QED

The arbitrage Pricing Theorem with Non Expected Utility Preferences

David Kelsey   Frank Milne

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

8-1992

Discussion Paper #866

The Arbitrage Pricing Theorem with Non
Expected Utility Preferences

by

David Kelsey
University of Birmingham

Frank Milne
Queen's University

August 1992

THE ARBITRAGE PRICING THEOREM WITH NON EXPECTED UTILITY PREFERENCES*

by

David Kelsey
Department of Economics,
The University of Birmingham,
Birmingham, B15 2TT,
United Kingdom.

and

Frank Milne
Department of Economics,
Queens University,
Kingston, Ontario K7L 3N6,
Canada.

26th August 1994

Address for correspondence,
David Kelsey, Department of Economics, University of Birmingham,
Birmingham, B15 2TT, United Kingdom.

1. INTRODUCTION

1.1 Arbitrage Pricing

Since the original work by Ross [34] there has been a number of papers clarifying the concept of the equilibrium Arbitrage Pricing Theory (APT). For a summary of this literature see Connor [9]. In an important contribution, Connor [8] constructed a two-date model with either finite or infinite assets, and consumers with risk-averse von Neumann-Morgenstern preferences to generate an equilibrium APT. Chamberlain [5] using the Martingale pricing technique generalised the Connor model to a multiperiod framework. Milne [30] using weak assumptions on induced preferences over assets proved an equilibrium APT for the finite asset case.

The aim of this paper is to extend the Milne approach in a number of directions:

a. Prove the Milne result when:

    (i) there are a countable number of assets;

    (ii) consumers have Gateaux[1] differentiable preferences over assets, rather than the stronger differentiability assumptions.

b. Use the above general result to explore the robustness of the APT theorem for two important classes of non-expected utility preferences.

These extensions are important in exploring the theoretical robustness of the APT result. The von Neumann-Morgenstern axiom

1

Abstract The arbitrage pricing theorem of finance shows that in certain circumstances the price of a financial asset may be written as a linear combination of the prices of certain market factors. This result is usually proved with von Neumann-Morgenstern preferences. In this paper we show that the result is robust in the sense that it will remain true if certain kinds of non expected utility preferences are used. We consider Machina preferences, the rank dependent model and non-additive subjective probabilities.

system is the most commonly used theory of choice in financial models. Unfortunately it has been criticized severely for weaknesses in empirical prediction. (For surveys of the available evidence see Machina [26] and Schoemaker [38]). A response to this empirical failing, has been the development of a number of alternative axiom systems that avoid some of the more obvious problems.

1.2 State-Dependent Preferences with Non-additive Subjective Probabilities

We shall discuss two general classes of non-expected utility models. The first class of models cover situations in which some of the relevant probabilities are not known in advance. Savage [36] produced an axiomatic theory of decision making in such situations, known as subjective expected utility theory. However it has been found in experiments, that people tend to deviate from the predictions of SEU, by preferring situations in which uncertainty is resolved with known probabilities to those in which it is resolved with unknown probabilities. (See Ellsberg [13], Anand [1]). Such behaviour is referred to as "uncertainty aversion".

This evidence can be explained if individuals have *non-additive* subjective probabilities. Choquet [7] has extended the Lebesgue integral to the case where the measure is not necessarily additive. It is possible to define an expected value with respect to a non-additive probability distribution to be a Choquet integral. Schmeidler [37] has shown that a small change in the usual axioms implies that a decision-maker will maximise expected utility with respect to a possibly non-additive

subjective probability distribution. (Other axiomatisations of non-additive probability can be found in Gilboa [17] and Wakker [40]). Individuals with non-additive probabilities can under certain conditions show a preference for situations in which uncertainty is resolved with known probabilities over situations in which it is resolved with unknown probabilities. Hence this theory is able to explain the evidence of Ellsberg [13], which contradicts subjective expected utility theory.

Our model generalises Connor's [8] model in two other respects. Firstly we allow different individuals to have different subjective probabilities (possibly non-additive). A special case of this is where all individuals are expected utility maximisers, but have different subjective probability distributions. Since there is no reason to expect different individuals to have the same subjective probability distribution this is, in itself, an important generalisation of the standard model.

Secondly we allow utility to be state-dependent, i.e. utility is $u_s(x)$ where s denotes the state and x denotes final wealth in that state. Thus the utility function may depend both on s and x. In this respect we generalise models in which utility depends only on wealth. With this class of preferences we need to impose an assumption which requires that individuals have common priors over the idiosyncratic portion of asset returns. This implies that all individuals agree which portfolios are diversified. Clearly it would not be possible to prove an APT result without such an assumption.

The usual motivation for considering state-dependent utility is when the resolution of the state uncertainty directly affects the decision-maker's utility function e.g. by affecting his state of health. To take an extreme case, suppose that in some of the states the decision-maker is dead. Then it is very unlikely that his value function for bequests in those states in which he is dead will be the same as his utility of wealth in those states in which he is alive. As this example suggests the state-dependent utility model has been principally applied to problems of health or life insurance.

In financial models, it is well-known (Fama and Miller [14] Chapter 8, Merton [28]) that the value function (or indirect utility function) from a multiperiod consumer's problem will be severely state-dependent. Therefore an APT theorem can be interpreted as a state-contingent pricing result. If the pricing theory is to be state-independent, except for the consumer's wealth, then further restrictions on the structure of the economy must be imposed. We will not discuss those types of conditions here.

