in equilibrium.

$$P_B^L(q_L^A) + kq_L^A = P_B^L(q_L - q_L^A).$$

The effect of an increase in k on a liquidity agent's trade at $k = 0$ is given by:

$$\frac{dq_L^A}{dk} = \frac{-q_L^A}{2P_B^{L'}},$$

The change in A's profits from liquidity trade is therefore given by

$$\frac{d}{dk}\left(\int_0^{q_L^A}(P_B^L(q) + kq)dq\right) = \frac{(q_L^A)^2}{2} + \left(\frac{-(q_L^A)}{2P_B^{L'}(q_L^A)}P_B^L(q_L^A)\right),$$

which is zero if the limit order schedule is linear, and strictly positive if it is convex. Since losses to informed trade are reduced by setting a steeper limit order schedule, provided that $P_B^L(\cdot)$ is not 'too' concave, A's profits are increased by setting a steeper schedule than B, so that no equilibrium can exist: each market maker always wants to set a less competitive schedule than the other, so that no equilibrium exists.

6.2 Elastic demand

In this section we show that when there are two agents setting limit order schedules, that the zero–expected profit limit order schedule cannot be an equilibrium outcome. Define $P_L(\cdot)$ to be the aggregate zero–profit limit order schedule: $P_L(\cdot)$ must generate zero expected profits for any given limit order quantity q, conditional on the fact that it is crossed against all market orders of at least size q. When there are two agents submitting limit orders, the zero–expected profit limit pricing schedule that each agent sets satisfies $P_S(\frac{q}{2}) = P_L(q)$. As a result, traders who submit market orders split their trades evenly between the agents who submit limit orders, so that the price for an order of size q remains $\int_0^q P_L(q)dq$.

We now show that $P_S(q)$ is not an equilibrium schedule: that is, agents submitting limit orders can do better by submitting less competitive limit order schedules. Since orders are handled independently, it is without loss of generality to focus on buy orders,

$q > 0$. Let agent B set a limit order schedule

$$P_B(q) = P_S(q)$$

and and let A set schedule

$$P_A(q) = P_S(q) + kq,\ k > 0.$$

The insider's orders $q_I^j(V)$, $j = A, B$ are again given by the solution to

$$P_j(q_I^j(V)) = V.$$

As before, suppose that the liquidity trader's reduced–form objective can be written as

$$max_{q_L^A, q_L^B}\ W(q_L^A + q_L^B \mid \phi) - \int_0^{q_L^A} (P_S(q) + kq)dq - \int_0^{q_L^B} P_S(q)dq,$$

The associated first order conditions are given by

$$\frac{dW(q_L^A + q_L^B \mid \phi)}{dq_L^B} = P_S(q_L^B).$$

and

$$\frac{dW(q_L^A + q_L^B \mid \phi)}{dq_L^A} = P_S(q_L^A) - kq_L^A.$$

Consider now liquidity and informed traders who would submit the same order were the two market makers to set the same schedules. The effects of marginal increase in market maker A's price schedule on their orders are given by:

$$\frac{dq_I^A}{dk} = \frac{-q_I^A}{P_S'} < 0,$$

$$\frac{dq_I^B}{dk} = 0,$$

$$\frac{dq_L^A}{dk} = \frac{q_L^A(W'' - P_S)}{(W'' - P_S')^2 - (W'')^2} < 0,$$

$$\frac{dq_L^B}{dk} = \frac{q_L^B W''}{(W'')^2 - (W'' - P_S')^2} > 0,$$

where we drop the arguments of the functions and the primes refer to the associated derivatives.

Since $W(\cdot)$ is strictly concave, at $q_L^A = q_I^A$, $\frac{dq_I^A}{dk} < \frac{dq_L^A}{dk}$. To see this, observe that

$$sign[\frac{d^2 q_L^A}{dkdW''}] = sign[(W''-P_S')^2-(W'')^2-2(W''-P_S')(W''-P_S'-W'')] = sign[-(P_S')^2] < 0.$$

The more concave is $W(\cdot)$, the less price elastic is liquidity trade with market maker A, and hence the less by which liquidity trade with A falls when A increases price. The argument now follows directly from that when market makers set average price schedules, liquidity trade is elastic, and traders can split orders.

