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Queen's Economics Department Working Paper No. 778

## Semiparametric Specification Testing

Miguel A. Delgado

Thanasis Stengos

Department of Economics  
Queen's University  
94 University Avenue  
Kingston, Ontario, Canada  
K7L 3N6

1-2013

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DISCUSSION PAPER #778

by

Miguel A. Delgado<sup>1</sup>

and

Thanasis Stengos<sup>2</sup>

March, 1990

1. Department of Economics, Indiana University, Bloomington, IN 47405
2. Department of Economics, University of Guelph, Guelph, Ontario N1G 2W1;  
Department of Economics, Queen's University, Kingston, Ontario K7L 3N6

### Abstract

We propose a specification test of a parametrically specified model against a weakly specified alternative. The latter is estimated using  $k$  nonparametric nearest neighbors ( $k$ -NN) in the context of an artificial regression. We derive the asymptotic distribution under the null hypothesis and under a series of local alternatives. Monte carlo simulations suggest that the test is quite powerful although it has a tendency to over-reject under the null hypothesis.

Key Words:  $k$ -nearest neighbor regression, artificial nesting, nonnested hypothesis.

## 1. Introduction

In this note we propose a specification test of a parametrically specified model, as dictated by economic theory, against a weakly specified alternative. Our test is based on an artificial nesting procedure for testing separate regressions, see Davidson and Mackinnon (1981) and Fisher and McAleer (1981). The performance of the alternative "under the truth" of the null forms the basis for a test of the latter. However, our procedure differs from theirs in that we are only interested in the performance of the null. In other words we do not want to make a statement about the validity of the alternative model, since the latter is not derived from some specific economic model. We also avoid the effect that a possibly misspecified parameterization of the alternative hypothesis would have on testing the validity of the null hypothesis. We use  $k$  nonparametric nearest neighbors ( $k$ -NN) to estimate the alternative model that enters the artificial regression.

In the next section we discuss the nature of the proposed test and we derive its distribution under the null hypothesis and under a series of local alternatives. We then proceed to investigate its small sample properties by means of a small monte carlo. Finally, we conclude.

## 2. The Specification Test

Suppose we observe a random sample  $\{(Y_i, X_i, Z_i), 1 \leq i \leq n\}$  from the  $R^p \times R^p \times R^r$  valued random variable  $\{Y, X, Z\}$ , having finite variance and conditional distribution  $F_{Y|X,Z}$  that is nondegenerate for all  $X, Z$  at which it is defined. Let  $E(\cdot)$  denote the mathematical expectation. The researcher faces the following competing hypotheses:

$$H_0: E(Y|X, Z) = X^T \beta_0 ; H_A: E(Y|X, Z) = E(Y|Z) \quad (1.1)$$

In other words  $H_0$  is completely parameterized. The alternative hypothesis  $H_A$  is only weakly specified. Hence, the functional form of  $E(Y|Z)$  does not take a specific parametric form. We can look at the composite hypothesis  $H_C$  as the weighted average of  $H_0$  and  $H_A$ :

$$H_C: E(Y|X, Z) = X^T \beta_0 (1-\theta) + \theta E(Y|Z) \quad (1.2)$$

Since  $E(Y|Z)$  is unspecified, we can estimate it by using a nonparametric estimator. In this paper we propose to use (k-NN) nonparametric weights. The use of these weights have been introduced in the semiparametric literature by Robinson (1987). In particular, the estimator of  $E_1$ , where  $E_1 = E(Y_1|Z_1)$ , is given by :

$$\hat{E}_1 = \sum_j Y_j \omega_{1j}(k) \quad (1.3)$$

where  $\omega_{1j}(k)$  are the weights based on the  $r$   $Z$ -regressors. For a positive integer  $k$  let  $C_i(k)$  be constants satisfying  $C_i(k) \geq 0$  ;  $C_i(k) = 0$ ,  $i > k$ ;  $\sum_{i=1}^k C_i(k) = 1$

$$\omega_{1j}(k) = 1(i \neq j) r_{ij}^{-1} \sum_{\tau=p_{ij}}^{p_{ij} + r_{ij} - 1} C_\tau(k) \quad (1.4)$$

