Queen's Economics Department Working Paper No. 867

# The Existence of Equilibrium and the Objective Function of the Firm

David Kelsey     Frank Milne

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

9-1992

Discussion Paper #867

# The Existence of Equilibrium and the Objective Function of the Firm

by

David Kelsey
University of Birmingham

Frank Milne
Queen's University

September 1992

# THE EXISTENCE OF EQUILIBRIUM AND THE OBJECTIVE FUNCTION OF THE FIRM*

BY

DAVID KELSEY

DEPARTMENT OF ECONOMICS,
THE UNIVERSITY OF BIRMINGHAM,
BIRMINGHAM B15 2TT, UK.

and

FRANK MILNE

DEPARTMENT OF ECONOMICS,
QUEENS UNIVERSITY,
KINGSTON, ONTARIO K7L 3N6,
CANADA.

22nd September 1992

**Abstract** We consider an economy in which firms' decisions are made by a collective decision of the shareholders. The main result shows that the simultaneous existence of an exchange equilibrium in the market for shares and a voting equilibrium in the internal decisions of firms. We present our results in a general framework, with a measure space of agents. Our framework covers the cases of incomplete markets and externalities between firms and shareholders. We show that a voting rule due to Kramer is a special case.

* We would like to thank Manimay Sengupta and the participants in a seminar at the University of Essex for comments and discussion.

**Keywords** Shareholder voting, incomplete markets, sophisticated voting, objective function of the firm.

**JEL Classification** D52,D70,L20.

Sadanand, A.B. and J.M. Williamson, (1991), "Equilibrium in a Stock Market with Shareholder Voting", *International Economic Review,* 32, 1-35.

Sen, A.K., (1970), *Collective Choice and Social Welfare,* North Holland, Amsterdam.

Sen, A.K., (1977), "Social Choice Theory: A Re-examination" *Econometrica,* 45, 217-245.



## 1 INTRODUCTION

This paper proves the existence of equilibrium when firms' production decisions are made by a collective choice of shareholders. In addition to the usual simultaneity problems in general equilibrium, the attractiveness of shares to individuals and hence the demand for shares depends upon production decisions, while firms' production decisions depend upon the outcome of shareholder voting, which is in turn determined by the demand for shares. Thus we need to look for a simultaneous exchange equilibrium in markets and a voting equilibrium in firms' internal decisions.

### 1.1 Background

When markets are complete and economic agents are price-takers, conflict between the interests of shareholders cannot arise. The Fisher Separation Theorem (see for instance Milne (1974)) shows that all shareholders will desire the firm to maximise its value with respect to any given contingent price system. With a complete set of contingent markets, individuals can allocate consumption across states of nature in any desired way. Hence the only effect of changes in the firm's production decision is to induce parallel shifts in shareholders' budget constraints. Naturally all shareholders desire their budget sets to be as large as possible, hence no disagreement arises.

This argument does not hold in general when markets are incomplete. A change in the firm's production plan causes shareholders' wealth to be reallocated between states of nature. In the absence of a complete set of insurance markets this would force individuals to reallocate their consumption across states of nature. If different shareholders have different subjective probabilities or different risk attitudes then they would have different preferences over production plans. In other words, when markets are incomplete a change in the production plan can change the *slope* as well as the position of a shareholder's budget line, neither budget set may be included in the other. Milne (1981) has shown that this disagreement may be so severe that the Arrow Impossibility Theorem can be proved, in which case the firm would have no well-defined objective function. Likewise when there are externalities between the firm and its shareholders, it is not in general possible for the shareholders to offset the effect of a change in the firm's production plan by trading in the market.

Thus production decisions are made by some collective choice rule which aggregates the preferences of shareholders and possibly other economic agents as well. This is important, since the social choice literature has demonstrated that, in general, we would not expect a group preference to be either complete or transitive. Sen (1977) has argued that social choice has considered two distinct problems, making a social welfare judgement and making a collective decision. He argues that welfare judgements are required to be transitive, but the paradoxes of social choice may be avoided in this case by using cardinal and interpersonally comparable utility. Collective decision procedures must be based on ordinal utilities, however it is possible to make a decision, (ie, pick a best element from a given set of alternatives) without having a transitive ordering of all alternatives. Since the objective function of the firm is a collective decision problem we shall adopt Sen's suggestion of relaxing transitivity.



REFERENCES

Caplin, A. and B. Nalebuff, (1991), "Aggregation and Social Choice: A Mean Voter Theorem", *Econometrica*, 59, 1-24

DeMarzo P.M., (1990), "Majority Voting and Corporate Control: The Rule of the Dominant Shareholder", Northwestern University working paper no. 86.

Dreze, Jaques, (1974), "Investment Under Private Ownership: Optimality, Equilibrium and Stability", in Jaques Dreze (ed), *Allocation Under Uncertainty*, Macmillan, New York.

Dreze, Jaques, (1985), "Uncertainty and the Firm in General Equilibrium Theory", *Economic Journal* 95, 1-20.