7 Conclusion

This paper demonstrates that when traders can split orders among risk–neutral market makers then competition *does not* lead to set schedules that earn them zero expected profits conditional on the order flow. Indeed, market makers set far less competitive schedules that earn significant profits and increase trading costs. If some (noise) traders have completely inelastic demands, then market makers want to set arbitrarily steep price schedules that earn them arbitrarily large profits at the expense of the noise traders, so that no equilibrium exists. This paper therefore documents an important lack of robustness of standard market microstructure models: competition among risk–neutral market makers *only* leads to zero-expected profit pricing if agents cannot split orders.

The paper also suggests that if there is a dominant exchange, then rules that prohibit traders from splitting orders, if enforceable, actually help the traders. In this instance, the introduction of an additional exchange between which traders can split can only be counterproductive. However, if there are multiple exchanges so that traders can split orders across exchanges, then additional exchanges lead to more competitive prices, as the exchanges compete for additional order share.

Lastly, the paper finds that limit order schedules set by a finite number of agents are necessarily uncompetitive and generate positive expected profits for the schedule setters.

8 Appendix

Proof to proposition 1:

Let market maker A's ask price schedule (also set by B) be given by

$$p_A(q) = 1 + aq.$$

Given this schedule, the insider observing V trades to maximize

$$max_{q_I} \; q_I(V - aq_I).$$

Solving, the informed agent buys

$$q_I = \frac{V}{2a}$$

of claims to the asset. Integrating over the possible innovations that the insider can observe, he earns expected profits

$$\int_0^1 \frac{V}{2a}(V - a\frac{V}{2a})dV = \int_0^1 \frac{V^2}{4a}dV = \frac{1}{12a}.$$

A liquidity trader who is not constrained by his endowment, trades to maximize

$$max_{q_L^A} \; Q - (1 + aq_L^A)q_L^A + \bar{\beta}q_L^A,$$

which has solution

$$q_L^A = \frac{\bar{\beta} - 1}{2a}.$$

When $\bar{\beta} \geq 2$, we will show that liquidity traders are always endowment constrained given zero–expected profit pricing.

We claim that the ask schedule that earns zero expected profits is given by

$$p(q) = 1 + \frac{q}{2}.$$

Given this price schedule, the informed trader trades

$$q_I = \frac{V}{2(\frac{1}{2})} = V.$$

We must now verify that market makers earn zero expected profits conditional on the order flow. That is, we verify that the expected value of the asset given an order of q is exactly $p(q) = 1 + \frac{q}{2}$. To see this, observe that the market maker's loss per share to an informed trader who buys claims to the risky asset $q_I = V > 0$ are

$$V - \frac{V}{2} = \frac{V}{2}.$$

Similarly, the market maker earns $\frac{V}{2}$ from a liquidity trader who also buys V.

For market makers to earn zero profits conditional on any order flow q, it must therefore be that conditional on that order flow the trader is equally likely to be informed as uninformed. Since informed purchases of the risky asset are given by V, which has a uniform [0,1] distribution, this means that the distribution of purchases of the risky asset by the liquidity trader must also be uniformly distributed on [0,1]. A liquidity trader with a first–date consumption endowment of Q purchases q claims to date–two consumption, where

$$Q = p(q)q = q + \frac{q^2}{2}.$$

Solving, for the number of claims that a liquidity trader buys,

$$q = -1 + \sqrt{(1 + 2Q)}.$$

For q to be distributed uniformly on [0,1], it must be that

$$Pr(-1 + \sqrt{(1 + 2Q)} \leq z) = z, \; z \in [0, 1]$$

or that

$$Pr(Q \leq \frac{(z + 1)^2 - 1}{2}) = z.$$

Using a change of variables, $y = \frac{(z+1)^2-1}{2}$, yields

$$Pr(Q \le y) = \sqrt{(2y+1)} - 1,$$

which has the density

$$f(Q) = \frac{1}{\sqrt{(2Q+1)}},$$

that we assumed governs the endowments of liquidity traders.

Proof to proposition 2

If a liquidity trader is constrained by his endowment Q, then he maximizes his purchases of claims to the risky asset by buying q_L^A from market maker A and $Q - q_L^A$ from B. His optimization problem is

$$max_{q_L^A, q_L^B} \; q_L^A + q_L^B$$

$$s.t. \; q_L^A(1 + aq_L^A) + q_L^B(1 + bq_L^B) = Q,$$

which has solution

$$q_L^A = \frac{-1 + a\sqrt{\frac{1}{a^2} + \frac{4Qb}{a(a+b)}}}{2a}, \quad q_L^B = \frac{-1 + b\sqrt{\frac{1}{b^2} + \frac{4Qa}{b(a+b)}}}{2b}.$$