where  $1(\cdot)$  is the indicator function and  $p_{ij}$  is 1 plus the number of  $Z$ 's closer to  $Z_i$  than  $Z_j$  and  $r_{ij}$  is 1 plus the number of  $Z$ 's equally near from  $Z_i$  as  $Z_j$ . To calculate distance we use the euclidean metric after we standardize by the sample standard deviation. Then we estimate  $(\theta, b^T)^T$ , where  $b = \beta_0(1-\theta)$ , by:

$$\begin{bmatrix} \hat{\theta}_n \\ \hat{b}_n \end{bmatrix} = A_n^{-1} \sum_i \begin{bmatrix} \hat{E}_i \\ X_i \end{bmatrix}, \text{ where } A_n = \sum_i \begin{bmatrix} \hat{E}_i & \hat{E}_i X_i^T \\ X_i \hat{E}_i & X_i X_i^T \end{bmatrix}$$

and  $\hat{E}_i$  is given by (1.3).

We will derive the asymptotic distribution of  $n^{1/2} \hat{\theta}_n$  under  $H_0$ . Hence we will be able to test the validity of  $H_0$  by simply testing the significance of  $\theta$ . Rejection of this simple hypothesis would imply that  $H_0$

is incorrectly specified. Note that in our context, we are not interested in reversing the order of  $H_0$  and  $H_A$  to test one against the other as it is traditionally the case with nonnested tests. In our framework  $H_A$  is not parametrically specified and hence it is not of specific interest to the researcher. In the event that one rejects the hypothesis that  $\theta$  is zero, then the researcher might want to parameterize  $H_0$  differently to account for the possible effect of the Z-variables on Y.

In order to derive the asymptotic distribution of  $n^{1/2}\hat{\theta}_n$  under  $H_0$ , we need the following regularity conditions :

- K.1  $\overline{\lim}_{n \rightarrow \infty} \max_i C_i(k) < \infty.$
- K.2  $nk^{-\nu/2} \rightarrow 0, kn^{-1} \rightarrow 0$  as  $n \rightarrow \infty$  for  $\nu > 2.$
- A.1  $\mathbb{E}(XX^T)$  is positive definite (p.d.).
- A.2  $\text{var}(Y|X, Y) = \sigma^2 > 0$  a. s.
- A.3  $\mathbb{E}\{\text{var}[\mathbb{E}(Y|Z)|X]\} > 0$  a. s.
- A.4  $\mathbb{E}\|X\|^\nu < \infty, \mathbb{E}|Y - X^T\beta_0|^{2\nu/(\nu-2)} < \infty,$  with  $\nu$  defined in K.2

Conditions K1 and K2 are sufficient for the consistency of k-NN weights, see Stone (1977). Robinson (1987) found that it was technically convenient to relate the rate of convergence of k to the moment conditions imposed on the regressors and residuals. Conditions A1 and A3 ensure that the asymptotic variance of  $n^{1/2}(\hat{\theta}_n \hat{b}_n)^T$  under  $H_0$  is positive definite. Note that A3 rules out situations where  $\mathbb{E}(Y|Z)$  only depends on X or Y is independent of X. The moment conditions in A4 are needed for the asymptotic analysis. It seems difficult to relax them without imposing a stronger rate of convergence in K2. These type of moment conditions were also needed in Newey (1989).

The following theorem is proved in the appendix:

**Theorem 1:** If K.1-K.2, A.1-A.4 hold then if under  $H_0: E(Y|X,Z) = X^T\beta_0$  a.s. we have that

$$(i) \quad n^{1/2}\hat{\theta}_n \xrightarrow{d} N\left[0, \sigma^2/E\{\text{var}[E(Y|Z)|X]\}\right].$$

$$(ii) \quad \hat{\sigma}^2 \left\{ n^{-1} \sum_1 \left[ \hat{E}_i - X_1^T \hat{\beta} \right]^2 \right\}^{-1} \xrightarrow{p} \sigma^2/E\{\text{var}[E(Y|Z)|X]\}.$$

$$\text{where } \hat{\beta} = \left[ \sum_1 X_1 X_1^T \right]^{-1} \sum_1 X_1 Y_1 \quad \text{and} \quad \hat{\sigma}^2 = n^{-1} \sum_1 \left[ Y_1 - X_1^T \hat{\beta} \right]^2.$$

The implication of the theorem is that the t-statistic on the significance of  $\theta$  under  $H_0$  in (1.2) has asymptotically a standard normal distribution. The above statistic is similar in spirit to the Davidson and MacKinnon (1981) J-test, where  $b$  and  $\theta$  are estimated jointly.