Dreze, Jaques, (1989), *Labour Management Contracts and Capital Markets,* Blackwell, Oxford.

Duffie, D. and W. Shafer, (1985) "Equilibrium in Incomplete Markets I", *Journal of Mathematical Economics*, 14, 285-300.

Grossman, S. and O. Hart, (1979), "A Theory of Competitive Equilibrium in Stock Market Economies", *Econometrica*,47,293-330.

Hart, O., (1975), "On the Optimality of Equilibrium When Markets are Incomplete", *Journal of Economic Theory,* 11,418-443.

Kahn M.A. and R. Vohra, (1984), "Equilibrium in Abstract Economies without Ordered Preferences and with a Measure Space of Agents", *Journal of Mathematical Economics*, 13, 133-142.

Kramer, G.H., (1972), "Sophisticated Voting over Multidimensional Choice Spaces", *Journal of Mathematical Sociology*, 2, 165-180.

McKelvey,R.D., (1979), "General Conditions for Global Intransitivities in Formal Voting Models", *Econometrica,* 47, 1085-1112,

Milne, F., (1974), "Corporate Investment and Finance Theory in General Equilibrium", *Economic Record,* 511-533.

Milne, F., (1981), "The Firm's Objective Function as Collective Choice Problem", *Public Choice,* 37, 473-486.

Plott, C.R., (1967), "A Notion of Equilibrium and its Possibility Under Majority Rule", *American Economic Review*, 57, 787-806.

Radner, R., (1972), "Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets", *Econometrica*, 40, 289-303.

Roberts, K.W.S., (1989), "The Theory of Union Behaviour: Labour Hoarding and Endogenous Hysteresis", London School of Economics Discussion Paper No. TE/89/209.

production plan which the first individual regards as inferior to the status quo.

### Notes on the text

1. A complete measure space satisfies the property that every subset of a set of measure zero is measurable.



## 1.2 Brief Description of Model

The model has one physical commodity. There are two time periods. In the second period there are *S* possible states of nature. Firms can produce *S* contingent commodities. The firm has to commit itself to a production plan in period 0. The firm's production plan is constrained to lie in a production possibilities set which is assumed to be compact and convex. We shall model the firm's decision by an abstract set of preferences. These preferences are not assumed to be complete or transitive but merely to satisfy weak continuity and convexity assumptions. Individuals have fixed endowments of shares in the firms and the consumption good, which is not storable. These they may exchange in a market in the first period. In the second period the true state of nature is revealed. Firms produce according to their previously decided production plans and allocate their output to individuals in proportion to their shareholdings. There is no trade in the second period.

We shall look for an equilibrium in the following sense. Firstly there is a competitive equilibrium in the market for shares and first period consumption. Each consumer maximises his or her utility given the prices prevailing and there is no excess demand, both for goods and shares. Secondly no firm has a production plan, which is preferred by the firm's collective choice rule to the status quo.

One might envisage this equilibrium could arise as result of the following kind of process. There is an auctioneer who announces prices. People then respond by stating demands at those prices. Assuming that everybody has their *desired* holdings of shares, a provisional vote is conducted on the production plans of the firms. (Note that the sum of desired shareholdings over all individuals may differ from the actual number of shares which the firm has issued.) If for some good supply is not equal to demand or some firm changes its production plan the auctioneer then announces the new production plans of the firms and a new set of prices. The individuals submit new demands for the shares and the consumption good. The auctioneer continues adjusting prices and the production plans of the firms continue to be revised by preliminary votes until we have reached a situation where supply equals demand for all goods and no firm's production plan can be changed by its decision rule. The agreed trades then take place at the equilibrium prices.

After proving our general existence result, we demonstrate that some familiar decision rules, are special cases. As is well known majority voting can be intransitive. Hence a voting equilibrium may fail to exist. The previous literature has resolved this problem in the following ways. The transitive closure of the majority preference is used, restrictions are placed on the profiles of individuals' preferences, a group of individuals is given veto power or restrictions are placed on the directions in which changes can be made (Kramer (1972)). All of these approaches could potentially be applied to shareholder voting.

However, the first is not suitable since McKelvey (1979) has shown the top-cycle of majority rule is likely to include the whole space, hence the transitive closure of majority rule will declare too many alternatives to be indifferent. In particular a point may be a convex combination of points collectively preferred to it. This non-convexity of preferences is a barrier to proving the existence of equilibrium.

The second approach is also difficult to apply to the shareholder problem. As Sen (1970) (chapter 10*) states the restrictions which must be placed on preferences are very strong.



particular kinds of labour to the firm, (eg. top-level management).

In the present paper we assume that individuals vote in line with their true preferences. (Individuals cannot make themselves better off by misrepresenting their preferences under the Kramer voting rule provided preferences are additively separable between each issue voted on. For details see Kramer (1972).) There is a second kind of strategic behaviour which we do not consider. We assume that individuals do not purchase shares in order to influence the firms' decisions, but only use shares to transfer consumption between time periods and states of nature. For consumers with measure zero such behaviour is not possible. We plan future research to investigate this. It seems plausible that our basic existence result could be extended to cover strategic behaviour. The competitive case provides a foundation for further study of these phenomena.