1.3 Gateaux Differentiable Preferences

The second class of models, assumes that consumers have common priors on a probability space of future consumption possibilities. We assume that consumers' preferences are complete, transitive and continuous over the probability space. By Debreu [10] there exists a continuous utility function, which represents the preferences. We assume that preferences are non-restricted (monotonicity is a sufficient condition), Gateaux differentiable and "diversification is desirable". Then we are

able to prove the APT result. Machina [25] has shown that preferences which are Frechet differentiable and satisfy two plausible hypothesis can explain much of the laboratory evidence which contradicts expected utility theory, while at the same time preserving many of the desirable features of that theory.

Quiggin [31] has proposed an alternative theory of decision making under uncertainty known as *rank-dependent expected utility* (RDEU). Related theories have been studied in Yaari [42] and Green and Jullien [21]. In the RDEU theory an individual is assumed to maximise a weighted sum of utilities. The weights are not probabilities but a transformation of probabilities based on the rank of the outcome. An example of this would be a very cautious person who always put more weight on the worst outcome than its probability would justify. This theory can explain laboratory evidence on choices under risk better than expected utility theory. Recently it has been shown that this theory can provide better explanations than expected utility theory for insurance purchase, Segal and Spivak [39] and the demand for lottery tickets, Quiggin [33].

RDEU preferences are not Frechet differentiable. However under mild assumptions they can be shown to satisfy the weaker assumption of Gateaux differentiability. By studying Gateaux differentiable preferences we can prove results on asset pricing for RDEU and Machina preferences simultaneously.

Dekel [11] has shown that with Frechet differentiable preferences it is no longer the case that risk aversion implies a preference for diversification. Hence, a priori, it is not clear whether the APT, which relies upon diversification

arguments can be proved with non-expected utility preferences.
Here we show how Dekel's argument can be reconciled with the APT.

1.4 The Dutch Book Argument

A common criticism of non-expected utility preferences is that,
in a sequence of trades, they may lead individuals to purchase
a portfolio which is dominated by their initial wealth. (In
models with objective probabilities, dominated refers to first-
order stochastic dominance and in models without objective
probabilities, it means state by state domination.) This is the
"Dutch Book Argument" of de Finetti [15] and Freedman and Purves
[16]. This question is especially important when considering the
application of non-expected utility theory to financial markets.
If it is possible to induce an individual to accept a Dutch Book
then this would represent an arbitrage opportunity for an
informed outsider. It is usual in financial analysis to assume
that any available arbitrage opportunity will be taken.

It is easy to see that the Dutch Book argument does not
apply to the preferences studied in the present paper. This is
because our model concerns only a single time period. Provided
that preferences are transitive it is not possible to construct
a Dutch Book within a single time period. This is because a non-
expected utility maximiser has a set of transitive preferences
over state-contingent consumption. If a Dutch Book were
possible, then by reinterpreting state-contingent commodities
as separate goods we could construct a Dutch Book against an
individual with non-separable, but transitive preferences in a
situation not involving uncertainty. But this is not possible
since it contradicts basic consumer theory. A similar argument,

for situations where the probabilities are known is made in Machina [27].

It would be useful to extend the model of the present paper to a multi-period context. Therefore we shall proceed to discuss whether the Dutch Book argument would apply to the preferences we use in a multi-period context. A problem here is that, at present, there is no generally agreed extension of these preferences to more than one time period. (Recently some progress has been made towards answering this question, see Gilboa and Schmeidler [19]).

With the second kind class of preferences, those which satisfy a general Gateaux differentiability assumption in the space of probability distributions, the question of whether Dutch Books are possible has been investigated by Green [20]. The latter paper provides a defence for non-expected utility preferences against the Dutch Book argument. Green shows that provided preferences are quasi-convex then it is not possible for an non-expected utility maximiser to be exploited by an outsider unless that outsider knows the individual's private information, such as the distribution of his endowment. Clearly there is no incentive to reveal such private information.

It is harder to discuss how the Dutch Book argument relates to the first class of preferences we consider. As far as we are aware there are no published articles which explicitly consider whether the Dutch Book Argument can be applied to preferences which derive from non-additive probabilities. The classic statements of the Dutch Book argument, de Finetti [15] and Freedman and Purves [16] make the assumption that an individual

7

is prepared to bet for and against any given event *at the same odds*. The preferences axiomatised in Schmeidler [37] do not satisfy this hypothesis. A crucial part of Schmeidler's explanation of the Ellsberg Paradox is violation of this hypothesis. Therefore these previous statements of the Dutch Book Argument do not apply to the non-additive probability model.

As noted above, since these preferences are transitive, a Dutch Book is not possible in the static model of this paper. However it would be useful to have a multi-period extension of the present model. Further research is required to establish whether Dutch Books would be possible in such an extension.

1.5 Outline of the Paper

The plan of the paper is as follows: section 2 sets out the basic APT model, section 3 discusses non-additive probabilities and state-dependent preferences and section 4 discusses preferences which are Gateaux differentiable over probabilities. The paper ends with some concluding comments.

2. THE ARBITRAGE PRICING THEOREM

In this section we shall give a proof of the Arbitrage Pricing Theorem for an asset exchange economy using linear algebra. The analysis is conducted at an abstract level. Later sections will consider particular models within this general framework.