Below we will derive the asymptotic distribution of  $n^{1/2} \hat{\theta}_n$  under a series of local alternatives. Hence, if the null hypothesis is false, the test statistic will have power to reject it. The proof is given in the appendix.

**Theorem 2** If K1,K2, A1-A4 hold and  $E(Y_1 | X_1, Z_1) = X_1^T \beta_0 (1 - n^{-1/2} \theta) + n^{-1/2} \theta E(Y_1 | Z_1)$  a.s. then

$$n^{1/2}\hat{\theta}_n \xrightarrow{d} N\left[\theta, \sigma^2/E\{\text{var}[E(Y|Z)|X]\}\right]$$

We can easily accommodate the case where  $H_0: E(Y|X,Z) = f(\beta_0, X)$ , where  $f(\cdot)$  is assumed to be a known parametric function and the case that  $Y$  is assumed to be multivariate. Conditional heteroskedasticity under  $H_0$  does not affect the normality result, but the asymptotic variance of the estimator of  $\theta$  and  $b$  will be different. In this case, one can obtain a consistent estimator of the asymptotic variance, see Eicker (1963) and White (1980).

Under heteroskedasticity of unknown form, an asymptotically efficient



estimator of  $\theta$  under  $H_0$  may be obtained by a semiparametric weighted least squares procedure. One can estimate  $\text{var}\{Y|X,Z\}$  from the squared residuals computed under the null hypothesis, see Carroll (1982), Robinson (1987) or Newey (1987). A pure nonparametric estimator of  $\text{var}\{Y|X,Z\}$  may alternatively be obtained from nonparametric estimators of  $E\{Y^2|X,Z\}$  and  $E\{Y|X,Z\}$ , see Delgado (1989). The method we propose is adaptive in the sense that the asymptotic variance of  $n^{1/2}\hat{\theta}_n$  is the same as the one that could have been obtained had the alternative hypothesis  $H_A$  been known or fully parameterized.

### 3. Monte Carlo Simulations

In this section we will investigate the small sample performance of the proposed test statistic by examining its size and power properties in the context of some monte carlo experiments. We take  $H_0$  to be linear as  $H_0: Y = \beta_0 + X_1\beta_1 + X_2\beta_2 + u_0$ . We consider two alternative hypotheses  $H_{1A}$  and  $H_{2A}$  for two sets of experiments, where  $H_{1A}: Y = \gamma_0 + (\gamma_1 Z_1 + \gamma_2 Z_2)^2 + u_1$  and  $H_{2A}: Y = \delta_0 + \exp\{\delta_1 Z_1 + \delta_2 Z_2\} + u_1$ . The parameters  $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2, \delta_0, \delta_1, \delta_2$  are set to unity. The  $X$ 's are generated as  $NID(0,1)$  variates and the error terms are generated independently of the regressors as  $N(0, \sigma^2)$ . By choosing different values of  $\sigma$  we control the fit of the data generating process. For instance, under  $H_0$  which is linear a  $\sigma$  of 0.33 corresponds to a squared correlation coefficient between  $y$  and the  $X$ 's of 0.9483, whereas a  $\sigma$  of 7 corresponds to one of 0.0392. We generate the  $Z$ 's as  $Z_i = \lambda X_i + v_i$ , where  $v_i$  is distributed as  $NID(0,1)$ ,  $i=1,2$ . By varying  $\lambda$  we control the correlation coefficient between the  $Z_i$ 's and the  $X_i$ 's. When  $\lambda$  is 1, the correlation coefficient between  $Z_i$  and  $X_i$  is 0.71, whereas when  $\lambda$  is 0.1, the latter is 0.1. We have used two  $k$ -NN estimates of  $E(Y|Z)$ , one with  $k=n^{1/2}$  and the

other with  $k=n^{2/3}$ . We have chosen sample sizes of  $n=25, 100, 1000$ , which correspond to very small and to moderate sizes for real world cross sections. All the programs were written in FORTRAN double precision and they were run on the VAX of Indiana University. The normal variates were generated by the G05DDF routine of NAG-13. The analysis of sample size  $n=1000$  proved to be quite expensive computationally. We only performed 250 replications in that case, whereas for  $n=25$  and  $n=100$  we performed 10000 and 2500 replications respectively. In both the size and the power experiments we consider also as a benchmark the t-statistic on the significance of  $\theta$  from the regression  $Y = X^T b + \theta E(Y|Z) + u$ , where  $E(Y|Z)$  takes the exact value from  $H_{1A}$  and  $H_{2A}$  respectively. The above t-statistic will outperform our statistic because it uses exact information that is unavailable to the researcher.

