Queen's Economics Department Working Paper No. 877

Dominance Relations Among Standardized Variable

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2-1993

Discussion Paper #877

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February 1993

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Latest revision: Feb. 11, 1993

Abstract

This paper examines stochastic dominance relations among discrete random variables defined on a common integer domain. While these restrictions are minimal, they lead both to new theoretical results and to simpler proofs of existing ones. The new results, obtained for dominance criteria of any degree, generalize an *SSD* result of Rothschild - Stiglitz to describe how for any dominance criterion a dominated variable is equal in distribution to a dominated variable plus perturbation terms. If the variables are comparable under *FSD* the perturbations are downward shift terms, while under *SSD (TSD)* all but two (three) of the perturbations are zero mean disturbance terms (noise). Under *SSD* the remaining perturbations are shift terms and under *TSD* noise and shift terms. However, under either *SSD* or *TSD* these remaining terms are identically zero if the variables to be compared have equal means. The paper also finds new proofs of well known results relating dominance criteria to preferences.

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1. Introduction and review of literature

The partial orderings induced by stochastic dominance have numerous uses: following Merton [1973], dominance arguments are commonly used in financial and economic theory to show that certain securities or portfolio values are incompatible with equilibrium. Dominance tests have also found management application, especially in portfolio selection. Bawa et. al. [1979] offer an early survey; recent studies include Falk-Levy [1989] and Ritchken-Kuo [1989].

Despite the extensive applications literature, there are few theoretical papers of direct relevance to the present work. Hadar-Russell [1969], Hanoch-Levy [1969], and Whitmore [1970] all relate classes of utility functions to particular dominance criteria. Rothschild-Stiglitz [1970, 1971] (hereafter RS) study *SSD* relations between pairs of comparable, equal mean variables, showing that the dominated variable is equal in distribution to the dominating variable perturbed by noise.[1]

This paper compares discrete random variables defined on a common domain, a minimally restrictive condition. Yet, the approach is fruitful: the paper finds new results for dominance criteria of any degree. In particular, it obtains new results for *FSD* and *TSD* and extends the RS results for *SSD*. In addition it provides new, easy proofs of known theorems relating dominance comparisons and preferences.

The paper shows that if two variables defined on the same domain are comparable under a given dominance criterion, the dominated variable is equal in distribution to the dominating variable plus perturbation terms. Under *FSD*, all perturbations are downward shift terms. Under *SSD* (*TSD*) all but two (three) are noise terms. The remaining perturbation terms can be combined into a shift term (*SSD*) or a noise and shift term (*TSD*). If the variables have equal means, there are no remaining perturbations under either *SSD* or *TSD*. The paper also shows that the perturbation terms for each dominance criterion relate naturally: to each other when different criteria are compared, and to preferences when a given criterion is selected.

The paper is organized as follows. Section 2 develops a convention for representing variables and uses it to prove two new lemmas which, when elaborated, lead to all the paper's results. Section 3 uses the lemmas to obtain *FSD* results, Section 4 obtains *SSD* and Section 5 *TSD* results. Section 6 concludes. An Appendix interprets standard dominance theorems using the paper's conventions.

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[1]Also, Ross [1978] discusses relations between dominance and the separation conditions of financial theory.

2. Theoretical Approach

This section defines the paper's notational conventions and develops an approach which unifies the paper's subsequent results. Later sections apply the approach and develop new proofs of existing results.

2.1 Representing variables

To standardize random variables while restricting their form only minimally, consider an approach due to Feller [1957]. For given values of n and k, define a family of random variables as all possible configurations of n indistinguishable balls placed in $k + 1$ cells. Each distinct ball - cell configuration represents a random variable, and the proportion of balls in a given cell indicates the probability with which the outcome (represented by the cell label) obtains. It is convenient to refer any such set of variables as an (n, k) - family.[2]

Let the outcomes represented by the cells be the integers $0, \ldots, k$, and index the cells using the same values. Then if X is a member of an (n, k) - family, its density function g_X is given by

$$g_X(j) = n_j \,/\, n \equiv x_j\,; \tag{2.1}$$

where $n_j \geq 0$ is the number of balls placed in the j'th cell,

$$j \in \{\, 0, 1, \ldots, k \,\} \equiv J_k\,,$$

and[3]

$$\Sigma_j\, n_j = n.$$

Thus, $x_j \geq 0$ is the probability of realizing outcome j. It is convenient to describe density functions[4] using *probability vectors:*

$$x \equiv (x_0, x_1, \ldots, x_k)'.$$

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[2]Taking the cell ordering into account, an (n, k) - family has $(n + k)! \,/\, n!k!$ members, where $!$ indicates a factorial. Continuous densities can be approximated to any desired degree of fineness by increasing n, k, or both. If the underlying densities are defined on an interval of fixed length, proportionately increasing the values of n and k increases the fineness of the densities' definition. Moreover, particular classes of densities (e.g.,unimodal densities) can be represented as subsets of the (n, k) - families.

[3]The n_j are sometimes called **occupancy numbers**; cf. Feller [1957].

[4]If $x_j = 0$ for some $j \in J_k$, the outcome is assumed not to occur.

The vector x satisfies $e'x = 1$, where e is a $(k + 1)$ - dimensional vector of ones.