At date 0, individuals trade assets, which give them claims to contingent consumption at date 1. There is a set $\{a_\alpha : \alpha \in \Omega\}$ of asset holdings, where Ω is some index set which may be finite or countable. The set A denotes the space of all portfolios of assets. A portfolio is a finite or infinite linear

8

combination of the basic assets. We do not make use of any topological structure on A.

Definition 2.1   A *factor structure* on A is a linear transformation T: A → F where F is a finite dimensional vector space. We shall use k to denote the dimension of F.

By deleting redundant factors, if necessary, we may assume that T is surjective. Ker(T) denotes the set of a ϵ A such that Ta = 0. Elements of Ker(T) will be called *idiosyncratic risk*.

Definition 2.2   An *asset exchange economy* is a set i = 1, 2,..., H of individuals each with endowment w_i ϵ A and preferences represented by the function V^i : A → $\mathbb{R}$. The preferences are assumed to satisfy non-satiation.

Remark   The preferences defined on A should be thought of as the induced preferences constructed from a basic set of preferences $\overline{V} : X \to \mathbb{R}$ defined over the space of contingent consumption bundles. Details of this construction can be found in Milne [29],[30].

Definition 2.3   A *competitive equilibrium* for an asset exchange economy is a linear functional p : A → $\mathbb{R}$ and a portfolio of assets a^*_i for each individual i such that a^*_i is the solution to

Max $V^i(a_i)$ subject to $p.a_i \leq p.w_i$ for $1 \leq i \leq H$

and $\sum\limits_{i=1}^{H} a_i^* = \sum\limits_{i=1}^{H} w_i$.

<u>Remark</u>   We are using the notation p.a for the value of the linear functional p on the vector a ϵ A. This notation does not imply the existence of an inner product on A.

<u>Definition 2.4</u>   A *constrained Pareto optimum* is an allocation $a_1^*, \ldots, a_H^*$ of one portfolio for each individual with there does not exist an allocation $a_1, \ldots, a_H$ such that

$V^i(a_i) \geq V^i(a_i^*)$, for $1 \leq i \leq H$; and

$V^j(a_j) > V^j(a_j^*)$ for some j and $\sum\limits_{i=1}^{H} a_i = \sum\limits_{i=1}^{H} w_i$.

<u>Theorem 2.1</u>   In an asset exchange economy every competitive equilibrium is a constrained Pareto Optimum.

We shall omit the proof of Theorem 2.1 since it is standard.

There is a subspace D of A called the subspace of *diversified portfolios*.

<u>Assumption 2.1</u>   We say that individual i is a *diversifier* if when Ta' = Ta" and a' ϵ D, a" ϵ A - D implies $V^i(a') > V^i(a")$.

This says that when choosing between two factor-equivalent portfolios a diversifier will always strictly prefer the one which is diversified.

<u>Assumption 2.2</u>   An asset exchange economy is said to be *insurable* if T(D) = F and the aggregate endowment lies in D i.e.,

$$\sum_{i=1}^{H} w_i \in D.$$

Together the assumptions, that the economy is insurable and that all individuals are diversifiers imply, in competitive equilibrium, everybody holds a diversified portfolio. This is proved in the following proposition.

<u>Proposition 2.1</u>   In an insurable asset exchange economy in which all individuals are diversifiers, any competitive allocation is diversified.

<u>Proof</u>   Let $a_1^*, \ldots, a_H^*$ be a competitive allocation. Since the economy is insurable we can find allocations $a_1', \ldots, a_H'$ such that a_h' is diversified and has the same factor structure as a_h^* , for $1 \leq h \leq H$.   The allocation $a_1', \ldots, a_H'$ is not necessarily feasible. However define $a_H = (a_H^* + \sum_{i=1}^{H-1} a_i^*) - \sum_{i=1}^{H-1} a_i'$. The term in brackets is the endowment of the whole economy, which is diversified by assumption, thus a_H is a sum of diversified portfolios and is itself diversified since D is a (vector) subspace.

Define $a_h = a_h'$ for $1 \leq h \leq H-1$. The allocation $a_1, \ldots, a_H$ is factor equivalent to $a_1^*, \ldots, a_H^*$. Since all individuals are diversifiers $a_1, \ldots, a_H$ is a Pareto improvement on $a_1^*, \ldots, a_H^*$ unless the latter allocation is itself diversified. But since

11

a competitive equilibrium is a constrained Pareto Optimum by

Theorem 2.1 it follows that $a_1^*, \ldots, a_H^*$ must be diversified.

■

<u>Definition 2.5</u>   A function $V : A \to \mathbb{R}$ is said to be *Gateaux differentiable* if for each a ϵ A there exists a linear functional $L_a : A \to \mathbb{R}$ such that,

$$L_a (a' - a) = \lim_{\lambda \to 0} \frac{1}{\lambda} (V(\lambda a' + (1-\lambda) a) - V(a)) .$$

<u>Assumption 2.3</u>   <u>Differentiability</u> Preferences are Gateaux differentiable.   We shall abbreviate this by saying that preferences are differentiable.

<u>Proposition 2.2</u>   Suppose that in an asset exchange economy all individuals are diversifiers and at least one individual has differentiable preferences, then the competitive equilibrium price of any portfolio which consists solely of idiosyncratic risk is zero.   (ie. a ϵ Ker(T) $\Rightarrow$ p.a = 0).