Table 1 presents the results of the size experiments. There is a tendency for our statistic to over-reject, although the proportion decreases as the sample size increases for  $\lambda=1$ , for the different choices of  $\sigma$ . The test performs quite poorly in the case of  $\lambda=0.1$ , but this is to be expected since the X's and the Z's are nearly orthogonal. In that case the denominator of the t-statistic approaches zero. The same problem exists in the case, when  $E(Y|Z)$  is completely parameterized as with the J-test of Davidson and MacKinnon (1981). The k-NN estimator with  $k=n^{1/2}$  performs better than the one with  $k=n^{2/3}$ . Tables 2 and 3 present the power results under  $H_{1A}$  and  $H_{2A}$  respectively. Except for the case of very small samples with a large  $\sigma$  or a small  $\lambda$ , the power results seem quite encouraging. Also as the sample size increases the results improve noticeably. In short the monte carlo results are mixed. The test displays a tendency to over-reject under the null, but it also seems to have

considerable power.

#### 4. Conclusions

In this note we have proposed a test statistic that tests a parametric formulation of a null hypothesis against a weakly specified alternative. The procedure we follow resembles in spirit the artificial nesting technique of Davidson and MacKinnon (1981). By means of a monte carlo we investigated the small sample properties of this test and we found them to be satisfactory with respect to power, but less so with respect to size. The test statistic is derived in the context of iid data and the extension to dependent data is left for future research.

Appendix: Proof of the Theorems

Proof of Theorem 1

Below we will present the lemmas that are used in the proof of the theorem. We define throughout as  $\tilde{E}_1 = \sum_j \mathbb{E}(Y_j | Z_j) \omega_{1j}$ , where  $\omega_{1j} = \omega_{1j}(k)$ . We need the following preliminary lemmas. Since the proofs to these lemmas constitute only minor modifications to the proofs of lemmas 1,7,8 and 9 in Robinson (1987), they are omitted.

Lemma 1: Let  $f(\cdot)$  be a Borel function such that  $\mathbb{E}|f(Z)|^p < \infty$ ,

for some  $p \geq 1$ . Then

$$\mathbb{E} \left\{ \sum_j |f(Z_j) - f(Z_1)|^p \omega_{1j} \right\} = o(1).$$

Lemma 2 : For any  $p \leq \nu$ ,  $\mathbb{E}|\hat{E}_1 - \tilde{E}_1|^p = O(k^{-p/2})$ .

Lemma 3 :  $\max_i |\hat{E}_1 - \tilde{E}_1| = O_p(n^{1/\nu} k^{-1/2})$ .

We have that

$$\begin{aligned} \begin{bmatrix} \hat{\theta}_n \\ \hat{b}_n \end{bmatrix} &= A_n^{-1} \sum_i \begin{bmatrix} \hat{E}_1 \\ X_i \end{bmatrix} \left\{ \begin{bmatrix} E_i & , X_i^T \end{bmatrix} \begin{bmatrix} \theta \\ b \end{bmatrix} + \varepsilon_i \right\} \\ &= A_n^{-1} \sum_i \begin{bmatrix} \hat{E}_1 \\ X_i \end{bmatrix} \left\{ \begin{bmatrix} (\hat{E}_1 & , X_i^T) + (E_i - \hat{E}_1, 0) \end{bmatrix} \begin{bmatrix} \theta \\ b \end{bmatrix} + \varepsilon_i \right\} \\ &= \begin{bmatrix} \theta \\ b \end{bmatrix} + A_n^{-1} \sum_i \begin{bmatrix} \hat{E}_1 \\ X_i \end{bmatrix} \left\{ \begin{bmatrix} E_i - \hat{E}_1, 0 \end{bmatrix} \begin{bmatrix} \theta \\ b \end{bmatrix} + \varepsilon_i \right\} \end{aligned}$$

where  $\varepsilon_i = Y_i - \mathbb{E}(Y_i | X_i, Z_i)$ . Under  $H_0: \theta=0$ ,

$$\begin{bmatrix} \hat{\theta}_n \\ \hat{b}_n \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} + A_n^{-1} \sum_i \begin{bmatrix} E_i \\ X_i \end{bmatrix} \varepsilon_i$$