2.2 Dominance Relations

For the rest of this paper, suppose all random variables are members of the same (n, k) - family.[5] It is then possible to use two linear operators to analyze dominance relations of any order. The first operator forms higher order cumulative summations of probability vectors (densities); the second, differencing operator is the first's inverse. The first order operators are denoted S and S^{-1}. S is a $(k+1) \times (k+1)$ matrix composed of ones on and below the main diagonal, and of zeroes above. It is easy to check that, given the form of S, S^{-1} exists. The higher order operators are $(S)^m \equiv S^m$ and $(S^{-1})^m \equiv S^{-m}$, where this paper considers the cases $m = 2, 3$. To illustrate, if $k = 3$,

$$S \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} ; \; S^2 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix} ; \; S^3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 6 & 3 & 1 & 0 \\ 10 & 6 & 3 & 1 \end{vmatrix} \qquad (2.2)$$

and

$$S^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix} ; \; S^{-2} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{vmatrix} ; \; S^{-3} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{vmatrix} . \qquad (2.3)$$

In keeping with earlier notation, the rows of S^m and S^{-m} are indexed from 0 to k.

S and S^{-1} are used for several purposes. First, if x is the probability vector of a random variable X, Sx is its vector of first order cumulative sums (corresponding to a distribution function), and $S^m x$ its vector of m'th cumulative sums. Second and conversely, if $S^m x \equiv x^{(m)}$, then $S^{-m} x^{(m)} = x$. As will be shown, this implies that the m'th order differencing operator S^{-m} relates the m'th order dominance criterion to the underlying densities. Third, S^{-m} will be applied to utilities to characterize the preferences underlying different dominance criteria.

To see how S standardizes relations between different orders of dominance, let

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[5]The standardized domain is not restrictive: any positive linear transformation can be applied to the domain without affecting the dominance relations between pairs of variables.

$$G_X^{(m)}(\cdot) \equiv S^m x. \tag{2.4}$$

Then A stochastically dominates B in the *m'th* degree iff

$$G_A^{(m)}(j) \leq G_B^{(m)}(j);\ j \in J_k; \tag{2.5}$$

$m \in \{1, 2, 3\}$, and[6] if $m = 3, E(A) \geq E(B)$ (cf. Hadar-Russell [1969], Hanoch-Levy [1969], Whitmore [1970]).

For present purposes, it is more convenient to write that A stochastically dominates B in the *m'th* degree iff

$$S^m(b - a) \equiv \alpha^{(m)} \geq 0; \tag{2.6}$$

where a and b are the probability vectors for A and B, $\geq$ is a vector inequality, and[7] if $m = 3, E(A) \geq E(B)$. To stress an underlying duality in the present treatment of densities and preferences, represent any von Neumann Morgenstern utility function U by the vector of values $u \equiv (u_0, \ldots, u_k)'$, where $u_j \equiv U(j)$. Thus for any X, $E\{\ U(X)\ \} \equiv u'x$.

It is now possible to state: i) relations between the density functions of any pair comparable in the *m'th* degree,[8] and ii) conditions on preferences for a pair to be comparable in the *m'th* degree.

Lemma 1. (Densities) Suppose A and B are members of the same (n, k) - family. Then A stochastically dominates B in the *m'th* degree; $m = 1, 2, \ldots, 3$ iff

$$b = a + S^{-m}\alpha^{(m)}, \tag{2.7}$$

and, if $m = 3, E(A) \geq E(B)$.

Proof: Condition (2.7) follows immediately from (2.6) and the existence of S^{-m}. ■

Lemma 1 reveals new aspects of dominance relations. As shown below, it is possible to write $S^{-m}\alpha^{(m)}$ as a sum of vectors, each representing a particular type of shift or noise term depending on the order of dominance being examined. Since $S^{-m}\alpha^{(m)}$ also relates the density

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[6]For reasons to be developed below, we do not require the means to be equal when $m = 2$.

[7]Note that $A \equiv B$ unless at least one component of α differs from zero.

[8]Lemma 1 also describes relations between pairs of variables that are not comparable, but we do not use this information below.

functions of a dominating and a dominated variable, it is then possible to show that if A dominates B (in any degree), B is equal in distribution to A plus a number of perturbation terms. As a result, (2.7) generalizes an *SSD* result of Rothschild - Stiglitz [1970] (RS) to dominance criteria of any degree. Additionally, (2.7) leads to simpler proofs of the RS results than those originally obtained.

The second lemma uses the difference operators S^{-m} to define classes of preferences. Let

$$\mathbf{U}^{(m)} \equiv \{ U \mid u'S^{-j} \leq 0; j = 1, \ldots, m \}; m = 1, 2, 3. \tag{2.8}$$

Condition (2.8) both defines the class of preferences to which a given dominance relation applies, and (more importantly) relates that preference class to the dominance criterion. As will later be shown, these relations are useful for understanding dominance criteria intuitively. When $m = 1$ (2.8) defines the class of monotone increasing utilities. In this case Lemma 1 establishes an equivalence between *FSD* on the one hand and downward shifts of probability associated with S^{-1} on the other. Moreoever, Lemma 2 shows through S^{-1} that differences in expected utility depend on the sign of the utilities' first differences, thus establishing a relationship that is a dual to the downward shifts of probabilities.

