# Soil resource and the profitability and sustainability of farms: A soil quality investment model

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# Soil resource and the profitability and sustainability of farms: A soil quality investment model

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## Abstract

There is a growing public concern for soils and the maintenance or enhancement of soil quality. Actually, soil resource plays a central role in issues regarding food security and climate change mitigation. Through their practices, farmers impact the physical, biological and chemical quality of their soils. However, in a strained economic environment, farmers face a trade-off between short term objectives of production and profitability, and a long term objective of soil resource conservation. In this article, we investigate the conditions under which farmers have a private interest to preserve the quality of their soil. We also characterize the optimal management strategies of soil quality dynamics. We use a simplified theoretical soil quality investment model, where farmers maximise their revenues under a soil quality dynamics constraint. In our production function, soil quality and productive inputs are cooperating production factors. In addition, productive inputs have a detrimental impact on soil quality dynamics. It appears that in some cases, farmers have a private and financial interest in preserving the quality of their soil at a certain level, since it is an endogenous production factor cooperating with productive inputs. However, situations can occur wherein the cooperative production benefits of soil quality and productive inputs are smaller than the marginal deterioration of soil quality due to productive inputs. In this case, one cannot draw conclusions about the existence of an equilibrium.



**Keywords:** optimal control, soil quality, endogenous production factor

**JEL Classification:** D90, Q10, Q24

**La ressource sol, et la rentabilité et la durabilité des exploitations agricoles : un modèle d'investissement dans la qualité des sols**

**Résumé**

Il y a un intérêt public croissant pour les sols et le maintien ou l'amélioration de la qualité des sols. En effet, la ressource sol a un rôle important en matière de sécurité alimentaire ou de réchauffement climatique. Par leurs pratiques, les agriculteurs impactent la qualité physique, biologique et chimique de leurs sols. Cependant, dans un contexte économique tendu, les agriculteurs font face à un arbitrage entre les objectifs de rendement et de rentabilité de court terme, et l'objectif de long terme qu'est la conservation de leurs sols. Dans cet article, nous étudions les conditions dans lesquelles les agriculteurs ont un intérêt privé à conserver la qualité de leurs sols. Nous caractérisons également les stratégies optimales de gestion de la dynamique de la qualité des sols. Nous utilisons un modèle théorique simplifié d'investissement dans la qualité des sols, où l'agriculteur maximise son profit sous contrainte de la dynamique de la qualité des sols. Dans la fonction de production, la qualité du sol et les intrants productifs sont des facteurs coopérants. De plus, les intrants productifs ont un impact négatif sur la dynamique de la qualité du sol. Il s'avère que dans certains cas, les agriculteurs ont un intérêt privé et financier à maintenir la qualité de leurs sols à un niveau donné, puisque la qualité du sol est un facteur de production endogène qui coopère avec les intrants productifs. Néanmoins, il y a des situations où les bénéfices de la coopération entre intrants productifs et qualité des sols en matière de production sont moins importants que la dégradation marginale de la qualité du sol causée par l'usage des intrants productifs. Dans ces cas, on ne peut conclure sur l'existence d'un équilibre.

**Mots-clés :** contrôle optimal, qualité du sol, facteur de production endogène

**Classification JEL :** D90, Q10, Q24

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## 1. Introduction

The importance of soil resource is increasingly recognized in the political sphere, and is at the center of the "4/1000 Initiative: Soil for Food Security and Climate", an international and multi-stakeholder voluntary action plan, presented at the 21st Session of the Conference of the Parties to the United Nations Framework Convention on Climate Change (COP21) in Paris on December 1$^{st}$, 2015. Soil resources are a common good. Moreover, soil quality plays an important role in food security and climate change mitigation, both as a production factor (*e.g.* through carbon sequestration) and a regulation factor (Lal, 2004). This explains the resurgence of public and political interest in soils.

Farmers impact the physical, chemical and biological quality of soil through their farming practices. These impacts can be positive or negative, depending on implemented farming practices, the location of the land, climate, and initial soil quality. Some farming practices are more likely to benefit soil quality than others (Chitrit and Gautronneau, 2011). For instance, agroecology practices related to the soil resource aim at maintaining or increasing soil quality. The concept of agroecology used here refers to an intensified use of natural processes and resources, including the soil resource, by implementing conservation practices (*e.g.* superficial tillage, cover crops, intercropping).