Roberts (1989) studies voting decisions by the members of a trades union. He discovers the possibility of a form of strategic voting. Suppose that an individual believes that a small wage rise would be desirable since although total employment may fall his or her own job was not at risk. The individual may not vote for a raise, since it may cause a new majority coalition to emerge which will vote for further wage rises. It is possible that this process could continue until the first individual's own job is at risk. Similar phenomena could be found within our model. An individual may not vote for the firm to adopt his or her most preferred production plan since to do so may cause changes in the pattern of share ownership. With the new ownership there would be a new majority among the shareholders. This process could, eventually, lead to a price-taking behaviour, honest voting and the assumption that individuals take production decisions as given when deciding share purchases, are best justified when each individual has measure zero. The previous literature has almost exclusively focused on incomplete markets, while we consider both externalities and incomplete markets. In addition we prove existence with a large class of procedures for the firm's internal decisions while previously only specific rules have been considered. A general model is advantageous, since we have little detailed analysis of firms' internal decisions.

**4.2 Possible Extensions**

It would be possible to extend our model to an economy with many goods along the lines of Radner (1972), who proceeds by placing bounds on the trades of agents. The disadvantage of this is that the bounds are arbitrary and the equilibrium which results may depend on them. If no such bounds are imposed it is not possible to prove existence for all economies since the budget correspondence may fail to be continuous (see Hart (1975)). It is likely, however, that existence could be proved for the generic case, using differential topology, as in Duffie and Shafer (1985).

An advantage of including more goods in the model is that it would be possible to consider decision rules for the firm which were based on other economic variables in addition to shareholdings. In particular if our model was extended to identify labour as a separate good it would be possible to include labour managed firms. (Dreze (1989) considers firms which are jointly managed by workers and shareholders.) If we model the firm's decisions by majority voting constrained by a veto to "the board of directors", in a multi-good model it would be possible to give board membership to those who supply



Unless a good argument can be found to the contrary it does not seem desirable to impose such restrictions in the shareholder's problem. An example of this approach can be found in Caplin and Nalebuff (1991), who show that an equilibrium will exist under 64% majority rule provided certain restrictions are imposed, both on the nature of preferences and on their distribution. So far we have not been able to apply this result to decision-making within firms. Caplin and Nalebuff require all voters to have the same utility function, apart from a term which is linear in a characteristic of the individual. However a shareholder's preferences will depend on the size of his or her shareholding. This relation is unlikely to be linear except in special cases. We hope in future research to be able to relax these restrictions, so that this voting system may be applied to firms' decisions. The other possibilities may all be applied to the shareholder problem. We shall consider each of them in turn.

Kramer (1972) has demonstrated the existence of equilibrium in a voting game, in which a vector in $\mathbb{R}^n$ is determined by voting on each coordinate in turn. This is in effect putting a restriction on the directions in which changes may be made. He shows that provided preferences are additively separable, the game so defined has an iterative dominance equilibrium. In section 3, we show that if firms' decisions are made by Kramer's voting rule then a simultaneous voting and exchange equilibrium will exist. We also demonstrate the existence of equilibrium when a subset of the shareholders have vetoes.

It should be emphasised that while the examples we use are mainly concerned with shareholder voting, there is no need in our general model to restrict attention to voting, instead an abstract collective choice rule or mechanism could be used. The participants in such a mechanism need not be restricted to shareholders. They could be other sets of individuals which are determined by economic variables. Examples would be all individuals who supply labour to the firm or individuals who supply particular kinds of labour to the firm, (eg. executives).

A possible objection to our framework is to argue that in practice few decisions of companies are made by a majority vote of shareholders. There are two possible responses to this. Firstly it could be the case that the decision-making procedures of large corporations are, in principle, democratic, however, in practice, small coalitions are able to exercise effective control by strategic behaviour. This is compatible with our model and indeed the model was developed to investigate such problems. Secondly it may be the case that firms' decisions are not, even in principle, made by a majority vote. It is likely that our existence result could be extended to such circumstances. The proofs we give do not rely on specific properties of majority rule. Any other decision procedure with suitable continuity properties could be substituted.

### 1.3 Outline of the Paper

In section 2 of the present paper we present a general model of a economy which allows for both incomplete markets and externalities in production. Firms' decisions are represented by a set of preferences, which are derived from an abstract collective choice rule. In section 3 of the paper we consider some decision rules for the firm which satisfy the assumptions of our general model. In section 4 we conclude by relating our research to other papers in this area and by outlining some possible extensions.



preferences cancel in pairs. The largest shareholder dominates because the other voters' preferences cancel. These symmetry conditions are clearly non-generic. De Marzo provides an example in which equilibrium can be shown to exist and to be robust to small changes in parameters. This shows that non-existence is not generic. However it does not establish the existence of equilibrium in any great generality.