<u>Proof</u>   By assumption there is an individual i with differentiable preferences.   Let a$*_i$ denotes i's competitive equilibrium allocation by Proposition 2.1, a$*_i$ ϵ D.   Note that since a ϵ Ker (T), T(a$*_i$ + αa) = Ta$*_i$.   Hence a$*_i$ is factor equivalent to a$*_i$ + αa.   Since i is a diversifier, α = 0 maximises V^i(a$*_i$+αa).   Hence $\frac{d}{d\alpha}$ V^i(a$*_i$+αa)$|_{\alpha=0}$ = 0 .   We must therefore have p.a = 0.   Since if a had either a positive or negative

price, utility could be increased by either short selling or buying a.

■

<u>Theorem 2.2 (The Arbitrage Pricing Theorem)</u> Let p be the competitive equilibrium price vector of an insurable asset exchange economy. Suppose that all individuals are diversifiers and at least one individual has differentiable preferences, then there exists a linear functional q on F such that

$$(\forall a \in A), \ p.a = q.Ta.$$

<u>Proof</u>   Define q : F → ℝ by q(f) = p.a where Ta = f. First note that q is well defined. Suppose Ta = Ta', then a − a' ∈ Ker(T). Hence by Proposition 2.6, p.(a−a') = 0, therefore p.a = p.a'. Thus q is well defined. It is a simple exercise in linear algebra to show that q($\alpha f_1 + \beta f_2$) = αq(f_1) + βq(f_2). Hence q is a linear functional on F and q(Ta) = p.a.

■

The proof of the APT given above depends on only two properties of preferences. These are assumption 2.1 that all individuals are diversifiers and that there is a single individual who obeys assumption 2.3 (differentiability). Connor [8] has shown that these assumptions will be true if all individuals are expected utility maximisers with strictly concave and differentiable von Neumann-Morgenstern utility functions. Milne [30] generalised that result to finite asset economics and a standard differentiability assumption on asset preferences. Therefore our result generalises both Milne and Connor. In sections 3 and 4 of this paper we provide two examples of non-

expected utility preferences which satisfy these assumptions. Thus the APT will automatically be satisfied.

3. STATE-DEPENDENT UTILITY AND NON-ADDITIVE PROBABILITIES

The analysis of section 2 was at a very abstract level. In sections 3 and 4 we shall put more structure on preferences and on asset returns. First we shall describe the assumptions that we shall make about preferences. The main innovation of this section is that we shall represent the decision-maker's beliefs by a non-additive subjective probability distribution.

In this section we shall use a framework similar to that of Anscombe and Aumann [2]. There is a set S of states of nature. We shall use Δ to denote the set of all probability distributions over $\mathbb{R}$. The elements of Δ are interpreted as gambles with objectively known probabilities whose outcomes are quantities of the physical commodity. An action is a function $y: S \to \Delta$. The space of all actions is denoted by Y. There are two kinds of uncertainty in this framework. The elements of Δ are objective probability distributions, known to all agents in the economy. The second kind of uncertainty is the state uncertainty. We do not assume that the probabilities of the states are known to individuals in advance and allow for the possibility that different individuals have different beliefs concerning the state uncertainty. As mentioned in the introduction, non-additive probabilities allow for the possibility that the decision-maker displays "uncertainty aversion".

Choquet [7] shows how an integral with respect to a non-additive measure can be defined. Using the Choquet integral it is possible to speak of the expected value of a utility function

14

with respect to a non-additive probability distribution. Schmeidler [37] has derived the existence of a non-additive subjective probability distribution within an axiomatic decision theory. Moreover he has shown that the decision-maker will maximise expected utility (defined as a Choquet integral). Schmeidler's axioms are similar to those of Anscombe and Aumann [2], except that he relaxes the independence axiom to the weaker assumption of comonotonic independence.

Definition 3.1 <u>Non-additive Probability</u> Let Σ be a non-empty algebra of subsets of S. A function $\mu: \Sigma \to \mathbb{R}$ is a *non-additive probability* if it satisfies the following three conditions:

i. $\mu(\phi) = 0$,

ii. $\mu(S) = 1$,

iii. $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

Definition 3.2 <u>Choquet Integral</u> Let $f = S \to \mathbb{R}$ be a bounded Σ-measurable function and let μ be a non-additive probability. Define the *Choquet Integral* of f with respect to μ by

$$\int f d\mu = \int_{0}^{\infty} \mu(x: f(x) \geq \alpha) \, d\alpha + \int_{-\infty}^{0} (\mu(x: f(x) \geq \alpha) - 1) \, d\alpha, \qquad (3.1)$$

where the integrals on the right hand side of (3.1) are Lebesgue integrals.

<u>Properties of the Choquet Integral</u> The Choquet integral satisfies the following properties.

$$\int \lambda f d\mu = \lambda \int f dv \ ,$$ (3.2)

$$\int \lambda + f dv = \lambda + \int f dv \ ,$$ (3.3)

$$\forall s \in S, \ f(s) \geq g(s) \Rightarrow \int f d\mu \geq \int g d\mu. \ ,$$ (3.4)

These three properties all follow directly from equation (3.1).

We assume that the decision maker has a state-dependent utility function $u_s(x)$, where x denotes wealth in state s. The fact that the utility of wealth can depend on the state is an implicit way of considering risks other than money risks. Thus we assume that the individual's preferences can be represented in the following form, $V(x) = \int E[u_s(x(s))] d\mu$, where $E \ u_s(x(s))$ denotes the expected value of u_s with respect to the objective probability distribution, μ is the non-additive subjective probability and the integral is a Choquet integral.