Domestic and international pressure to reduce agricultural support is growing, while the price of energy, fertilizers and other crop production costs are expected to increase. Under these circumstances, farmers have fewer levers to increase revenues (or limit losses) and the role of soil quality becomes increasingly important. However, farming practices aiming to preserve or improve soil quality imply more complex agricultural procedures than the ones used in conventional agriculture, and require farmers to invest in innovation and research (Ghali, 2013). Hence, there seems to be a trade-off between the short-term production and profitability objective and the long term soil conservation objective. Due to the complexity of the mechanisms and dynamics involved in the relationships between soil quality, farming practices and production, the modelling of this trade-off is needed to derive and characterize the set of objective and rational decisions taken by farmers.

The present paper focuses on farmer's decision making process regarding the management of his/her soil quality. In the economic literature, optimal control models have been used to understand farmers' motivation to invest or not in conservation practices (Saliba, 1985; Barbier, 1998), since there may be a conflict between their profitability and sustainability objectives (Segarra and Taylor, 1987; Barbier, 1990; Quang *et al.*, 2014).

Our work is based on the theoretical optimal control models of McConnell (1983), Saliba (1985), Barbier (1990) and Hediger (2003). Although these dynamic models incorporate soil quality, they consider only the dynamics related to soil erosion (soil loss) and soil depth (McConnell, 1983; Saliba, 1985; Barbier, 1990; Hediger, 2003). Considering only one characteristic of soil quality, such as soil depth, is quite reductive with regard to the many aspects of soil quality that may impact soil productivity (*e.g.* physical, chemical, biological). Barbier (1990) and Hediger (2003) provide some interesting analyses of the optimal equilibrium, but do not thoroughly discuss the dynamics of this equilibrium and the optimal paths. The present paper aims to fill in this gap.

The objective of this article is to determine under which conditions farmers have a private interest in maintaining or increasing soil quality, as they maximize their revenue under a soil quality dynamics constraint. To do so, we propose a soil quality investment model where soil quality dynamics refer to the crop production system. We discuss extensively the equilibrium of the model and the dynamics of this equilibrium. Our discussion is structured around two main points: (1)how the optimal equilibrium is characterized, and under which conditions the optimal path to attain the equilibrium corresponds to agroecology practices; and (2) the reaction of the system to changes in the economic environment (crop prices, time preferences, costs). Soil quality is included in the model through the physical, chemical and biological attributes of soil, such as soil depth, soil acidity and soil flora and fauna, all being affected by farming practices. We adopt an infinite time planning horizon and investigate how changes in soil quality affect the sustainability of farms. Sustainability is defined here as the farmer's ability to maintain his/her production and profitability throughout time. Besides, it allows to introduce a discount rate, assimilated here to farmer's time preferences.

In section 2 we present the functional relationships between soil quality, agricultural practices and crop production and analyse the optimal soil quality investment model. Section 3 discusses the comparative statics and dynamics of our model. While comparative statics allow to determine how the steady state is impacted by a change in parameters, local comparative dynamics provide information as to how the approach path is affected. The two approaches are complementary (Caputo, 2005). Finally, in section 4 we present some stylized facts to illustrate the dynamics of our model.

## 2. The farmer's decision making process

### 2.1. Soil quality, agricultural practices and crop production

In the soil quality investment model proposed below, farmers maximise their profit. The profit is equal to the crop yield multiplied by the crop price, minus the costs of farming practices, which are subject to soil quality dynamics.

Two types of farming practices are identified: (1) the use of productive inputs $m$ (*e.g.* fertilizers, pesticides); and (2) conservation practices $u$ considered as an investment in soil quality. Investments in soil quality $m$ correspond to the additional costs induced, for instance, by incorporating green manure within the crop rotation, leaving crop residues, or adopting superficial tillage or no-tillage. These extra costs also encompass the costs induced by a more complex management of the production system.