Sadanand and Williamson (1991) prove the existence of a situation which is a voting equilibrium (in the sense of Kramer(1972)) in firm's decisions and such that it is not possible to bring about a Pareto improvement by reallocating shares and first period consumption. This is not a competitive equilibrium in the usual sense. While the second fundamental theorem guarantees that any Pareto Optimum will be a competitive equilibrium for *some* initial endowment, a given Pareto Optimum will not be a competitive equilibrium for an arbitrary endowment unless lump-sum transfers between shareholders are made. Such transfers are contrary to the spirit of a non-cooperative model. In the present paper we prove the existence of a state of the economy which is simultaneously a competitive equilibrium in markets and a voting equilibrium in firms' internal decisions, for arbitrary initial endowments. Put more strongly Sadanand and Williamson show that *there exists* an initial endowment which will lead to an equilibrium, while we show that such an equilibrium exists *for all* endowments.

Dreze considers a modified version of majority rule where some players have a veto and proves existence of a simultaneous voting and exchange equilibrium. We extend Dreze and the other literature in the following ways. Our results apply to continuous as well as discrete distributions of individuals. This is important since many of the implicit assumptions such as

<u>Definition 3.5</u> A *Kramer voting and exchange equilibrium* is a state $z$ of the economy such that no firm wishes to change its production plan in an allowable direction,

$P_s^f(z) \bigcap \beta_s^f(z) = \emptyset$, $1 \le s \le S$, $1 \le f \le F$ all consumers are maximising their utility subject to their budget constraints,

$\int \theta^{*\alpha f}\, d\alpha \le 1$ for $0 \le f \le F$, and $\int x^\alpha d\alpha \le \int \bar{x}^\alpha d\alpha$ .

<u>Corollary 3.2</u> A Kramer voting equilibrium exists.

<u>Proof</u> This can be shown by adapting Theorem 2.1 along the lines indicated in the proof of Theorem 2.2 and using Propositions 3.2 and 3.3.

■

It is possible to generalise the analysis of this subsection since it would be possible to define a similar voting rule for any other (not necessarily linear) coordinate system for $\mathbb{R}^S$. However, it would be necessary to ensure that preferences are quasi-concave in each coordinate.

**4 CONCLUSION**

**4.1 Related Literature**

Models of shareholder voting in general equilibrium have been studied previously by Dreze (1985),(1989), De Marzo (1990) and Sadanand and Williamson (1991).

De Marzo (1990) has shown that in an equilibrium the firm's decisions will coincide with the preferences of the dominant shareholder, the individual with the largest shareholding. He does not prove existence of equilibrium. If, equilibrium does not typically exist the dominant shareholder property may fail to be robust. De Marzo's proof relies upon Plott's (1967) symmetry conditions which imply that voters'



**2. EQUILIBRIUM IN AN ABSTRACT ECONOMY**

In this section we present a general model, which allows simultaneously for the possibility of incomplete markets and externalities between firms and individuals. We shall model firms' decisions as being made by an abstract set of preferences. In section 3 we shall give examples of decision rules for firms, which fit this framework.

**Market Structure** There are two time periods $t = 0,1$. There are $S$ states of nature $1 \le s \le S$. In period 0, the state of nature is unknown. The true state is revealed before the beginning of period 1. There is one physical commodity which is non-storable. There are $F$ firms $1 \le f \le F$. At time $t = 0$ there are markets in the physical commodity and the shares of the firms. Let $q^f$, $1 \le f \le F$ denote the price of shares in firm $f$, and $q^o$ the price of the physical commodity. Let $q$ denote the vector $\langle q^o, q^1, \ldots, q^F\rangle \in \mathbb{R}^{F+1}$ We shall normalise prices so that they lie in the unit simplex $\Delta \sqsubseteq \mathbb{R}^{F+1}$.

**Consumers** There is a complete[1] finite measure space $\langle A, \Omega, \mu\rangle$ of consumers. We shall denote a generic consumer by $\alpha \in A$. It is assumed that $L^1(\mu)$ is separable. Note that this includes the possibilities of a discrete set of consumers, a continuum of atomnless consumers or some combination of the two. This gives useful generality when modelling the stock market, since one could think of private investors as a continuum of agents while institutional investors could be modelled as point masses.

Individual $\alpha$ is assumed to have a utility function of the form $u\langle \alpha, x_o^\alpha, x_1^\alpha, \ldots, x_S^\alpha, y^1, \ldots, y^F\rangle = u^\alpha\langle x^\alpha, y\rangle$ ,where $x_o^\alpha$ (resp. $x_s^\alpha$ ) denotes the consumption of individual $\alpha$ in period 0 (resp. period 1 state s), $x^\alpha$ denotes the vector $\langle x_o^\alpha, \ldots, x_S^\alpha\rangle$ , $y^f$ denotes the production plan of firm $f$ and $y$ denotes the vector $\langle y^1,\ldots,y^F\rangle$. The presence of the firms' production plans in the utility function allows for the possibility of externalities between the firm and the individuals.