<u>Assumption 3.1</u>   Each state specific utility function u_s is assumed to be strictly increasing.

<u>Assumption 3.2</u>   <u>Risk Aversion</u> Each state-specific utility function is assumed to be strictly concave.

<u>Remark</u>   In Kelsey and Nordquist [22] it is argued that this is the correct definition of risk-aversion when utility is state-dependent.

We shall now describe our assumptions on asset returns. Our basic probability space is [0,1] with the σ-algebra of Lebesgue measurable sets and Lebesgue measure. This assumption can be made without serious loss of generality. We shall restrict the objective probability distributions to be square integrable hence the space X is the space of all functions from S to $L^2[0,1]$.

Let $f_1,\ldots, f_k$ be k linearly independent members of X. We assume that returns on asset α are given by $x_\alpha = \sum_{i=1}^{k} \beta_{i\alpha} f_i + e_\alpha$, where e_α satisfies $E(e_\alpha) = 0$ and $E(e_\alpha \mid f_i) = 0$, $1 \le i \le k$. In this interpretation F is the vector subspace of X spanned by $f_1,\ldots, f_k$ and T is the function, $T(x_\alpha) = \sum_{i=1}^{K} \beta_{i\alpha} f_i$. Since the x_α span A, this defines T on the whole of A. Portfolios are assumed to be absolutely summable i.e. if γ_α denotes the amount of asset α held in portfolio a then $\sum_{\alpha \in \Omega} |\gamma_\alpha|$ exists. Further we assume the existence of a function $h \in L^2[0,1]$ such that $h \ge |e_\alpha|$ for all $\alpha \in \Omega$. The last two assumptions will always be satisfied when there is a finite number of assets.

The space D, of diversified portfolios is defined to be the linear span of $f_1,\ldots,f_k$. While it is not possible for an individual to buy the f_i directly, there are two ways in which he may be able to buy the f_i as a combination of the available assets. Firstly if there is a linear dependency between the e_α than an investor could eliminate the idiosyncratic risk by buying an appropriate finite linear combination of assets. If the e_α have finite second moments this would imply that the covariance matrix of a finite set of the e_α is singular. Secondly an investor may be able to eliminate the idiosyncratic risk by

holding infinitesimal amounts of each asset. This second possibility can only occur with there are an infinite number of assets.

Assumption 3.3   The idiosyncratic risk is assumed to be state-independent.

Remark   This implies that the idiosyncratic risk is described only by objective probabilities. On the other hand we need no assumption about the factor risk, which can depend on both known and unknown probabilities in any combination. A motivation for assumption 3.3 can be given as follows. The factor risks are things which affect the economy as a whole, such as the price of major exports, interest rates, war or earthquakes. The behaviour of these variables will depend, in part on risks for which relative frequencies exist but also on some events which are non-repeatable. The idiosyncratic risk represents firm-specific factors. A good example of this might be the chance that an individual firm would have a bad management. This would be a risk which could be expected to be independent of the state. If the economy contained a large number of firms it may be possible to observe the frequency of management failure and hence it could be viewed as an objective probability[2].

Lemma 3.1   Suppose that a and b are factor equivalent portfolios where b is diversified and a is not, then a - b is a mean preserving increase in risk on b.

<u>Proof</u>   In the case where there are a finite number of assets $a - b = \sum_{i=1}^{n} \gamma_i e_i$ It is immediate that $E(a-b) = 0$ and $E(a-b|b) = 0$ and hence $a - b$ is a mean preserving increase in risk on b.   In the countable case $a - b = \sum_{i=0}^{\infty} \gamma_i e_i$.   Define $f_n = \sum_{i=1}^{n} \gamma_i e_i$.   Let $A = \sum_{i=0}^{\infty} |\gamma_i|$ which exists by assumption. For $x \in [0,1]$ $|e_i(x)| \le h(x)$, hence $|\gamma_i\, e_i(x)| < |\gamma_i|\, h(x)$.   It follows that the series $\sum_{i=0}^{\infty} \gamma_i e_i(x)$ converges pointwise to a limit f.   Moreover $|f(x)| \le Ah(x) = H(x)$.   Clearly H is an integrable function and $|f_n(x)| \le H(x)$.   We may apply the Dominated Convergence Theorem (Weir [41]) to deduce that,

$$\int f = \lim_{n \to \infty} \int f_n = 0 \; , \quad \text{hence } E(a-b) = 0.$$

For fixed $y \in \mathbb{R}$, the sequence of functions, $\max \{y - b(x) - f_n(x), 0\}$ converges pointwise to $\max\{y - b(x) - f(x), 0\}$.

Moreover $|\max \{y - b(x) - f_n(x), 0\}| \le |y| + |b(x)| + H(x)$, which is integrable.   Since $-\max \{y-t, 0\}$ is a concave function of t and $b + f_n$ is a mean preserving increase on b, we have,

$$-\int \max \{y - b(x),\, 0\}\, dx \ge -\int \max \{y - b(x) - f_n(x),\, 0\}\, dx \; .$$

By the Dominated Convergence Theorem,

$$-\int \max \{y - b(x),\, 0\}\, dx \ge -\int \max \{y - b(x) - f(x),\, 0\}\, dx.$$

Let G and H denote respectively the distribution functions of b and $b + f$.   Then we have

$$-\int_0^y (y-t)\, dG(t) \ge -\int_0^y (y-t)\, dH(t) \; .$$

Integrating by parts we obtain,

$$yG(y) - tG(t)\Big|_o^y + \int_o^y G(t)\,dt \le yH(y) - tH(t)\Big|_o^y + \int_o^y H(t)\,dt$$

which implies, $\int_o^y G(t)\,dt \le \int_o^y H(t)\,dt.$ But this is the integral form of the Rothschild-Stiglitz conditions for b + f to be a mean-preserving increase in risk on b^3.