Similarly when $m = 2$, (2.8) defines the set of increasing concave utilities.[9] Lemma 1 establishes an equivalence between *SSD* on the one hand, and perturbations that are all (except for one shift term) zero mean spread variables associated with S^{-2} on the other. Then lemma 2 shows through S^{-2} that differences in expected utility depend on the sign of the utilities' second differences, again establishing a dual relation. Finally for $m = 3$, lemma 1 establishes an equivalence between *TSD* and a third order perturbation term[10] associated with S^{-3}, and the dual relation through S^{-3} relates the sign of the utilities' third differences to differences in expected utility.[11]

Lemma 2. (Preferences): Suppose A and B are members of the same (n, k) - family and that A stochastically dominates B in the *m'th* degree. Suppose moreover that, if $m = 3$, $E(A) \geq E(B)$. Then A is preferred to B iff $u'a \geq u'b$ for all $u \in \mathbf{U}^{(m)}$.

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[9]We require both monotonicity and concavity for studying second degree dominance because, unlike Rothschild-Stiglitz [1970,1971], we do not require equality of the variables' means. When the means are equal, monotonicity is unnecessary.

[10]As will be seen, however, for most purposes it is convenient to reinterpret the third order perturbation as sums of first and second order perturbations.

[11]Similar arguments can be made for $m > 3$, but in the interests of brevity we do not develop them here.

Proof (Sufficiency): Suppose A dominates B in the $m'th$ degree and consider the expected utility difference $-u'(b - a)$. Then from (2.7)

$$-u'(b - a) = -u'(S^{-m}\alpha^{(m)}) = -(u'S^{-m})\alpha^{(m)}, \tag{2.9}$$

and $\alpha^{(m)} \geq 0$ by dominance. But since the first $k - (m - 1)$ components of $u'S^{-m}$ can be chosen arbitrarily[12] it follows that $u'a \geq u'b$ only if $-u'S^{-m} \geq 0$; i.e., only if $u \in U^{(m)}$.

(Necessity): The converse uses the argument of Hadar-Russell [1969]. Suppose A does not dominate B, in which case at least one component of $\alpha^{(m)}$ is negative. But then, since the first $k - (m - 1)$ components of u^* can still be chosen arbitrarily, it is possible to find some $u^* \in U^{(m)}$ such that $-((u^*)'S^{-m})\alpha^{(m)} < 0$; i.e., a utility for which B is preferred to A. As shown in detail in Theorems A1 through A3, the approach is to define a utility function whose $m'th$ order differences are zero except at one point where the inequality defining dominance is reversed. For such a choice, the expected utility of B exceeds that for A, creating the desired contradiction.[13] ■

The importance of Lemma 2 rests on applying the differencing operators S^{-m} to u rather than to α. This change in emphasis leads to easy proofs of well known results for *FSD, SSD,* and *TSD,* as indicated by the necessity proof of Lemma 2 and as discussed further below.

3. First degree dominance

By (2.6), A dominates B by *FSD* if and only if

$$S(b - a) = \alpha^{(1)} \equiv \alpha \geq 0. \tag{3.1}$$

where a and b are the probability vectors for A and B respectively.[14] By (2.7),

$$b - a = S^{-1}\alpha \tag{3.2}$$

and $S^{-1}\alpha$ takes the particular form

$$S^{-1}\alpha = (\alpha_0 , -\alpha_0 + \alpha_1 , \ldots, -\alpha_{k-2} + \alpha_{k-1} , -\alpha_{k-1} + \alpha_k)'. \tag{3.3}$$

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[12] S^{-m} takes the $m'th$ differences of $k + 1$ integers, which leaves $k + 1 - m$ terms to be chosen arbitrarily. For example the first differences of a_1 , a_2 , and a_3 are $a_2 - a_1$, $a_3 - a_2$.

[13] This argument is standard in the literature. For convenience, it is developed at greater length, for *FSD, SSD* and *TSD* respectively, in Appendix Theorems A1, A2, and A3.

[14] Note that $A \equiv B$ unless at least one component of α differs from zero.

In the rest of this section it will be helpful to write

$$S^{-1}\alpha \equiv \delta_0 + \ldots + \delta_k$$

where

$$S^{-1}\alpha \equiv \begin{vmatrix} \alpha_0 \\ -\alpha_0 + \alpha_1 \\ 0 \\ 0 \\ \ldots \\ -\alpha_{k-1}+\alpha_k \end{vmatrix} \equiv \begin{vmatrix} \alpha_0 \\ -\alpha_0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{vmatrix} + \ldots + \begin{vmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ \alpha_{k-1} \\ -\alpha_{k-1} \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \\ \alpha_k \end{vmatrix} . \tag{3.4}$$

Denoting the rows of S by S_j , it follows from $S_k b = S_k a = 1$ that $\alpha_k = 0$. Hence in (3.4) the extreme right-hand vector $\delta_k \equiv 0$.

Conditions (3.2) to (3.4) will next be used to show that (since $\alpha \geq 0$), the dominated variable B is equal in distribution to the dominating variable A perturbed by a set of downward shift terms. The downward shift terms are found by observing from the right hand side of (3.4) that S^{-1} assigns a given value α_j , $j \in J_{k-1}$, only to the outcomes $A = j + 1$ and $A = j$.