<u>Assumption 2.1</u> The utility function will be assumed to satisfy the following properties;

a. The utility function $u$ is measurable,

b. $u$ is continuous in $x$ and $y$,

c. $u$ is increasing in $x$ and strictly increasing in $x_o^\alpha$

d. $u$ is strictly quasi-concave in $x$.

Individual $\alpha$ is assumed to have a consumption set $X^\alpha$, which is a compact and convex subset of $\mathbb{R}^{(S+1)}$. Let $X = \times_{\alpha \in A} X^\alpha$. Individual $\alpha$ is assumed to have endowments $\overline{x}_s^\alpha$ and $\overline{\theta}^{\alpha f}$ of consumption good in state $s$ and of shares in firm $f$, respectively. Both endowments are assumed to be measurable functions of $\alpha$. The endowment of the consumption good is assumed to lie in the interior of the consumption set. Individual $\alpha$'s demand for shares in firm $f$ is constrained to lie in the set $\Theta^\alpha = \{\theta^\alpha : k\overline{\theta}^\alpha \le \theta^\alpha \le K\}$, where $0 \le k \le \frac{1}{2}$ and $K$ is some constant sufficiently large to ensure that $\Theta^\alpha$ is non-empty for almost all $\alpha$. Let $\Theta = \times_{\alpha \in A} \Theta^\alpha$. Individual $\alpha$'s budget constraint is, $q_o x_o^\alpha + \sum_{f=1}^{F} q_f\, \theta^{\alpha f} \le q_o \overline{x}_o^\alpha + \sum_{f=1}^{F} q_f\, \overline{\theta}^{\alpha f}$.

**Firms** There are $F$ firms, $1 \le f \le F$. Each firm $f$, controls a vector $y^f \in \mathbb{R}^{S+1}$. The vector $y^f$ is constrained to lie in a compact convex subset $Y^f$ of $\mathbb{R}^{S+1}$. Define $Y = \times_{f=1}^{F} Y^f$.



Suppose that $\alpha$ is such that $\zeta^\alpha \ge 0$, then there exists $\lambda^\alpha : 0 < \lambda^\alpha < 1$ such that $y^f = \lambda^\alpha y_o + (1-\lambda^\alpha) y^\alpha$. By strict quasi-concavity we know that $\alpha$ prefers $y^f$ to $y_o$. Since $y_o \in P^{fs}(z)$ we may deduce that strictly less than half of the shares are owned by individuals with $\zeta^\alpha \ge 0$. However by similar reasoning we can also show that strictly less than half of the shares in firm $f$ are owned by individuals with $\zeta^\alpha \le 0$. This is a contradiction. The result follows.

■

<u>Remark</u> The conditions given in sub-sub-section 3.1.2 are sufficient for preferences to be strictly quasi-concave in $y_s^f$.

<u>Proposition 3.3</u> The graph $\Gamma$ of $P_s^f(z)$ is an open set.

<u>Proof</u> Let $\neg\Gamma$ denote the complement of $\Gamma$. We shall show that $\neg\Gamma$ is closed. Let $\langle z_n, \hat{y}_{sn}^f\rangle$ be a sequence of points from $\neg\Gamma$, which converges to a limit $\langle z, \hat{y}_s^f\rangle$. Define,

$J_n = \{\alpha : u^\alpha(z_n^{-fs}, \hat{y}_{sn}^f) \le u^\alpha(z_n)\}$ and $J = \{\alpha : u^\alpha(z^{-fs}, \hat{y}_s^f) < u^\alpha(z)\}$.

Let $\chi_n$ (resp. $\chi$) denote the indicator function of the set $J_n$ (resp. $J$). By continuity of $u$ if $\alpha \in J$, then for all sufficiently large $n$, $\alpha \in J_n$. Hence $\liminf \chi_n \ge \chi$. By Fatou's lemma,

$$\int \theta^{\alpha f} \chi(\alpha)\, dv(\alpha) \le \int \liminf \theta_n^{\alpha f} \chi_n(\alpha)\, dv(\alpha) \le \liminf \int \theta_n^{\alpha f} \chi_n(\alpha)\, dv(\alpha) \le \tfrac{1}{2}. \qquad (3.1)$$

The last inequality follows from the fact that $\langle z_n, \hat{y}_{sn}^f\rangle \in \neg\Gamma$. Equation (3.1) implies that $\langle \overline{z}, \overline{y}^f\rangle$ is an element of $\neg\Gamma$. The result follows.

■

$P^f_s(z) = \{\hat{y}^f \in \beta^f_s(y^f) : \nu(\Pi(\hat{y}^f_s, z)) > \tfrac{1}{2}\}$ where $\nu$ denotes the measure on A defined by $\nu(B) = \int_B \theta^\alpha d\alpha / \int_A \theta^\alpha d\alpha$ .

<u>Definition 3.4</u> A state $z$ of the economy is a *Kramer voting equilibrium* for firm $f$ if $P^f_s(z) \bigcap \beta^f_s(z) = \emptyset$ , for $1 \leq s \leq S$.