■

<u>Proposition 3.1</u>   Any risk averse individual with a state-dependent utility function and non-additive subjective probabilities, strictly prefers a portfolio with zero idiosyncratic risk to a factor equivalent portfolio with non-zero idiosyncratic risk.

<u>Proof</u>   Let x* denote the returns from a portfolio with zero idiosyncratic risk and let x' be the returns from a factor equivalent portfolio with non-zero idiosyncratic risk. Then

E u_s(x'(s)) = E u_s(x*(s) + (x'(s) - x*(s))) < E u_s(x*(s)).

Since u_s is strictly concave and since x* and x' have the same factor structure the addition of x' - x* represents a mean preserving increase in risk on x*, by Lemma 3.1. It follows by the monotonicity property of the Choquet integral (3.4) that

$\int E[u_s(x*(s))]\,d\mu > \int E[u_s(x'(s))]\,d\mu ,$ and hence the investor prefers the portfolio with zero idiosyncratic risk.

■

<u>Assumption 3.4 Differentiability</u>   An individual is said to have differentiable preferences if for all s ϵ S and all w ϵ R, u_s(w) is a differentiable function of w.

To show that the proof of the APT is valid for these preferences we need to show that

$$\frac{d}{d\lambda} V(a+\lambda a') \Big|_{\lambda = 0} \tag{3.5}$$

exists when a' is a portfolio that consists solely of idiosyncratic risk. (The existence of the derivative is required to prove proposition 2.2).

By assumption the idiosyncratic risk consists only of objective probabilities. The preferences which we are considering are linear and hence differentiable in objective probabilities. If in addition it is assumed that there is at least one individual i who satisfies assumption 3.4 then the derivative (3.5) will exist.

Corollary 3.1   The Arbitrage Pricing Theorem for State-Dependent Preferences with Non-additive Probabilities   In an insurable asset exchange economy, which satisfies assumption 3.3, where all individuals have state-dependent preferences with non-additive subjective probabilities satisfying assumptions 3.1 (monotonicity), 3.2 (risk aversion) and at least one individual's preferences satisfy assumption 3.4 (differentiability), then if p_α denotes the competitive equilibrium price of asset α there exists a linear functional q on F such that;

$$p_\alpha = \sum_{i=1}^{k} \beta_{i\alpha} \, q(f_i) \, .$$

4. GATEAUX DIFFERENTIABLE PREFERENCES OVER PROBABILITIES OF FUTURE WEALTH

In this section we shall show that the APT holds for risk-averse individuals with preferences which are Gateaux differentiable in probabilities over future wealth. As noted in the introduction this is a general form of preferences which has two special cases among the more prominent non-expected utility theories. The rank dependent model is Gateaux differentiable under certain assumptions and Machina preferences are always Gateaux differentiable.

In this section we shall use the same assumptions about assets as in section 3 with the additional restriction that there is only one state. This implies that all the risks can be described by objective probability distributions. Hence the basic objects of choice are elements of Δ, probability distributions over $\mathbb{R}$. We shall assume that an investors' preferences can be represented by a function $\overline{V} : \Delta \to \mathbb{R}$. We shall assume that $\overline{V}$ is Gateaux differentiable.

Chew, Karni and Safra [6] show that this implies that there exists a function $U_G : \mathbb{R} \to \mathbb{R}$ such that,

$$L_G(G'-G) = \int U_G \, d(G'-G).$$

This means that for small changes in the probability distribution the investor acts as if he is maximising the expected value of U_G. For this reason U_G is referred to as the *local utility function*.

<u>Proposition 4.1</u>   Let $\bar{V}$ be a Gateaux differentiable preference function such that the local utility functions U_G are strictly concave for all $G \in \Delta$ then $V(G') > V(G'')$ whenever G'' differs from G' by a mean preserving increase in risk.

<u>Proof</u>   Let G^1 be a mean-preserving increase in risk from G^0. Define $G^\alpha = (1-\alpha)G^0 + \alpha G^1$ and $\psi(\alpha) = V(G^\alpha)$. Then $\psi'(\alpha) = \int U_{G^\alpha} d(G^1 - G^0)$. Since G^1 is a mean-preserving increase in risk from G^0 and U_{G^α} is concave,

$\int U_{G^\alpha} dG^0 > \int U_{G^\alpha} dG^1$, which implies $\phi(1) < \phi(0)$. Hence G^0 is preferred to G^1.
∎

<u>Remark</u>   This result is essentially a Corollary of Theorem 1 of Chew, Karni and Safra [6]. The latter result was stated within the framework of the rank dependent model, however the proof only uses Gateaux differentiability and does not depend on any specific features of the rank dependent model. If instead it was assumed that the local utility functions were weakly concave we could prove that the decision-maker will never strictly prefer a mean preserving increase in risk. Proposition 4.1 motivates the following definition.

<u>Definition 4.1</u>   We shall say that an individual with Gateaux differentiable preferences over future wealth is *risk averse* if all his local utility functions are strictly concave.