There are two production factors, soil quality and productive inputs. The crop production function is represented by $y = \phi(q, m)$. Similarly to McConnell (1983), Barbier (1990) and Hediger (2003), soil quality $q$ is composed of endogenous attributes $s$ and exogenous attributes $a$. Exogenous attributes, such as soil type or other site-specific attributes, are fixed, while endogenous attributes vary across farming practices. When the farmer invests in soil quality, he/she invests in the endogenous attributes of soil quality. Contrary to McConnell (1983), Barbier (1990) and Hediger (2003), we consider that soil quality is endogeneous not only with respect to soil depth (the physical dimension of soil quality), but also with respect to soil acidity and soil fauna and flora, as examples of the chemical and biological dimensions of soil quality. We choose these three soil characteristics because they are positively correlated soil quality, are positively impacted by conservation practices $u$, and cooperate with productive inputs $m$.

In reality, there are numerous dimensions of soil quality (physical, chemical or biological) that may have a positive or negative impact on crop production. Moreover, a soil quality characteristic can be positively affected by one practice, and negatively by another practice implemented at the same time. This is a simple model, where the effects of so-called productive inputs and conservation inputs are simplified and exaggerated in order to focus on the qualitative discussion of trade-offs faced by the farmer.

The crop production per hectare $y$ depends on soil quality $q$ and on productive inputs $m$. The latter have a direct impact on production. The production function is twice continuously differentiable ($C^{(2)}$). Let $t$ denote time. Since soil quality exogenous attributes $a$ are fixed, the crop production function can be written as:[1]

$$y(t) = \phi(q(s(t), a), m(t)) = f(s(t), m(t)) \tag{1}$$

$$f_s > 0, f_m > 0, f_{ss} < 0, f_{mm} < 0, \tag{2}$$

$$f_{sm} = f_{ms} > 0, f_{ss}f_{mm} - (f_{ms})^2 > 0 \tag{3}$$

Crop production $f$ is increasing with soil quality ($f_s > 0$) and with productive inputs ($f_m > 0$). There are diminishing marginal returns for both soil quality and productive inputs. Thus, the higher the soil quality, the smaller the observed increase in production ($f_{ss} < 0$). Similarly, the more productive the used inputs, the lower their positive impact ($f_{mm} < 0$). We assume that soil

[1]For reading simplicity, soil quality endogenous attributes are referred to as soil quality in the rest of the article.

quality and productive inputs are cooperating production factors, that work as a team (Alchian and Demsetz, 1972).[2] This means that soil quality amplifies the positive impact of productive inputs and *vice versa* ($f_{sm} > 0$). Hence, crop yield is higher when in addition to the natural production factor (soil quality), the farmer uses fertilizers.

Soil quality changes over time with a natural soil degradation factor $\delta$ and a natural soil formation factor $g$. This approach is similar to McConnell (1983) and Hediger (2003), who account for soil loss/erosion and natural soil regeneration in their models. Barbier (1990) focuses exclusively on soil degradation and assumes that the use of productive inputs accelerates soil erosion.

We analyse both the detrimental effects of productive inputs and the beneficial effects of conservation practices on changes in soil quality. The soil degradation factor $\delta$ is not exogenous: It depends on productive inputs $m$, that are assimilated with farming practices damaging soil quality. For instance, pesticides may have non-desirable negative effects on soil flora and fauna, fertilizers may increase soil acidity (Verhulst *et al.*, 2010), both leading to a decrease in soil productivity.

The farmer can invest in soil quality by adopting conservation practices $u$ that increase the soil regeneration rate $g$. The soil quality dynamics function is $C^{(2)}$ with:

$$\dot{s}(t) = -\delta(m(t))s(t) + g(u(t)) \tag{4}$$

$$\delta_m > 0, \delta_{mm} > 0, g_u > 0, g_{uu} < 0 \tag{5}$$

The natural soil formation factor $g$ depends positively on conservation practices $u$ that increase soil quality ($g_u > 0$). For instance, leaving crop residues on the soil surface decreases erosion (Cutforth and McConkey, 1997; Malhi and Lemke, 2007) and increases the abundance and diversity of soil fauna, while more complex crop rotation decreases the pest and disease pressure (Cook and Haglund, 1991). The positive impact on soil quality of a conservation practice decreases with the number of implemented practices (*i.e.* of farming activities) ($g_{uu} < 0$). On the other hand, the use of productive inputs leads to soil degradation ($\delta_m > 0$). This negative effect on soil quality is amplified by the more intensive use of productive inputs ($\delta_{mm} > 0$).