The last observation means that conditions (3.2) to (3.4) can be related to a theorem of Rothschild-Stiglitz [1970]. The RS findings, applying to *SSD*, can be expressed as

$$B \simeq A + \Delta, \tag{3.5}$$

where $\simeq$ means equality in distribution; cf. Ingersoll [1987, pp. 71-72, p. 120]. To find similar relations for *FSD*, find some $j \in J_k$ for which $\alpha_j > 0$. From (3.2) and (3.4), it is evident that δ_j transfers probability from outcome $j + 1$ to outcome j. Interpreting the transfer of probability in terms of the conditional random variable $\Delta | (A = j + 1)$, we have[15]:

[15]Even if $\alpha_i < 0$ for some $i \in J_k$, so there is no first degree dominance, it is possible to define similar shift variables, which in this case represent a transfer of probability toward the next higher outcome. The fact that the only permissible shifts of probability are in a downward direction if variables are to be ranked for all monotone increasing utilities forms the basis of dominance proofs through contradiction; cf. Lemma 2 and Appendix Theorem A1.

$$\Delta|(A = j+1) = \begin{cases} -1; \text{ with prob } \alpha_j \,/\, a_{j+1} \\ 0; \text{ with prob } (a_{j+1}-\alpha_j) \,/\, a_{j+1} \end{cases} \quad . \tag{3.6}$$

Division by a_{j+1} converts the joint probabilities α_j and a_{j+1} - α_j to conditional probabilities.[16]

Next denote the conditional random variables in (3.6) by Δ_j, $j \in J_k$, and let Δ be equal in distribution to the combined effect of the Δ_j . Then

Theorem 1: A dominates B by FSD if and only if:

$$B \simeq A + \Delta,$$

where

$$E(\Delta_j | A) \leq 0; j \in J_k.$$

Proof: Given the above information, necessity and sufficiency are both immediate. ■

Table 1 gives an example of the FSD perturbations, for brevity omitting δ_0 and δ_3 which in this case are both identically zero.

[16]If a_{j+1} - α_j < 0, the conditional probability can be defined using $a_{j+1} + \alpha_{j+1}$ - α_j / $a_{j+1} + \alpha_{j+1}$. The numerator of this latter quantity is non-negative because the probability vector b is constrained to have non-negative components.

Table 1. Example of Perturbations for *FSD*

Outcomes	a	δ_1	δ_2	b
0	0	0	0	0
1	0	2/7	0	2/7
2	3/7	-2/7	1/7	2/7
3	4/7	0	-1/7	3/7

The columns headed δ_1 and δ_2 indicate the redistribution of (unconditional) probability attributable to α_1 and α_2. In this case, $\alpha = (1/7)(0, 2, 1, 0)'$. Moreover,

$$\Delta | (A = 2) = \begin{cases} -1; \text{ with prob } 2/3 \\ 0; \text{ with prob } 1/3 \end{cases}$$

and

$$\Delta | (A = 3) = \begin{cases} -1; \text{ with prob } 1/4 \\ 0; \text{ with prob } 3/4. \end{cases}$$

The principal Theorem of *FSD* can now be proved using Lemma 2. Consider the class of monotone non-decreasing utilities $U^{(1)} \equiv M$.

Theorem 2: A dominates B by *FSD* iff $u'a \geq u'b$ for any $U \in M$.

Proof (Sufficiency): Suppose A dominates B by *FSD*, so that $\alpha_j \geq 0, j \in J_k$. But

$$-u'(b - a) = -u'S^{-1}\alpha,$$

and by definition of S^{-1},

$$-u'S^{-1} = -(u_0 - u_1, u_1 - u_2, \ldots, u_{k-1} - u_k, u_k)'.$$

Now since the first k components of $-u'S^{-1}$ can be chosen arbitrarily, $u'(b - a) \geq 0$ only if $u \in M$.

(Necessity): The converse can be established as a special case of the methods used in Lemma 2; cf. Appendix Theorem A1. ■

While Hadar-Russell [1969] show that $E(A) \geq E(B)$ is necessary for A to dominate B by *FSD*, it is not sufficient for members of the same (n, k) - family. For if $E(A) = E(B)$, then

$$E(A) - E(B) = (0, 1, \dots, k)' \cdot (b - a) =$$

$$-(k+1)[\ (b_0 + \dots + b_k) - (a_0 + \dots + a_k)\ ] + (0, 1, \dots, k)' \cdot (b - a) = \qquad (3.7)$$

$$= e'\alpha = 0.$$

But the last equality in (3.7) cannot hold unless either $\alpha_j < 0$ for some $j \in J_k$ or $\alpha_j = 0$ for all $j \in J_k$. The first case is ruled out by *FSD*, and the second is possible only if the variables are identical. In the case of strict dominance, $E(A) > E(B)$ is a necessary condition because $E(\Delta | A = j) < 0$ for at least one value of $j \in J_k$.