Let

$$V_0 = \mathbb{E} \begin{bmatrix} [\mathbb{E}(Y|Z)]^2 & \mathbb{E}(Y|Z)X^T \\ X\mathbb{E}(Y|Z) & XX^T \end{bmatrix} = \begin{bmatrix} \mathbb{E}[\mathbb{E}(Y|Z)^2] & \beta_0^T \mathbb{E}(XX^T) \\ \mathbb{E}(XX^T)\beta_0 & \mathbb{E}(XX^T) \end{bmatrix}$$

We need to prove that

$$(a.1) \quad n^{1/2} \begin{bmatrix} \hat{\theta}_n \\ \hat{b}_n - b \end{bmatrix} \xrightarrow{d} N \left\{ 0, \sigma^2 V_0^{-1} \right\}$$

which implies that

$$n^{1/2} \hat{\theta}_n \xrightarrow{d} N \left[ 0, \sigma^2 / \mathbb{E}\{\text{var}[\mathbb{E}(Y|Z)|X]\} \right]$$

Note that under  $H_0$   $\mathbb{E}\{\text{var}[\mathbb{E}(Y|Z)|X]\} = \mathbb{E}[\mathbb{E}(Y|Z)^2] - \beta_0^T \mathbb{E}(XX^T)\beta_0$ .

We prove (a.1) from

$$(a.2) \quad n^{-1/2} \sum_1 \begin{bmatrix} E_1 \\ X_1 \end{bmatrix} \varepsilon_1 \xrightarrow{d} N(0, \sigma^2 V_0)$$

$$(a.3) \quad n^{-1} A_n - V_0 = o_p(1)$$

$$(a.4) \quad n^{-1/2} \sum_1 (\hat{E}_1 - E_1) \varepsilon_1 = o_p(1)$$

By A1-A4,  $\begin{bmatrix} E_1 \\ X_1 \end{bmatrix} \varepsilon_1$  are iid with zero mean and finite variance

and (a.2) follows from the Lindenberg-Levy Central Limit Theorem.

We conclude (a.3) from,

$$(a.5) \quad n^{-1} \sum_1 \left[ E_1^2 - \mathbb{E}(E_1^2) \right] = o_p(1)$$

$$(a.6) \quad n^{-1} \sum_1 \left[ E_1 X_1 - \mathbb{E}(E_1 X_1) \right] = o_p(1)$$

$$(a.7) \quad n^{-1} \sum_1 \left[ X_1 X_1^T - \mathbb{E}(X_1 X_1^T) \right] = o_p(1)$$

$$(a.8) \quad n^{-1} \sum_i \left[ \hat{E}_i^2 - \tilde{E}_i^2 \right] = o_p(1)$$

$$(a.9) \quad n^{-1} \sum_i \left[ \tilde{E}_i^2 - E_i^2 \right] = o_p(1)$$

$$(a.10) \quad n^{-1} \sum_i \left[ (\hat{E}_i - \tilde{E}_i) X_i \right] = o_p(1)$$

$$(a.11) \quad n^{-1} \sum_i \left[ (\tilde{E}_i - E_i) X_i \right] = o_p(1)$$

(a.5) to (a.7) follow from the law of large numbers (LLN). In order to prove (a.8) note that its left side is bounded by

$$\max_i |\hat{E}_i - \tilde{E}_i| \cdot n^{-1} \sum_i |\hat{E}_i - \tilde{E}_i| = o_p(1),$$

using lemmas 2 and 3 above. Also (a.9) follows from lemma 1. The left side of (a.10) is bounded by

$$\left[ n^{-1} \sum_i (\hat{E}_i - \tilde{E}_i)^2 \right]^{1/2} \left[ n^{-1} \sum_i \|X_i\|^2 \right]^{1/2} = O_p(k^{-1/2})$$

using Markov's inequality and lemma 2. The left side of (a.11) is bounded by

$$\left[ n^{-1} \sum_i (\tilde{E}_i - E_i)^2 \right]^{1/2} \left[ n^{-1} \sum_i \|X_i\|^2 \right]^{1/2} = o_p(1)$$

using lemma 1. Finally we conclude (a.4) from

$$(a.12) \quad n^{-1/2} \sum_i \left[ (\hat{E}_i - \tilde{E}_i) \varepsilon_i \right] = o_p(1)$$

$$(a.13) \quad n^{-1/2} \sum_i \left[ (\tilde{E}_i - E_i) \varepsilon_i \right] = o_p(1)$$