<u>Proposition 4.2</u>   If an investor has Gateaux differentiable preferences and is risk averse then he is a diversifier in the sense of assumption 2.1.

<u>Proof</u>   Let a ϵ D be a diversified portfolio and let a' be an undiversified portfolio which is factor equivalent (ie Ta = Ta'). Then by lemma 3.1 a' - a is a mean preserving increase in risk on a and hence by Proposition 4.1, V(a) > V(a').

∎

<u>Remark</u>   Proposition 4.2 has shown that risk-averse Gateaux differentiable preferences satisfy one of the assumptions used in section 2 to prove the APT.   The other assumption was differentiability.   Specifically that: if a ϵ D and a' ϵ Ker(T), $\frac{d}{d\lambda} V(a + \lambda a') \big|_{\lambda = 0}$ exists.   (4.1)
Gateaux differentiability of V is not in itself sufficient to ensure that this derivative exists.   The problem is that we have required V to be Gateaux differentiable with respect to probabilities.   As λ varies both the probabilities and the outcomes associated with the portfolio a + λa' change.   To ensure that the derivative (4.1) exists it is necessary that the local utility function itself be differentiable with respect to its argument.   This is analogous to assuming that an expected utility maximiser has a differentiable von Neumann-Morgenstern utility function. With the additional assumption that there is at least one individual whose local utility functions are differentiable, it follows by the analysis of section 2 that the arbitrage pricing theorem will hold.   This is stated below.

<u>Corollary 4.1</u> <u>The Arbitrage Pricing Theorem for Gateaux Differentiable Preferences</u>   In an insurable asset exchange economy suppose that all individuals have risk-averse Gateaux differentiable preferences and at least one individual has

continuously differentiable local utility functions, if p denotes the competitive equilibrium price functional, then there exists a linear functional q on the space of factors such that p.a = q.Ta.

5. CONCLUSION

In this paper we have shown that the APT is true for risk-averters with Gateaux differentiable preferences, individuals with state-dependent preferences and non-additive subjective probabilities as well as for individuals with state-independent von Neumann-Morgenstern preferences. Since the proof of the APT in section 2 only relies upon a preference for diversification and the existence of a single individual with differentiable preferences, the APT will remain true if the economy consists of a mixture of these three kinds of individuals.

One of the most widely discussed non-expected utility theories which we have not considered so far is regret theory, Bell [3], Loomes and Sugden [23]. Most existing versions of regret theory only consider choices between two alternatives. This makes them unsuitable for application to financial markets.

Recently Quiggin [32] has proposed a theory of choices over larger sets which agrees with the usual regret theory when applied to choices over pairs. In Quiggin's extension of regret theory the decision maker maximises the expected utility function $U(a(s),\beta^s)$, where β^s denotes the largest pay off in state s of any available action. Thus preferences may be represented in the following form, $c \, P \, d \Leftrightarrow E \, U(c(s),\beta^s) > E \, U(d(s),\beta^s)$,   where P denotes strict preference.

Assume now that the set of available actions is fixed. Then we may define $u_s(x) = U(x, \beta_s)$. The decision maker will maximise $E\, u_s(x)$. Hence this extension of regret theory may be seen as a special case of the state-dependent utility model considered in section 3 of the present paper. Therefore the APT result also applies to this extension of regret theory.

If an expected utility maximiser is risk averse and is indifferent between two assets he will (weakly) prefer a portfolio consisting of a mixture of the two assets to a portfolio containing only one of the assets. Dekel [11] has shown that this result does not generalise to Machina preferences. The reason that he obtained this result is that in his example the diversified portfolio does not second order stochastically dominate either asset. The proof of the APT in the present paper only relies on individuals diversifying in order to achieve second order stochastically dominating distributions. Thus Dekel's result is compatible with our own.

Notes on the text

1. The Gateaux derivative is a generalisation of the directional derivative on $\mathbb{R}^n$ to general vector spaces over $\mathbb{R}$. Frechet differentiability is a generalisation of the total derivative in $\mathbb{R}^n$ to a normed vector space. If a function f is Frechet differentiable at a point, then the Gateaux derivative exists at that point and is equal. The converse is false in general. For a discussion, see Chapter 7 Luenberger [24].

2. There are two implications of assumption 3.3. Firstly, all individuals have the *same* subjective probabilities for the idiosyncratic risk. Secondly, that subjective probabilities for the idiosyncratic risk are *additive*. The first part of this assumption is crucial for our result. If individuals did not agree on the probability distribution of the idiosyncratic risk then they would not agree which portfolios were diversified. It would not necessarily be the case that the price of a portfolio which contains only idiosyncratic risk would be zero. Hence the APT result would not hold, in general.

We conjecture that if the first part of assumption 3.3 held but the second did not, then the APT result would continue to be valid. We have not investigated this case in detail since the resulting preferences would be very similar to the rank dependent model which is investigated in section 4 of the present paper.

3. The latter part of this proof is based the analysis of Rothschild-Stiglitz [35].

REFERENCES

1. Anand, P., Analysis of Uncertainty as Opposed to Risk: An Experimental Approach, *Agricultural Economics*, 4 (1990), 145-163.