### 2.2. Optimal soil quality investments

The farmer, owner of his land, maximises his discounted profit over an infinite planning horizon. The profit depends on crop yield, crop prices and the costs of farming practices. The constant marginal cost of productive inputs $m$ is denoted by $c_m$ and the constant marginal cost associated

[2] Alchian and Demsetz (1972) introduce the term *team production* to denote productive processes where inputs generate a higher output together than separately. The output is yielded by the team of inputs $(s, m)$, and cannot be written as the sum of separable outputs of each input.

with conservation practices $u$ is denoted by $c_u$. The marginal costs encompass labour costs and energy costs associated with each activity. The price of the crop, $p$, is constant. Farmer's profit can be written as:

$$\pi(t) = pf(s(t), m(t)) - c_m m(t) - c_u u(t) \tag{6}$$

In addition, the farmer is constrained by the dynamics of soil. Here, the discount rate $r$, comprised between 0 and 1, is interpreted as the time preference of the farmer. When $r$ is equal to 0, the farmer has the same preference for present and future revenues; when $r$ is equal to 1, the farmer only values present revenues. Hence, the farmer faces the following optimisation problem:[3]

$$\underset{m,u}{\text{Max}} \int_0^{T\to\infty} e^{-rt}[pf(s(t), m(t)) - c_m m(t) - c_u u(t)]dt \tag{7}$$

$$\text{subject to:} \quad \dot{s}(t) \tag{8}$$

The current value hamiltonian of this problem can be written as:

$$\tilde{H}(m, u, s, \mu) = pf(s(t), m(t)) - c_m m(t) - c_u u(t) + \mu\dot{s} \tag{9}$$

According to the maximum principle, the optimal paths of $m$, $u$, $s$ and $\mu$ satisfy:

$$\tilde{H}_m = pf_m - c_m - \mu\delta_m s = 0 \tag{10}$$

$$\tilde{H}_u = -c_u + \mu g_u = 0 \tag{11}$$

$$\dot{\mu} - r\mu = -\tilde{H}_s \Leftrightarrow \dot{\mu} = r\mu - pf_s + \delta(m)\mu = \mu(r + \delta(m)) - pf_s \tag{12}$$

Condition (10) indicates that marginal revenues obtained from using more productive inputs must be balanced with their marginal damages on soil quality, expressed in soil quality marginal value. Condition (11) states that conservation practices $u$ should be implemented, such that the costs of conservation inputs $c_u$ are equal to the benefits generated in terms of soil quality marginal value. The costate variable $\mu$, which is the implicit value of soil quality, has a rate of change $\dot{\mu}$ that depends on the discount rate $r$, the degradation rate $\delta$, the current soil quality implicit value $\mu$, the crop price $p$, and the influence of soil quality on crop yield $f_s$ (condition (12)). The implicit value of soil quality grows at the rate of discount and degradation, minus the contribution of soil quality to current profits. In addition, the rate of change of the costate variable, $\dot{\mu}$, also depends on productive inputs, by intensifying the soil degradation rate.

The two conditions (10) and (11), respectively related to productive inputs and conservation practices, are always true at equilibrium and on the optimal paths leading to the equilibrium (when both exist).

[3] Appendix A also contains a particular case of our theoretical model where farmers do not consider soil quality dynamics in their decision making process, which leads to a long-term unsustainable degradation of their soils.

In addition, when (10) and (11) are combined, we obtain:

$$\frac{pf_m - c_m}{\delta_m s} = \frac{c_u}{g_u} = \mu \tag{13}$$

Equation (13) states that, at equilibrium and on the optimal paths, soil quality marginal value is equal to the ratio between the marginal revenues obtained from the use of productive inputs and their marginal damages on soil quality. This ratio must be equal to the ratio between soil conservation costs and the marginal soil restoration.

Along the optimal time paths of the state and costate variables $s$ and $\mu$, management intensity must continuously be adjusted to satisfy at any time, respectively for each case, the first-order condition (10). Similarly, soil quality investment must satisfy (11).