(a.12) follows from Chebyshev's inequality, noting that

$$\mathbb{E} \left[ n^{-1/2} \sum_i \left[ (\hat{E}_i - \tilde{E}_i) \varepsilon_i \right] \right]^2 = C_1 + C_2$$

where

$$\begin{aligned} C_1 &= \mathbb{E} \left\{ |\hat{E}_1 - \tilde{E}_1|^2 |\varepsilon_1|^2 \right\} \\ &\leq \left\{ \mathbb{E} |\hat{E}_1 - \tilde{E}_1|^\nu \right\}^{2/\nu} \left\{ \mathbb{E} |\varepsilon_1|^{2\nu/(\nu-2)} \right\}^{(\nu-2)/\nu} = o(k^{-1}) \end{aligned}$$

by Hölder's inequality and lemma 2, and

$$\begin{aligned} C_2 &= \mathbb{E} \left[ n^{-1} \sum_{i \neq j} (\hat{E}_i - \tilde{E}_i) (\hat{E}_j - \tilde{E}_j) \varepsilon_i \varepsilon_j \right] \\ &\leq \left[ \mathbb{E} \left[ n^{-1} \sum_{i \neq j} (\hat{E}_i - \tilde{E}_i)^2 \varepsilon_j^2 \right] \right]^{1/2} \left[ \mathbb{E} \left[ n^{-1} \sum_{i \neq j} (\hat{E}_j - \tilde{E}_j)^2 \varepsilon_i^2 \right] \right]^{1/2} \end{aligned}$$

Each of these two expectations is by Hölder's inequality bounded by

$$\left[ \mathbb{E} |\varepsilon_1|^{2\nu/(\nu-2)} \right]^{(\nu-2)/\nu} \left[ n \mathbb{E} |\hat{E}_1 - \tilde{E}_1|^\nu \right]^{2/\nu} = o(n^{2/\nu} k^{-1})$$

using lemma 2. Also (a.13) follows from Chebyshev's inequality,

since by lemma 1,

$$\mathbb{E} \left[ \left| n^{-1/2} \sum_i (\tilde{E}_i - E_i) \varepsilon_i \right|^2 \right] = \mathbb{E} \left[ |\tilde{E}_1 - E_1|^2 \varepsilon_1^2 \right]$$

$$= \sigma^2 \mathbb{E} |\tilde{E}_1 - E_1|^2 = o(1).$$

It is straightforward to show that  $\hat{\sigma}^2 - \sigma^2 = o_p(1)$  under  $H_0$ . Hence part (ii)

of the theorem follows from

$$(a.14) \quad n^{-1} \sum_i (\hat{E}_i^2 - E_i^2) = o_p(1)$$

$$(a.15) \quad n^{-1} \sum_i \left[ \hat{\beta}^T X_i X_i^T \hat{\beta} - \beta_0^T \mathbb{E}(X X^T) \beta_0 \right] = o_p(1)$$

$$(a.16) \quad n^{-1} \sum_i \left[ \hat{E}_i X_i^T \hat{\beta} - \beta_0 \mathbb{E}(XX^T) \beta_0 \right] = o_p(1)$$

(a.14) follows from (a.8) and (a.9) above, whereas (a.15) follows from the LLN noting that  $\hat{\beta}$  is  $n^{1/2}$ -consistent under  $H_0$ . Using this fact and (a.10) and (a.11) we obtain (a.16). ■

Proof of Theorem 2

Let  $b_0 = \beta_0(1-n^{-1/2}\theta)$ . From the proof of Theorem 1,

$$\begin{bmatrix} n^{1/2} \hat{\theta}_n - \theta \\ n^{1/2} (\hat{b}_n - b_0) \end{bmatrix} = [n^{-1} A_n]^{-1} n^{-1/2} \sum_i \begin{bmatrix} \hat{E}_i \\ X_i \end{bmatrix} \left\{ \begin{bmatrix} \hat{E}_i - E_i, 0 \end{bmatrix} \begin{bmatrix} \theta n^{-1/2} \\ b_0 \end{bmatrix} + \varepsilon_i \right\}$$