The Kramer voting rule is not defined when almost all individuals have zero demand for the shares. As a first step we can ensure this does not occur by placing lower bounds on shareholdings in the sets of allowable shareholdings $\Theta^\alpha$ . This assumption could be dispensed with by considering a sequence of economies such that this restriction does not hold in the limit. (For details of this argument see Dreze (1989) p.129).

<u>Proposition 3.2</u> Assume that all individuals induced preferences over $y^f_s$ are strictly quasi-concave. If $z = \langle x, y, \theta, q \rangle$ then $y^f_s \notin co(P^f_s(z))$ .

<u>Proof</u> By continuity and strict quasi-concavity of the utility function each individual $\alpha$ has a unique optimal production plan $y^\alpha \in \beta^f_s(y^f)$ . By definition of $\beta^f_s(y^f)$ we may write $y^\alpha = y^f + \zeta^\alpha e_s + \xi^\alpha e_o$ where $e_s$ (resp. $e_o$) denotes the unit vector in direction $s$ (resp. o).
Suppose, if possible, that the result is false. Then there exist $y_o, y_1 \in \beta^f_s(y^f)$ and $\lambda : 0 < \lambda < 1$ such that $y^f = \lambda y_1 + (1-\lambda) y_o$ . We may write $y_j = y^f + \zeta_j e_s + \xi_j e_o$ for $j = 0,1$. Without loss of generality we may assume that $\zeta_o < 0$, $\zeta_1 > 0$.



<u>Definition 2.1</u> We define a *state of the economy* to be a four-tuple $z = \langle x, y, \theta, q \rangle$ , where $x \in X$, $y \in Y$, $\theta \in \Theta$, and $q \in \Delta$.

Note, the term "state" is being used in two distinct ways, to denote a state of nature and to denote a state of the economy. This should not cause any confusion.

Firm $f$ has to decide the value of $y^f$. We shall assume that this decision can be described by a correspondence $P^f(z) : Z \rightarrow Y^f$ , which will be assumed to have open graph and to satisfy $y^f \notin co(P^f(z))$ where $z = \langle x, y, \theta, q \rangle$.

<u>Definition 2.2</u> A *Simultaneous Voting and Exchange Equilibrium* is a state of the economy $z^* = \langle x^*, y^*, \theta^*, q^* \rangle$ such that, $y^* \notin P^f(z^*)$, for $1 \leq f \leq F$, for almost all $\alpha \in A$, $x^*$, $\theta^*$ maximises $u^\alpha$ subject to individual $\alpha$'s budget constraint, $\int \theta^{*\alpha f} d\alpha \leq 1$ for $1 \leq f \leq F$, and $\int x^\alpha d\alpha \leq \int w^\alpha d\alpha$ .

To prove the existence of equilibrium we shall use the following result on the existence of equilibrium in an abstract economy, which is proved in Kahn-Vohra (1984).

<u>Theorem 2.1</u> (Kahn-Vohra) Let an abstract economy satisfy the following assumptions.

1. $\langle W, \Omega, \upsilon \rangle$ is a complete finite measure space such that $L^1(\nu)$ is separable.

2. $X$ is an integrably bounded measurable correspondence such that for all $w \in W$, $X(w)$ is nonempty, convex and compact.

3. $B$ is a correspondence such that
   a. for all $x$ in $L^1(\mu, X(.))$ , the graph of $B(.,x)$ belongs to $\Omega \times \mathcal{B}(\mathbb{R}^l)$,
   b. for all $w \in W$ and for all $x$ in $L^1(\mu, X(.))$, $B(w,x)$ is a nonempty closed and convex subset of $X(w)$.

c. for all $w \in W$, $B(w,.)$ is a continuous correspondence.

4. $P$ is a correspondence such that

a. the graph of $P(..)$ belongs to $\Omega \times \mathfrak{B}(L^1\langle\mu, X(.)\rangle) \times \mathfrak{B}(\mathbb{R}^l)$

b. $\forall w \in W$, the graph of $P(w,.)$ is open in the set $X(w) \times L^1(\mu, X(.))$,

c. for almost all $w \in W$, for all $x \in L^1(\mu, X(.))$, $x(w) \notin co(P(w,x))$,

then there exists a Nash equilibrium for the abstract economy, ie there exists $x^* \in X$ such that for almost all $w \in W$, $x^* \in B(w,x^*)$ and $B(w,x^*) \cap P(w,x^*) = \emptyset$.

<u>Theorem 2.2</u> Under the above assumptions a simultaneous voting and exchange equilibrium exists.

<u>Proof</u> We shall add an additional agent, agent 0, who will play the role of the auctioneer. The preferences of agent 0 are defined as follows;

$$P^o(z) = \{q' \in \Delta : (q'_o - q_o)\int (x^\alpha_o - \overline{x}^\alpha_o)\, d\alpha + \sum_{f=1}^{F} (q'_f - q_f)\left(\int \theta^{\alpha f} d\alpha - 1\right) > 0\}.$$

Define the preferred set, $P^\alpha(z)$ of individual $\alpha$ by $P^\alpha(z) = \{\hat{x}^\alpha \in X^\alpha : u^\alpha(\hat{x}^\alpha, y) > u^\alpha(x^\alpha, y)\}$, where $z = \langle x, y, \theta, q\rangle$. Let $\langle A^*, \Omega^*, \mu^*\rangle$ be the measure space derived from $\langle A, \Omega, \mu\rangle$ by adding $F+1$ point masses corresponding to the auctioneer and the $F$ firms. Each of these point masses is given measure 1. By the Kahn-Vohra Theorem there exists a Nash equilibrium $z^*$.