2. Anscombe, F.J. and R.J. Aumann, A Definition of Subjective Probability, *Ann. Math. Statis.*, 34 (1963), 199-20.

3. Bell, D., Regret in Decision Making Under Uncertainty, *Operations Research*, 30 (1982), 961-981.

4. Border, K. and Segal, U., "Dutch Book Arguments and Subjective Probability" working paper, (1990).

5. Chamberlain, G., Asset Pricing in Multiperiod Securities Markets, *Econometrica*, 56 (1988), 1283-1300.

6. Chew, S.H., Karni, E. and Z. Safra, Risk Aversion in the Theory of Rank-Dependent Probabilities, *J. of Econ. Theory*, 42 (1987), 370-381.

7. Choquet, G., Theory of Capacities, *Annuales Institut Fourier*, 5 (1953-4), 131-295.

8. Connor, G., A Unified Beta Pricing Theory, *J. of Econ. Theory*, 34 (1984), 13-31.

9. Connor, G., Notes on the Arbitrage Pricing Theory, in S Bhattacharya and G Constantinedes (eds), "Theory of Valuation: Frontiers of Modern Financial Theory", Rowen and Littlefield, Totowa, N.J. (1989).

10. Debreu, G., Continuity Properties of Paretian Utility, *Int. Econ. Rev.*, 5 (1964), 285-293.

11. Dekel, E., Asset Demands Without the Independence Axiom, *Econometrica*, 57 (1989), 163-169.

12. Dow, J. and Werlang S.R.C., Uncertainty Aversion and the Optimal Choice of Portfolio, *Econometrica,* 60 (1992), 197-204.

13, Ellsberg, D (1961), "Risk, Ambiguity and the Savage Axioms", *Quart. J. of Econ.*, 77 (1963), 643-669.

14. Fama, G. and Miller, M. "The Theory of Finance", Holt, Rhinehart and Winston, New York (1972).

15. de Finetti, B., "Theory of Probability", Wiley, New York (1974).

16. Freedman, D.A. and R.A. Purves, Bayes' Method for Bookies, *Ann. of Math. Statis.*, 34 (1969), 199-20

17. Gilboa, I., Expected Utility with Purely Subjective Non-additive Probabilities, *J. of Math. Econ.* 16 (1987), 65-88.

18. Gilboa, I. and Schmeidler, D., Maxmin Expected Utility with a Non-unique Prior, *J. of Math. Econ.,* 18 (1989), 141-153.

19. Gilboa, I. and Schmeidler, D., "Updating Ambiguous Beliefs" *J. of Econ. Theory,* forthcoming (1991).

20. Green, J., Making Book Against Oneself: The Independence Axiom and Non-linear Utility Theory, *Quart. J. of Econ.*, 102 (1987), 785-796.

21. Green, J. and Jullien, B., Ordinal Independence in Non-linear Utility Theory, *J. of Risk and Uncertainty*, 1 (1988), 355-88.

22. Kelsey, D. and Nordquist, G., A More General Measure of Risk Aversion When Utility is State-Dependent, *Oxford Econ. Pap.*, 43 (1991), 59-74.

23. Loomes, G. and Sugden, R., Some Implications of a More General Form of Regret Theory, *J. of Econ. Theory*, 41 (1987), 270-287.

24. Luenberger, D., "Optimization by Vector Space Methods", Wiley, New York (1969).

25. Machina, M., Expected Utility Analysis Without the Independence Axiom, *Econometrica*, 50 (1982), 277-323.

26. Machina, M., "The Economic Theory of Individual Behaviour Toward Risk: Theory, Evidence and New Directions" Technical Report, No 433, Center for Research on Organisational Efficiency Stanford, (1983).

27. Machina, M., Dynamic Consistency and Non-Expected Utility Models of Choice under Uncertainty, *J. of Econ. Lit.*, 27 (1989), 1622-1668.

28. Merton, R., "An Intertemporal Capital Asset Pricing Model, *Econometrica*, 41 (1973), 867-886.

29. Milne, F., Induced Preferences and the Theory of the Consumer, *J. of Econ. Theory*, 24 (1981), 205-217.

30. Milne, F., Arbitrage and Diversification in a General Equilibrium Asset Economy, *Econometrica*, 56 (1988), 815-840.

31. Quiggin, J., A Theory of Anticipated Utility, *J. of Econ. Behaviour and Organisation*, 3 (1982), 323-334.

32. Quiggin, J., "Regret Theory: The General Case", University of Maryland working paper, (1989).

33. Quiggin, J., On The Optimal Design of Lotteries, *Economica* forthcoming, (1991).

34. Ross, S., The Arbitrage Theory of Capital Asset Pricing, *J. of Econ. Theory*, 17 (1976), 315-329.

35. Rothschild, M. and Stiglitz, J., Increasing Risk I: A Definition, *J. of Econ. Theory*, 2 (1970), 315-329.

36. Savage, L.J., <u>Foundations of Statistics</u>, Wiley, New York, (1954).

37. Schmeidler, D., Subjective Probability and Expected Utility without Additivity, *Econometrica*, 57 (1989), 571-587.

38. Schoemaker, P., The Expected Utility Model; its Variants, Evidence and Limitations, *J. of Econ. Lit.*, 20 (1982), 529-563.

39. Segal, U. and Spivak A., First-Order versus Second-Order Risk Aversion, *J. of Econ. Theory* 28 (1990), 111-25.

40. Wakker,P., Continuous Subjective Expected Utility with Non-additive Probabilities, *J. of Math. Econ.* 18 (1989), 1-27.

41. Weir, A.J., "Lebesgue Integration and Measure", CUP, Cambridge, (1973).

42. Yaari, M., The Dual Theory of Choice Under Risk, *Econometrica*, 55 (1987), 95-115.