Given the proof of Theorem 1, for the proof of theorem 2 it suffices to prove that

$$(b.1) \quad n^{-1} \sum_i \hat{E}_i \left[ \hat{E}_i - E_i \right] = o_p(1)$$

$$(b.2) \quad n^{-1} \sum_i X_i \left[ \hat{E}_i - E_i \right] = o_p(1)$$

(b.1) follows from

$$(b.3) \quad n^{-1} \sum_i \hat{E}_i \left[ \hat{E}_i - \tilde{E}_i \right] = o_p(1)$$

$$(b.4) \quad n^{-1} \sum_i \hat{E}_i \left[ \tilde{E}_i - E_i \right] = o_p(1)$$

We prove (b.3) noting that its left side is bounded by

$$\max_i |\hat{E}_i - \tilde{E}_i| \cdot n^{-1} \sum_i \hat{E}_i = o_p(1)$$

by lemma 1,2,3 and the LLN. (b.4) is bounded by

$$\left[ n^{-1} \sum_i \hat{E}_i^2 \right]^{1/2} \left[ n^{-1} \sum_i [\hat{E}_i - E_i]^2 \right]^{1/2} = o_p(1)$$

by lemma 1, (a.7), (a.8) and (a.11). The proof of (b.2) is identical to the proof of (b.1). ■



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Table 1\*

Size Results: Proportion of Rejections when  $H_0$  is true

n =25; Number of Replications = 10000;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	0.3886	0.2610	0.4861	0.3715	0.5533	0.4406
	0.5376	0.4262	0.6458	0.5476	0.7009	0.6200
	0.1596	0.0631	0.1599	0.0644	0.1805	0.0729
	0.2034	0.0924	0.2112	0.1046	0.2241	0.1083
$\sigma = 2$	0.4330	0.3130	0.4626	0.3479	0.4849	0.3661
	0.6369	0.5413	0.6518	0.5568	0.6710	0.5828
	0.1621	0.0660	0.1661	0.0687	0.1858	0.0781
	0.2046	0.0945	0.2142	0.1089	0.2277	0.1157
$\sigma = 7$	0.4083	0.2683	0.3778	0.2506	0.3705	0.2410
	0.6330	0.5222	0.5736	0.4600	0.5519	0.4441
	0.1805	0.0797	0.2085	0.1053	0.2379	0.1274
	0.2298	0.1168	0.2689	0.1609	0.2937	0.1783

n =100; Number of Replications = 2500;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	0.2688	0.1460	0.4652	0.3316	0.6156	0.5056
	0.3968	0.2548	0.5740	0.4572	0.7332	0.6500
	0.0852	0.0244	0.0936	0.0236	0.1020	0.0280
	0.1404	0.0504	0.1496	0.0484	0.1484	0.0552
$\sigma = 2$	0.3132	0.1956	0.4088	0.2936	0.4960	0.3716
	0.4660	0.3496	0.5688	0.4680	0.6632	0.5668
	0.0856	0.0248	0.0956	0.0240	0.1036	0.0288
	0.1416	0.0504	0.1540	0.0504	0.1524	0.0564
$\sigma = 7$	0.2668	0.1556	0.2860	0.1744	0.2832	0.1708
	0.4464	0.3328	0.4240	0.3100	0.4304	0.3224
	0.0852	0.0240	0.0996	0.0272	0.1140	0.0348
	0.1448	0.0560	0.1708	0.0620	0.1640	0.0700

\*The first row corresponds to the number of rejections, under  $H_0$ , when  $E(Y|Z)$  is estimated by k-NN, where  $k = n^{1/2}$ . The second uses  $k = n^{2/3}$ . The third row corresponds to the benchmark t-statistic, when the added regressor is given by  $1+\exp\{Z_1+Z_2\}$  and the fourth when the added regressor is given by  $1+(Z_1+Z_2)^2$ , i.e when  $E(Y|Z)$  is perfectly known.