The correspondence $B$ is interpreted as the budget set of an individual, the price simplex for the auctioneer or the production set of a firm. In the case of firms, if it is



we hope to be able to relax it in future research. Many firms have a divisional structure, which implies that in any single decision it is only possible to change a few parameters. Although a division would not typically be responsible for deciding output in a single state of nature, it is possible that our analysis could be extended to the case where firms' decisions are made along divisional lines.

The above restrictions on changes in production can result in the firm's strategy set failing to satisfy the convexity assumption of the Kahn-Vohra Theorem. We can overcome this problem by modelling the firm as having $S$ preferences one for each direction of allowable change. As we shall show below each of these "directional preferences" have a strategy set which satisfies the assumptions of the Kahn-Vohra Theorem.

<u>Definition 3.2</u> Define a correspondence $\beta^f_s : Y^f \to Y^f$ by $\beta^f_s(y^f) = \{\tilde{y}^f \in Y^f : \tilde{y}^f_\sigma = y^f_\sigma, 1 \le \sigma \le S, \sigma \ne s\}$, ie. $\beta^f_s(y^f)$ is the set of feasible production plans of firm f, which only differ from $y^f$ in period 0 and period 1, state s.

The set $\beta^f_s(y^f)$ is the counterpart of the correspondence $B(.,.)$ for the preferences of firm $f$ in direction $s$ in the Kahn-Vohra framework. It inherits the properties of being closed and convex from the set $Y^f$.

<u>Definition 3.3</u> The *s-preference* of firm $f$ is a correspondence $P^f_s(z) : Z \to \beta^f_s(y^f)$ defined as follows. Let

$\Pi(\hat{y}^f_s, z) = \{\alpha \in A : u^\alpha(z^{-f}, \hat{y}^f_s) > u^\alpha(z)\}$ and individuals with positive shareholdings. This criterion has been studied previously by Dreze (1974) and Grossman and Hart (1979).

### 3.1.2 Sufficient Conditions for Strict Quasi-Concavity

The above analysis relies upon the assumption that at least one of the individuals with a veto has preferences which are strictly quasi-concave in $y^f$. If there are no externalities between firms and individuals this would be implied by strict quasi-concavity of the utility function. In the presence of externalities induced preferences will be strictly quasi-concave in $y^f$ provided that the utility function takes the additively separable form $u^\alpha(x^\alpha, y) = v^\alpha(x^\alpha) + w^\alpha(y)$, where $v^\alpha$ and $w^\alpha$ are strictly concave.

### 3.2 The Kramer Voting Rule

The Kramer voting rule enables a group of individuals to choose a vector in $\mathbb{R}^n$ by voting over each component in turn. The outcome of the Kramer voting rule is independent of the order in which decisions are taken, however, it does depend upon the choice of bases for the coordinate system of $\mathbb{R}^n$.

Sadanand and Williamson (1991) have analysed the existence of equilibrium when firm's decisions are made by this rule. They assume that at each stage in the voting process the output of one firm in one state of nature is determined. We shall continue to make this assumption. One implication of this is that it is not possible to make coordinated changes in the production plans of more than one firm. This seems reasonable and is in keeping with the spirit of non-co-operative models of firms. A second implication of this assumption is that it is not possible for a single firm to simultaneously introduce changes in more than one state of nature. This implication seems less easy to defend and



possible that outputs in some states are negative, then extra restrictions must be placed on the choice of production plan to ensure that no shareholder is bankrupt in any state of nature. The correspondence $B(f,.)$ will still be closed and convex with these restrictions. For details see Sadanand and Williamson (1991) p.14.

The Kahn-Vohra theorem implies that in state $z^*$, all firms are maximising their preferences and almost all individuals are maximising utility. It is now sufficient to prove that excess demand is non-positive. Since individuals' utility is strictly increasing in first period consumption its price will be strictly positive. Therefore all individuals' budget constraints will hold with equality. Substituting from the budget constraint into the auctioneer's preferences we obtain,

$$P^o(z) = \{q' \in \Delta : q'_o \int (x^\alpha_o - \overline{x}^\alpha_o)\, d\alpha + \sum_{f=1}^{F} q'_f \left(\int \theta^{\alpha f} d\alpha - 1\right) > 0\}.$$

Therefore $P^o(x) \cap \Delta = \emptyset$ implies that $\int (x^\alpha_o - \overline{x}^\alpha_o)\, d\alpha \le 0$, and $\int \theta^{\alpha f} d\alpha \le 1$, $1 \le f \le F$. The result follows.