Consequently, management intensity and soil quality investment can be represented as an implicit function of soil quality $s$ and soil marginal value $\mu$:

$$\frac{\partial u}{\partial s} = -\frac{\tilde{H}_{us}}{\tilde{H}_{uu}} = -\frac{0}{\mu g_{uu}} = 0 \tag{14}$$

$$\frac{\partial u}{\partial \mu} = -\frac{\tilde{H}_{u\mu}}{\tilde{H}_{uu}} = -\frac{g_u}{\mu g_{uu}} > 0 \tag{15}$$

According to (15), the implementation of soil conservation practices increases with the soil marginal value. However, a change in soil quality does not trigger a change in soil conservation practices (14).

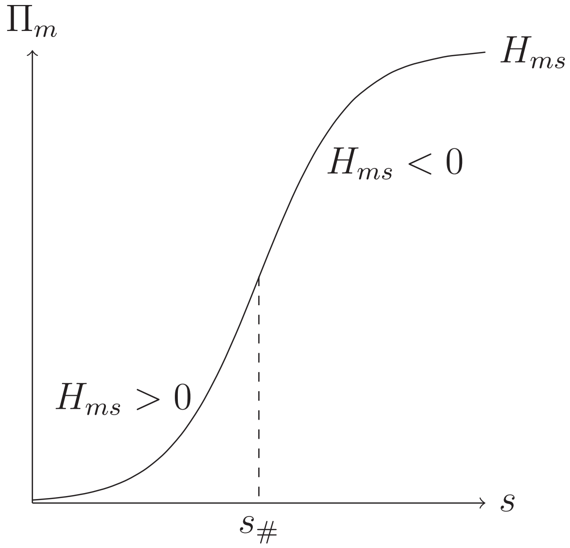
$$\frac{\partial m}{\partial s} = -\frac{\tilde{H}_{ms}}{\tilde{H}_{mm}} = -\frac{pf_{ms} - \mu\delta_m}{pf_{mm} - \mu\delta_{mm} s} \gtreqless 0 \tag{16}$$

$$\frac{\partial m}{\partial \mu} = -\frac{\tilde{H}_{m\mu}}{\tilde{H}_{mm}} = -\frac{-\delta_m s}{pf_{mm} - \mu\delta_{mm} s} < 0 \tag{17}$$

When productive inputs negatively impact soil quality, the use of productive inputs decreases with the marginal value of soil (17). However, the sign of the relationship between productive inputs use and soil quality is ambiguous (16). Indeed, on one hand productive inputs and soil quality are cooperative production factors. On the other hand, the use of productive inputs deteriorates soil quality. Hence, the sign associated with the implicit function of $m$ is undetermined. More specifically, it is the sign of $H_{ms} = pf_{ms} - \mu\delta_m$ that is ambiguous.

Actually, when deciding the amount of productive inputs to be used and the soil quality to be restored, one has to consider the costs and benefits of organizing such a cooperation. When $H_{ms} > 0$, the situation is favorable to the cooperation between productive inputs and soil quality. In this case $\frac{p}{\mu} > \frac{\delta_m}{f_{sm}}$, *i.e.* the ratio between crop price and soil quality value is higher than the ratio between the damages of $m$ on soil quality and the cooperating effect. In the

Figure 1: Soil quality threshold and marginal productivity of productive inputs




opposite case $(H_{ms} < 0)$, we have $\frac{p}{\mu} < \frac{\delta_m}{f_{sm}}$, and it is more difficult to derive conclusions.

Two cases can be distinguished:

1. The case where $H_{ms} > 0$, which can also be written as $pf_{ms} > \mu\delta_m$. In this case the use of productive inputs produces more benefits in terms of revenues than losses in terms of soil quality marginal value.
2. The case where $H_{ms} < 0$, which can also be written as $pf_{ms} < \mu\delta_m$. This is the opposite of case 1. It corresponds to a situation when the marginal damages on soil quality caused by productive inputs are higher than the marginal benefits in terms of productivity.