If the liquidity trader is unconstrained by his endowment, then he trades to maximize

$$max_{q_L^A, q_L^B} \; Q - (1 + aq_L^A)q_L^A - (1 + bq_L^B)q_L^B + \bar{\beta}(q_L^A + q_L^B),$$

which has solution

$$q_L^A = \frac{\bar{\beta} - 1}{2a}, \quad q_L^B = \frac{\bar{\beta} - 1}{2b}.$$

Hence, a liquidity trader trades the quantity

$$min\left\{\frac{-1 + a\sqrt{\frac{1}{a^2} + \frac{4Qb}{a(a+b)}}}{2a}, \; \frac{\bar{\beta} - 1}{2a}\right\}$$

with market maker A, and

$$min\left\{\frac{-1+b\sqrt{\frac{1}{b^2}+\frac{4Qa}{b(a+b)}}}{2b}, \ \frac{\bar{\beta}-1}{2b}\right\}$$

with market maker B. For first–period endowment $Q \geq \frac{(\bar{\beta}^2-1)(a+b)}{4ab}$, the liquidity trader's optimal purchases of claims to second–period consumption are not limited by his endowment.

Market maker A's objective is then to maximize profits given correct conjectures about B's ask price schedule:

$$max_a \ \frac{-1}{12a}+\int_0^{\hat{Q}}\left(\frac{-1+a\sqrt{\frac{1}{a^2}+\frac{4Qb}{a(a+b)}}}{2a}\right)^2 af(Q)dQ+\int_{\hat{Q}}^{\frac{3}{2}}\left(\frac{\bar{\beta}-1}{2a}\right)^2 af(Q)dQ,$$

where $\hat{Q}=\frac{(\bar{\beta}^2-1)(a+b)}{4ab}$, and $f(Q)=\frac{1}{\sqrt{1+2Q}}$.

Differentiating with respect to a and setting $a=b$ in the first order conditions in order to solve for the symmetric equilibrium price schedule yields an expression that cannot be solved analytically for a. However, we can solve for a numerically for different values of $\bar{\beta}$. The results are graphed in figure 1.

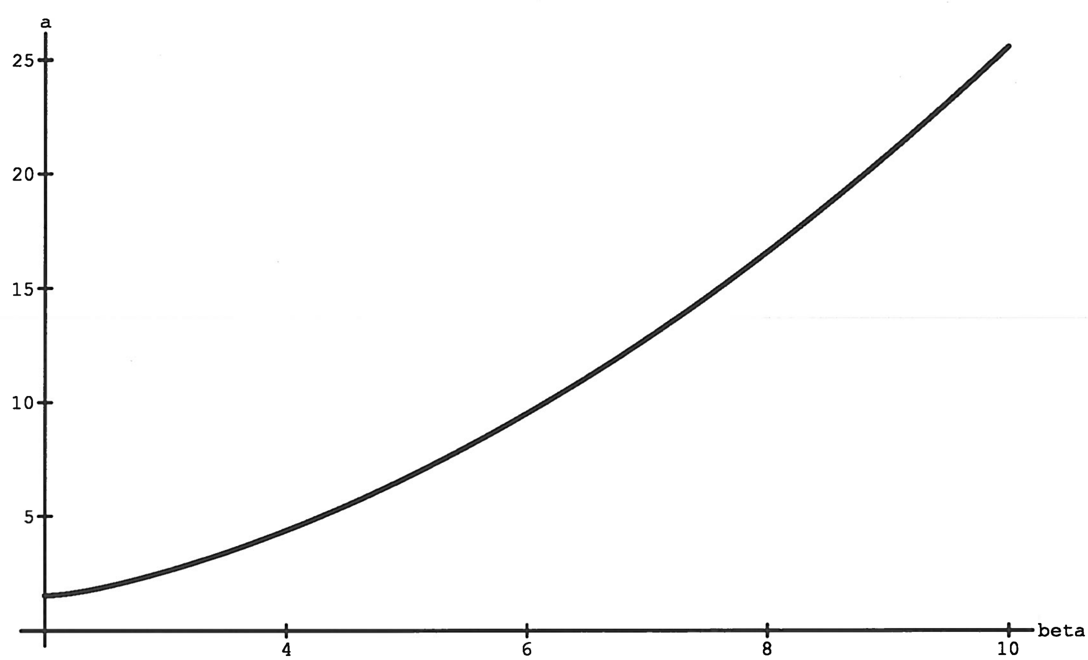

Figure 1: This figure graphs the slope of the equilibrium pricing function for different values of $\bar{\beta}$.

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