4. Second Degree Dominance

By (2.6), A stochastically dominates B by *SSD* if and only if

$$S^2(b - a) = \alpha^{(2)} \equiv \beta \geq 0. \qquad (4.1)$$

Since $\beta = S\alpha$, $\beta_k = \beta_{k-1} + \alpha_k$, and since Section 3 showed $\alpha_k = 0$, it follows immediately from (4.1) that $\beta_k = \beta_{k-1}$. However, the relation is actually more specific:

Lemma 3:

$$\beta_k = \beta_{k-1} = [E(A) - E(B)]. \qquad (4.2)$$

Proof: The properties of S developed in Section 2 imply

$$\beta_{k-1} = S^2_{k-1}x = (k, k-1, \dots, 1, 0)'x =$$

$$k(e'x) - (0, 1, \dots, k-1, k)'x =$$

$$k - E(X). \qquad (4.3)$$

Hence

$$S^2_{k-1}(b - a) = [k - E(B)] - [k - E(A)] = [E(A) - E(B)],$$

and moreoever

$$\beta_k = e'x + \beta_{k-1} =$$

$$[1 + k - E(B)] - [1 + k - E(A)] = [E(A) - E(B)]. \qquad \blacksquare$$

We wish next to apply (2.7) with a view to interpreting the relations between A and

B. Rewrite (4.1) as

$$(b - a) = S^{-2}\beta, \tag{4.4}$$

where $S^{-2}\beta$ takes the particular form

$$S^{-2}\beta = (\beta_0 , -2\beta_0 + \beta_1 , \beta_0 - 2\beta_1 + \beta_2 , \ldots, \beta_{k-2} - 2\beta_{k-1} + \beta_k)'. \tag{4.5}$$

For the rest of this section, it is helpful to write

$$S^{-2}\alpha \equiv \delta_0 + \ldots + \delta_k$$

where

$$S^{-2}\beta = \begin{vmatrix} \beta_0 \\ -2\beta_0 + \beta_1 \\ \beta_0 - 2\beta_1 + \beta_2 \\ 0 \\ \ldots \\ \beta_{k-2} - 2\beta_{k-1} + \beta_k \end{vmatrix} = \begin{vmatrix} \beta_0 \\ -2\beta_0 \\ \beta_0 \\ 0 \\ \ldots \\ 0 \end{vmatrix} + \ldots + \begin{vmatrix} 0 \\ 0 \\ \ldots \\ \beta_{k-2} \\ -2\beta_{k-2} \\ \beta_{k-2} \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ \beta_{k-1} \\ -2\beta_{k-1} \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \\ \beta_k \end{vmatrix}. \tag{4.6}$$

Suppose $\beta_j > 0$ for some $j \in J_{k-2}$. From (4.6), it is evident that β_j transfers probability from outcome $j + 1$ equally to outcomes j and $j + 2$. Interpreting this transfer of probability along the lines of Section 3,

$$\Delta|(A = j+1) = \begin{cases} -1; \text{ with prob } \beta_j \,/\, a_{j+1} \\ 0; \text{ with prob } (a_{j+1} - 2\beta_j) \,/\, a_{j+1}. \\ 1; \text{ with prob } \beta_j \,/\, a_{j+1} \end{cases} \tag{4.7}$$

That is, (4.7) defines the (conditional) probability density of the random variable $\Delta | A = j + 1$ which, for any $\beta_j > 0$, effectively shifts probability from outcome[17] $A = j + 1$ equally to $A = j$ and to $A = j + 2$. Finally, if $E(A) = E(B)$, $\beta_{k-1} = \beta_k = 0$, and the two right hand vectors in (4.6) are both identically zero. If instead $E(A) > E(B)$, then $\beta_{k-1} = \beta_k > 0$, and summing δ_{k-1} and δ_k shows that probability is transferred from outcome k to outcome

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[17]As in n.14, if $a_{j+1} - 2\beta_j < 0$, the conditional probability can be specficied using $a_{j+1} + \beta_{j+1} - 2\beta_j$.

$k - 1$ by a shift variable (as with *FSD*).

The Rothschild-Stiglitz [1970] results can now be generalized in the:

Theorem 3: A dominates B in the second degree if and only if

$$B \simeq A + \Delta,$$

where Δ is equal in distribution to the combined effects of $(\Delta | A = j + 1) \equiv \Delta_j$, the latter are noise terms such that

$$E(\Delta_{j-1} | A = j) = 0; j\text{-}1 \in J_{k-1},$$

and Δ_k is a shift term such that

$$E(\Delta_k | A = k) \leq 0.$$

Proof: Both necessity and sufficiency follow immediately from the examination of the conditional random variables. ■

The Rothschild-Stiglitz [1970] result itself can then be stated as the:

Corollary: If A and B are members of an (n, k) - family and if $E(A) = E(B)$, then A dominates B by *SSD* if and only if $B \simeq A + \Delta$, where

$$E(\Delta_{j-1} | A = j) = 0; j \in \{1, \ldots, k\}.$$

Proof: If $E(A) = E(B)$ then

$$\beta_k = \beta_{k-1} = [E(A) - E(B)] = 0$$

by (4.4). Then both necessity and sufficiency are immediate as before. ■

With arbitrarily described random variables, the RS necessity proof of the foregoing corollary is considered complex; cf. Huang and Litzenberger [1988]. The structure imposed by restricting the variables to an (n, k) - family simplifies the demonstration. Table 2 gives an example of the perturbations, as before omitting identically zero variables.