■

## 3 THE FIRMS' DECISION RULE

In this section we give some examples of decision rules for the firm which satisfy the assumptions of our general model.

### 3.1 Veto Based Decision Procedures

Suppose that there is a group of individuals $G^f(z)$ who have veto power over the decisions of firm $f$. A veto implies that the production plan of firm $f$ cannot be changed if any individual $g \in G^f(z)$ objects.

Definition 3.1 Define the *veto correspondence* $V^f(z): Z \to Y^f$ by $V^f(z) = \{\hat{y}^f \in Y^f : \forall g \in G^f(z),\ u^g(z^{-f}, \hat{y}^f) > u^g(z)\}$, where $\langle z^{-f}, \hat{y}^f \rangle$ denotes that state of the economy, which agrees with $z$ expect that the production plan of firm $f$ has been changed from $y^f$ to $\hat{y}^f$.

Since group $G^f(z)$ of individuals has veto $P^f(z) \subset V^f(z)$. If $y^f \notin co(P^g(z))$ for some $g \in G^f(z)$, then $y^f \notin co(P^f(z))$ and hence the convexity assumption of Theorem 2.2 is satisfied.

Assumption 3.1 We assume that $G^f(z)$ is a finite set of individuals, all of whom have positive measure.

To be able to apply the Kahn-Vohra existence theorem we need to establish that $P^f$ has open graph. Assumption 3.1 is necessary since if $G^f(z)$ has measure zero then $V^f(z)$ will not be continuous. If the set of individuals is finite, Assumption 3.1 imposes no restriction. Therefore this analysis generalises the model of Dreze (1989), which assumes a finite set of individuals throughout. In addition to prove the graph of $V^f(z)$ is open we also need the following assumption.

Assumption 3.2 For each $z \in Z$ and each $f$: $1 \le f \le F$, there is an open set $U$ such that $z \in U$ and for all $z' \in U$, $G^f(z') \subseteq G^f(z)$.

This is a continuity assumption and is needed for technical reasons. Examples which satisfy this assumption are giving veto to all individuals who own more than a certain fraction of the firm or the $k$ largest shareholders in the firm, with ties settled by the union rule. Assumption 3.2 and these examples were previously used by Dreze (1989).

Proposition 3.1 Under assumptions 3.1 and 3.2 the correspondence $V^f(z)$ has open graph.



Proof Let $\Gamma$ denote the graph of $V^f(z)$. Suppose that $\langle z^*, \bar{y}^f \rangle \in \Gamma$. Then by definition of $\Gamma$, $\forall g \in G$, $u^g\langle z^{*-f}, \bar{y}^f \rangle > u^g\langle z^* \rangle$. Since $G^f(z^*)$ is finite, we may write $G^f(z^*) = \{g_1, \ldots, g_m\}$ for some integer $m$. By continuity of $u^{g_i}$, there exists $\epsilon_i$ such that $\|z^* - \hat{z}\| < \epsilon_i$, $\|\bar{y}^f - \hat{y}^f\| < \epsilon_i$ implies that $u^{g_i}\langle \hat{z}^{-f}, \hat{y}^f \rangle > u^{g_i}\langle \hat{z} \rangle$. By assumption 3.1, there exists $\epsilon^*$ such that $\|z^* - \hat{z}\| < \epsilon^*$ implies $G^f(z') \subseteq G^f(z)$. Let $\epsilon = \min\{\epsilon^*, \epsilon_1, \ldots, \epsilon_m\}$ then if $\|z^* - \hat{z}\| < \epsilon$, $\|\bar{y}^f - \hat{y}^f\| < \epsilon$, $\forall g \in G^f(\hat{z})$, $u^g\langle \hat{z}^{-f}, \hat{y}^f \rangle > u^g\langle \hat{z} \rangle$. The result follows.
■

Corollary 3.1 Let $D^f : Z \to Y^f$ be any measurable correspondence with open graph. Assume that assumptions 3.1 and 3.2 hold. Then if $P^f(z) = V^f(z) \cap D^f(z)$, a simultaneous voting and exchange equilibrium will exist.

Proof This is a corollary of Theorem 2.2 and Proposition 3.1.
■

Remark As an example let $D^f$ be the set of production plans preferred by a majority of shareholders. This case is considered by Dreze (1989), DeMarzo (1990), who interpret the set of individuals with veto as the board of directors.

### 3.1.1 The Pareto Extension Rule

Suppose that a group $G$ of individuals have to make a decision for a set $\Lambda$ of alternatives. The Pareto extension rule is defined by, $P(\lambda) = \{\lambda' \in \Lambda : \forall g \in G,\ u^g(\lambda') > u^g(\lambda)\}$. In this case all individuals in the group $G$ have veto power. Suppose that a firm's decisions are made by the Pareto extension rule applied to the set $G^f(z)$, then sufficient conditions for the existence of equilibrium will be satisfied provided assumptions 3.1 and 3.2 are satisfied. An example would be the set of all individuals with positive shareholdings. In this case the firm's decisions would be Pareto optimal among the set of