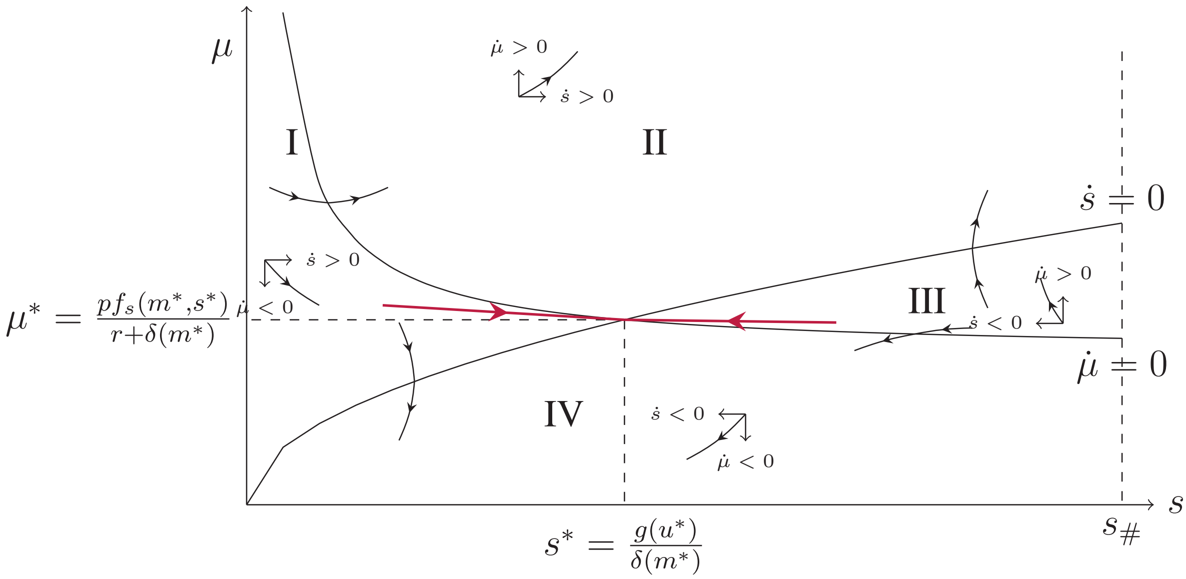
$H_{ms}$ can be rewritten using condition (10), such that:

$$pf_{ms} - \mu\delta_m \gtreqless 0 \Longleftrightarrow \frac{\partial \Pi_m}{\partial s} \gtreqless \frac{\Pi_m}{s} \tag{18}$$

where $\Pi_m/s$ is the marginal profit of productive inputs $m$ per unit of soil quality, and $\partial\Pi_m/\partial s$ is the marginal profit of productive inputs $m$ for one additional unit of soil quality.

There can be a threshold value of soil quality, $s_{\#}$, below which soil quality is sufficiently low for the cooperating marginal productivity of $m$ and $s$ to exceed the marginal damages caused by $m$. Above this threshold, marginal damages are more important than the marginal cooperative productivity. In this case, the shadow value of soil quality $\mu$ is higher than below the threshold $s_{\#}$ (see Figure 1).

Figure 2: Phase diagram when productive inputs damage soil quality




### 2.3. Phase diagram and stability properties of the optimization problem

The solution of our optimal control problem is characterized by a long-term optimum, only when $H_{ms} > 0$. The long term equilibrium is represented in a phase diagram (see Figure 2). The steady state is reached when $\dot{m} = \dot{u} = \dot{s} = \dot{\mu} = 0$. The evolution from an initial state $s_0$ to the steady state $(s^*, \mu^*)$ is achieved through a stable transition path. The stability properties of the problem and the determination of the long term equilibrium are described in Appendix B.

The phase diagram (Figure 2) corresponds to the situation where soil quality is below some soil quality threshold $s_{\#}$. It corresponds only to the case where the damages caused by the use of productive inputs are overcompensated by its cooperating benefits with soil quality in terms of revenue (see Figure 1). For the case above the threshold $s_{\#}$, one cannot determine the existence of an equilibrium, nor represent the phase diagram.

The steady state can only be achieved by pursuing one of the optimal trajectories. The optimal trajectories are represented in the phase diagram by the two directed lines going toward the steady state $(s^*, \mu^*)$ (see Figure 2).

When the initial soil quality is low ($s_0 < s^* < s_{\#}$), the optimal trajectory is located in region I. On this path, the soil quality increases, while the marginal soil quality value decreases. In addition, the intensity use of productive inputs increases with soil quality (from (16)), and conservation practices decrease with the soil quality marginal value (from (15)). Actually, it is a situation where soils have a low productivity. To improve this situation, investments in soil conservation are made, that diminish as soil quality improves and its value decreases. Indeed, on this optimal path, the higher soil quality, the lower its marginal value, and the more effective conservation practices are. Thus, as soil quality increases, less investments in conservation practices are required to increase soil quality (see condition (11)). Since productive inputs and soil quality are cooperative production factors, the farmer adjusts the use of productive inputs to the higher soil quality.