Table 2. Example of Perturbations for *SSD*

Outcomes	a	δ_1	b
0	1/7	0	1/7
1	2/7	1/7	3/7
2	4/7	-2/7	2/7
3	0	1/7	1/7

The column headed δ_1 indicates how δ_1 corresponds to the redistribution of (unconditional) probability associated with β_1. In this case, $\beta = (1/7)(0,\ 1,\ 0,\ 0)'$, and

$$\Delta | (A = 2) = \begin{cases} -1; \text{ with prob } 1/4 \\ 0; \text{ with prob } 1/2 \\ 1; \text{ with prob } 1/4. \end{cases}$$

The next Theorem is a direct consequence of Lemma 2. Consider the class of non-decreasing concave utilities $\mathbf{U}^{(2)} \equiv V$. Then

Theorem 4: A dominates B by *SSD* iff $u'a \geq u'b$ for any $U \in V$.

Proof (Sufficiency): Suppose A dominates B, in which case $\beta_j \geq 0, j \in J_k$. But

$$-u'(b - a) = -u'(S^{-2}\beta) = -(u'S^{-2})\beta,$$

and

$$-u'S^{-2} = -(u_0 - 2u_1 + u_2, u_1 - 2u_2 + u_3, \ldots, u_{k-1} - 2u_k, u_k)'.$$

Now the first $k - 1$ components of $-u'S^{-2}$ can be chosen arbitrarily, and $\beta \geq 0$ by *SSD*. Hence $-u'(b - a) \geq 0$ only if $U \in V$.

(Necessity): The converse is established as a special case of Lemma 2; cf. Appendix Theorem A2. ■

Corollary: If $E(A) = E(B)$, the utilities need only be concave; the monotonicity restriction can be dropped.[18]

Hadar-Russell [1969] show that $E(A) \geq E(B)$ is necessary for A to dominate B by

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[18]Cf. the discussion in Huang and Litzenberger [1988, Chapter 2].

SSD. For members of an *(n, k)* - family, $E(A) < E(B)$; i.e., the logical contradiction of $E(A) \geq E(B)$, implies $\beta_k < 0$ by (4.2). But since second degree dominance is then not possible, the necessity of $E(A) \geq E(B)$ follows.

5. Third Degree Dominance

By (2.6), *A* stochastically dominates *B* in the third degree if and only if:

$$S^3(b - a) = \alpha^{(3)} \equiv \gamma \geq 0 \tag{5.1}$$

and[19]

$$[E(A) - E(B)] \geq 0; \tag{5.2}$$

cf. (2.5). Since (cf. (2.3))

$$\gamma_{k-1} = \gamma_{k-2} + \beta_{k-1}\ , \gamma_k = \gamma_{k-1} + \beta_k\ ,$$

and since Section 4 established that $\beta_k = \beta_{k-1}$,

$$\gamma_{k-2} - 2\gamma_{k-1} + \gamma_k = 0. \tag{5.3}$$

In this case

$$(b - a) = S^{-3}\gamma \tag{5.4}$$

takes the specific form

$$(b - a) = \tag{5.5}$$
$$(\gamma_0, -3\gamma_0+\gamma_1, 3\gamma_0-3\gamma_1+\gamma_2\ , -\gamma_0+3\gamma_1-3\gamma_2+\gamma_3\ , \ldots, -\gamma_{k-3}+3\gamma_{k-2}-3\gamma_{k-1}+\gamma_k)'.$$

As before, it is useful to take

$$S^{-3}\alpha \equiv \delta_0 + \ldots + \delta_k$$

where now

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[19]Whitmore [1970] shows $E(A) \geq E(B)$ is necessary if *A* is to dominate *B* by TSD. This result is interpreted below.

$$S^{-3}\beta \equiv \begin{vmatrix} \gamma_0 \\ -3\gamma_0 + \gamma_1 \\ 3\gamma_0 - 3\gamma_1 + \gamma_2 \\ 0 \\ \dots \\ \gamma_{k-3} - 3\gamma_{k-2} + 3\gamma_{k-1} - \gamma_k \end{vmatrix} \equiv \begin{vmatrix} \gamma_0 \\ -3\gamma_0 \\ 3\gamma_0 \\ -\gamma_0 \\ \dots \\ 0 \end{vmatrix} + \dots + \begin{vmatrix} 0 \\ 0 \\ \dots \\ -3\gamma_{k-2} \\ 3\gamma_{k-2} \\ -\gamma_{k-2} \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 3\gamma_{k-1} \\ -\gamma_{k-1} \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ -\gamma_k \end{vmatrix} . \qquad (5.6)$$

The relations between densities, which can be studied in exactly the same way as for *FSD* and *SSD*, are mean zero terms for *0, 1, ..., k-3*. More particularly, since

$$\begin{vmatrix} 1 \\ -3 \\ 3 \\ -1 \end{vmatrix} \equiv \begin{vmatrix} 1 \\ -2 \\ 1 \\ 0 \end{vmatrix} + \begin{vmatrix} 0 \\ -1 \\ 1 \\ 0 \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ 1 \\ -1 \end{vmatrix} . \qquad (5.7)$$

the probability redistribution effects of $\gamma_0, \dots, \gamma_{k-3}$ can each be viewed as the sum of a noise term and two shift terms whose combined effect is that of a (mean preserving) noise reducing term. Finally, returning to (5.6) the effects of γ_{k-2}, γ_{k-1}, and γ_k together represent a noise and a shift term with a non-positive conditional mean. (If $E(A) = E(B)$ then $\gamma_k = \gamma_{k-1} = \gamma_{k-2} = 0$, as shown below.) These observations lead to

Theorem 5: *A* dominates *B* by *TSD* if and only if

$$B \simeq A + \Delta,$$

where Δ is equal in distribution to the combined effects of the conditional variables defined by the right hand side of (5.6). The first $k - 2$ variables can be interpreted as combinations of a noise and two shift terms using (5.7) and the methods of Sections 3 and 4. The remaining three variables combine to form a noise and a shift term, the latter with a non-positive conditional mean.