Table 1 (continued)

n =1000; Number of Replications = 250;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	0.2000	0.0720	0.4800	0.3320	0.7200	0.6560
	0.2520	0.1080	0.5240	0.3520	0.8320	0.7960
	0.0520	0.0120	0.0720	0.0280	0.0560	0.0360
	0.1480	0.0600	0.1360	0.0520	0.1560	0.0640
$\sigma = 2$	0.2440	0.1120	0.4640	0.3320	0.6800	0.5800
	0.2920	0.1640	0.5160	0.4040	0.7800	0.7160
	0.0520	0.0120	0.0720	0.0280	0.0600	0.0360
	0.1480	0.0560	0.1400	0.0520	0.1600	0.0600
$\sigma = 7$	0.2280	0.1200	0.3040	0.2040	0.3720	0.2240
	0.3000	0.2080	0.4120	0.3080	0.5520	0.4240
	0.0520	0.0120	0.0760	0.0280	0.0760	0.0280
	0.1480	0.0560	0.1360	0.0520	0.1560	0.0600

Table 2\*

Power Results: Proportion of Rejections when  $H_{1A}$  is true  
 $n = 25$ ; Number of Replications = 10000;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	0.9830	0.9645	0.9818	0.9614	0.9809	0.9572
	0.9418	0.9022	0.9422	0.9039	0.9420	0.9013
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 2$	0.8885	0.8140	0.7251	0.5948	0.6541	0.5137
	0.8212	0.7490	0.6887	0.5823	0.6359	0.5166
	0.9998	0.9993	0.9952	0.9862	0.9910	0.9716
$\sigma = 7$	0.3701	0.2311	0.3143	0.1882	0.3090	0.1826
	0.4772	0.3522	0.4628	0.3381	0.4436	0.3266
	0.8933	0.7747	0.7047	0.5297	0.6514	0.4724

$n = 100$ ; Number of Replications = 2500;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 2$	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 7$	0.9428	0.8884	0.5556	0.4120	0.4536	0.2892
	0.9536	0.9112	0.6000	0.4680	0.4972	0.3528
	1.0000	1.0000	0.9908	0.9648	0.9716	0.9068

\*The first row corresponds to the number of rejections, under  $H_{1A}$ , when  $E(Y|Z)$  is estimated by  $k$ -NN, where  $k = n^{1/2}$ . The second uses  $k = n^{2/3}$ . The third row corresponds to the benchmark  $t$ -statistic, when the added regressor is given by  $1+(Z_1+Z_2)^2$ , i.e when  $E(Y|Z)$  is perfectly known.

Table 2 (continued)

n =1000; Number of Replications = 250;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 2$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 7$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3\*

Power Results: Proportion of Rejections when  $H_{2A}$  is true  
 $n = 25$ ; Number of Replications = 10000;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	0.8896	0.8241	0.9857	0.9655	0.9940	0.9844
	0.8439	0.7653	0.9841	0.9603	0.9959	0.9864
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 2$	0.8194	0.7194	0.8483	0.7501	0.8430	0.7304
	0.7734	0.6664	0.8615	0.7671	0.8621	0.7584
	0.9990	0.9966	0.9938	0.9850	0.9911	0.9776
$\sigma = 7$	0.5189	0.3912	0.4106	0.2649	0.3757	0.2395
	0.5537	0.4310	0.4925	0.3589	0.4735	0.3359
	0.9402	0.8861	0.8288	0.7140	0.7904	0.6553

$n = 100$ ; Number of Replications = 2500;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	0.9960	0.9992	1.0000	1.0000	1.0000	0.9996
	0.9980	0.9980	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 2$	0.9996	0.9992	1.0000	1.0000	1.0000	0.9996
	0.9980	0.9976	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 7$	0.9900	0.9808	0.8920	0.8276	0.8176	0.7116
	0.9932	0.9840	0.9128	0.8684	0.8508	0.7772
	1.0000	1.0000	0.9972	0.9952	0.9964	0.9844

\*The first row corresponds to the number of rejections, under  $H_{2A}$ , when  $E(Y|Z)$  is estimated by  $k$ -NN, where  $k = n^{1/2}$ . The second uses  $k = n^{2/3}$ . The third row corresponds to the benchmark  $t$ -statistic, when the added regressor is given by  $1 + \exp\{Z_1 + Z_2\}$ , i.e  $E(Y|Z)$  is perfectly known.

Table 3 (continued)

n =1000; Number of Replications = 250;

	$\lambda = 1$		$\lambda = 0.43$		$\lambda = 0.1$	
	5%	1%	5%	1%	5%	1%
$\sigma = 0.33$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 2$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\sigma = 7$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000