Proof: Both necessity and sufficiency are established as before. ■

Corollary: If $E(A) = E(B)$ then $\beta_{k-1} = \beta_k = 0$, which in turn implies that

$$\gamma_{k-2} = \gamma_{k-1} = \gamma_k,$$

and $\Delta_{k-2} \equiv 0$.

Proof: By the paper's conventions,

$$(k+1)e'(b - a) = (k+1, \ldots, k+1)'(b - a) = 0. \tag{5.8}$$

Moreover, $E(A) = E(B)$ implies

$$(0, 1, \ldots, k)'(b - a) = 0. \tag{5.9}$$

Subtracting (5.9) from (5.8) gives

$$(k+1, k, \ldots, 1)'(b - a) = 0 = \gamma_k - \gamma_{k-1},$$

the last equality following from definition of γ_k and γ_{k-1}. But by equal means and $\beta_k = \beta_{k-1}$, $\gamma_k = \gamma_{k-1}$. It therefore follows that $\gamma_k = \gamma_{k-1} = \gamma_{k-2} = 0$. ■

Table 3 gives examples of the TSD perturbations, with the now customary omission of identically zero variables.

Table 3. Example of Perturbations for *TSD*

Outcomes	a	δ_1	δ_2	b
0	0	0	0	0
1	1/7	1/7	0	2/7
2	2/7	-3/7	1/7	0
3	1/7	3/7	-3/7	1/7
4	1/7	-1/7	3/7	3/7
5	2/7	0	-1/7	1/7

The columns headed δ_1 and δ_2 indicate how γ_1 and γ_2 correspond to a redistribution of (unconditional) probability. In this case, $\gamma = (1/7)(0, 1, 1, 0, 0, 0)'$. The conditional distributions can be written using (5.6) and the methods of previous sections.

The next Theorem is a direct consequence of Lemma 2. Consider $U^{(3)} \equiv R$, the class of non decreasing concave utilities whose third difference is non-negative; i.e.,

$$u_{j+3} - 3u_{j+2} + 3u_{j+1} - u_j \geq 0, \tag{5.8}$$

$j \in J_{k-3}$. Condition (5.7) is the discrete analogue of a necessary condition for decreasing absolute risk aversion; cf. Whitmore [1970]. Then

Theorem 6: A dominates B by *TSD* iff $u'a \geq u'b$ for any $U \in R$.

Proof (Sufficiency): Suppose A dominates B, in which case $\gamma_j \geq 0, j \in J_k$. But

$$-u'(b - a) = -u'S^{-3}\gamma,$$

and

$$-u'S^{-3} = -(u_0 - 3u_1 + 3u_2 - u_3, u_1 - 3u_2 + 3u_3 - u_4, \ldots, u_{k-1} - 3u_k, u_k)'.$$

Now the first $k - 2$ components of $-u'S^{-3}$ can be chosen arbitrarily, and $\gamma \geq 0$ by *TSD*. Hence $-u'(b - a) \geq 0$ only if $U \in R$.

(Necessity): The converse is established as a special case of Lemma 2; cf. Appendix Theorem A3. ■

To show why $E(A) \geq E(B)$ is necessary for *TSD*, consider the restricted case where $E(X) = j; j \in J_k$. Analogous to condition (4a) in Whitmore [1970] it can be verified that

$$S^3_k x = [h(E(X), k) + \sigma^2(X)/2], \tag{5.9}$$

where

$$h(E(X), k) = (k + 2 - E(X))! \;/\; (k - E(X))!2!$$

and $\sigma^2(X)$ is the variance of X. But then

$$S^3_k(b - a) = [h(E(B), k) - h(E(A), k)] + (1/2)[\sigma^2(B) - \sigma^2(A)],$$

and it is clear that $S^3_k(b - a) \geq 0$ cannot generally hold unless $E(A) \geq E(B)$; cf. Whitmore [1970, eqns (4a) and (5a)].

6. Conclusions

For any two comparable members of a given (n, k) - family, this paper shows that ranking by a dominance criterion is equivalent to perturbing the dominating variable, with the type of perturbation depending on the criterion employed. For *FSD*, the perturbation terms' coefficients are given by 1 and -1; for *SSD* by the binomial expansion of $(1 - 1)^2$; and for *TSD* by $(1 - 1)^3$. For *FSD*, the perturbations have non - positive means. For *SSD*, the first $k - 1$ perturbations are noise terms, and the *k'th* perturbation is zero if $E(A) = E(B)$. For *TSD*, the first $k - 2$ perturbations are symmetric noise terms, and the remaining perturbations are zero if $E(A) = E(B)$. Finally, the results generalize immediately to orders of dominance greater than three.

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Appendix. Interpretation of Proofs in the Literature

This Appendix gives, using the paper's notation, the standard proofs in the literature for relations between a dominance criterion and preferences. The text of the paper offers shorter proofs.

Theorem A1: A dominates B by FSD iff $u'a \geq u'b$ for any $U \in M$.

Proof (Sufficiency): By the paper's conventions, A dominates B is equivalent to $\alpha_j \geq 0$, $j \in J_k$. But then

$$u'a - u'b = -u'(b - a) = -u'(S^{-1}\alpha) =$$

$$-(u_0\alpha_0 + u_1(\alpha_1 - \alpha_0) + \ldots + u_{k-1}(\alpha_{k-1} - \alpha_{k-2}) - u_k\alpha_{k-1}) =$$

$$= -(u'S^{-1})\alpha = -((u_0 - u_1)\alpha_1 + \ldots + (u_{k-1} - u_k)\alpha_k) \geq 0$$

by monotonicity of U.

Proof (Necessity): The argument uses the same method of contradiction as in Hadar-Russell [1969]. Suppose A does not dominate B by FSD. Then $\alpha_j < 0$ for some $j \in J_k$. Next, choose $U^* \in M$ such that

$$u_i^* = 0; i = 0, 1, \ldots, j, \text{ and}$$

$$u_i^* = 1; i = j+1, \ldots, k.$$

But then

$$(u^*)'b - (u^*)'a = u^{*\prime}(S^{-1}\alpha) = -\alpha_j > 0,$$

the desired contradiction. ■

Theorem A2: A dominates B by SSD iff $u'a \geq u'b$ for any $U \in V$.

Proof (Sufficiency): By the preceding theorem, A dominates B is equivalent to using the perturbations in Table 2. Consider first the cases $j \in J_{k-2}$, and recall $\beta_j \geq 0$ by SSD. If $\beta_j = 0$, it has no effect and can be ignored. If $\beta_j > 0$, probability is moved from an outcome j of A and distributed equally between the outcomes $j - 1$ and $j + 1$ of B. Such transformations cannot increase a concave utility.

Now consider the remaining case, $j = k-1$. If $\beta_{k-1} = \beta_k > 0$, the net effect is to reduce a_k and increase a_{k-1}. This transformation cannot increase expected utility, since any $U \in V$ is non-decreasing.

Hence both cases imply $u'a \geq u'b$.

Proof (Necessity): The converse again uses the Hadar Russell [1969] method of contradiction. Suppose A does not dominate B by SSD. Then $\beta_j < 0$ for some $j \in J_{k-1}$. Choose $U^* \in V$ such that

$$u^*_{i+1} - u^*_i = 1;\ i = 0, 1, \ldots, j, \text{ and}$$

$$u^*_{i+1} - u^*_i = 0;\ i = j+1, \ldots, k-1.$$

But then

$$u^*_{j+2} - 2u^*_{j+1} + u^*_j = -1$$

while all other perturbations create utility differences of zero. The desired contradiction follows immediately. ■

Theorem A3: A dominates B by TSD iff $u'a \geq u'b$ for any $U \in R$.

Proof (Sufficiency): Let $U \in R$. By the preceding theorem, A dominates B is equivalent to using the perturbations in Table 3 with $\gamma_j \geq 0, j \in J_{k-2}$. First, consider any $j \in J_{k-3}$ for which $\gamma_j > 0$. The associated perturbation is one for which

i) probability is moved from an outcome j of A and distributed equally between the outcomes $j - 1$ and $j + 1$ of B; and

ii) the same amount of probability as in i) is moved equally from the outcomes j and $j + 2$ of A to the outcome $j + 1$ of B.

The first such perturbation reduces any concave utility, the second increases it. But, the increase is less than the decrease by decreasing marginal utility. The combination means $u'a \geq u'b$.

Finally, consider $j = k - 2$. If $\gamma_{k-2} > 0$, probability is

i) shifted from outcome k of A to outcome $k - 1$ of B; and

ii) shifted according to the condition $\gamma_{k-2} - 2\gamma_{k-1} + \gamma_k = 0$

Both i) and ii) imply decreases in utility, and in this case also $u'a \geq u'b$.

Proof (Necessity): The converse again uses the Hadar Russell [1969] method of contradiction. Suppose A does not dominate B by TSD, and choose $U^* \in R$ such that its second differences satisfy:

$$u^*_i - 2u^*_{i+1} + u^*_{i+2} = -1;\ i = 0, 2, \ldots, j,$$

and

$$u^*_i - 2u^*_{i+1} + u^*_{i+2} = 0;\ i = j+1, \ldots, k\text{-}2.$$

By hypothesis, $\gamma_j < 0$ for some $j \in J_k$.

First, consider the cases $j \in J_{k-3}$. Then

$$u^*_i - 3u^*_{i+1} + 3u^*_{i+2} - u^*_{i+3} = 1;\ i = j;$$

$$u^*_i - 3u^*_{i+1} + 3u^*_{i+2} - u^*_{i+3} = 0;\ i \neq j;\ i \in J_{k-3};$$

from which it follows that $\gamma_j < 0$ increases the utility of B relative to A.

Finally, consider $j = k - 2$. If $\gamma_{k-2} < 0$ utility increases result from a positive shift term and a reduction of dispersion.

The desired contradiction follows in either case. ■