When the initial soil quality is high ($s^* < s_0 < s_\#$), the optimal trajectory is located in region III. Along this path, soil quality decreases, while the marginal soil quality value increases. Besides, from (16) and (15), when soil quality decreases, the use of productive inputs decreases, and when the marginal soil quality value increases, more conservation practices are implemented. In this situation, the quality of soil is "too" high compared to the equilibrium with a high soil productivity. Hence, the optimal strategy for the farmer is to let soil deteriorate until it attains the equilibrium level of soil quality. Doing so, the farmer diminishes the use of productive inputs, while implementing conservation practices. Indeed, at some point, the impact of soil deterioration on productivity is such that soil quality investments become necessary.

All strategies differing from these two optimal strategies will turn away from the steady state equilibrium.

For instance, initial conditions $(s_0, \mu_0)$ can be such that the farmer is located in region I, with a $\mu_0$ placing him/her above the unique optimal path of region I. Recall that $\mu_0 = \frac{c_u}{g_u(u_0)}$. Hence, such a case may correspond to a situation where $g_u(u_0)$ is small and $u_0$ is large. Since we are not on an optimal path, it is a case where $u_0 > u^*$, *i.e.* where investments in soil conservation are higher than what the optimum would require. At first, the strategy followed by the farmer would be similar to the optimal one. However, at some point, the path followed by the farmer will cross the $\dot{\mu} = 0$ locus, and will enter the region II. In region II, the trajectory followed is to increase both soil quality $s$ and soil quality marginal value $\mu$, by using more and more productive inputs $m$ and investing increasingly in soil quality conservation practices $u$. Thus, in this region, the paths followed correspond to an over-production, which is clearly an unsustainable strategy.

Initial conditions $(s_0, \mu_0)$ can also be such that the farmer is in region I, but with a small $\mu_0$. In this case, $g_u(u_0)$ is high, hence $u_0$ is small and investment in soil conservation are lower than what would require the optimum ($u_0 < u^*$). Once again, at first, the strategy followed corresponds to the optimal one. However, when following this non-optimal path, the $\dot{s} = 0$ locus is crossed. The farmer arrives then in region IV, where it is no longer optimal to maintain soil quality. In region IV, both soil quality and soil marginal quality decrease, along with management intensity and soil quality investments. This corresponds to a situation of under-production, where soil quality is depleted until it is totally degraded.

A similar discussion can be done for initial conditions placing the farmer in region III. Initial conditions can be such that the non-optimal path followed will cross the $\dot{s} = 0$ locus, leading into region II and its unsustainable over-production, where investment in soil quality is above optimal. Initial conditions can also be such that the non-optimal path followed will cross the

$\dot{\mu} = 0$ locus and enter region IV and its unsustainable under-production, where investment in soil quality is lower than optimal.

Optimal levels and strategies to attain a steady state equilibrium have been described. The next sections present the static and dynamics comparative of our problem.

## 3. Statics and dynamics comparative of the optimization problem

### 3.1. Impacts of a change in parameters on the equilibrium

A comparative statics analysis of the optimization problem allows us to determine how endogenous variables of our model differ from their value at the steady state equilibrium, for different values of exogenous parameters (Léonard and van Long, 1992). In our case, the endogenous variables that characterize the optimal steady state are productive inputs $m$, conservation practices $u$, soil quality $s$, and soil quality implicit value $\mu$. In what follows, we present the change in optimal values for a change in a given parameter, all other parameters remaining constant.

When the damages caused by the use of productive inputs are overcompensated by its cooperating benefits with soil quality in terms of revenue ($H_{ms} > 0$), our comparative statics analysis yields the following results (see Appendix B for the computational details):

$$m = m(\overset{-}{c_m}, \overset{-}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{19}$$

$$u = u(\overset{-}{c_m}, \overset{-}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{20}$$

$$\mu = \mu(\overset{-}{c_m}, \overset{+}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{21}$$

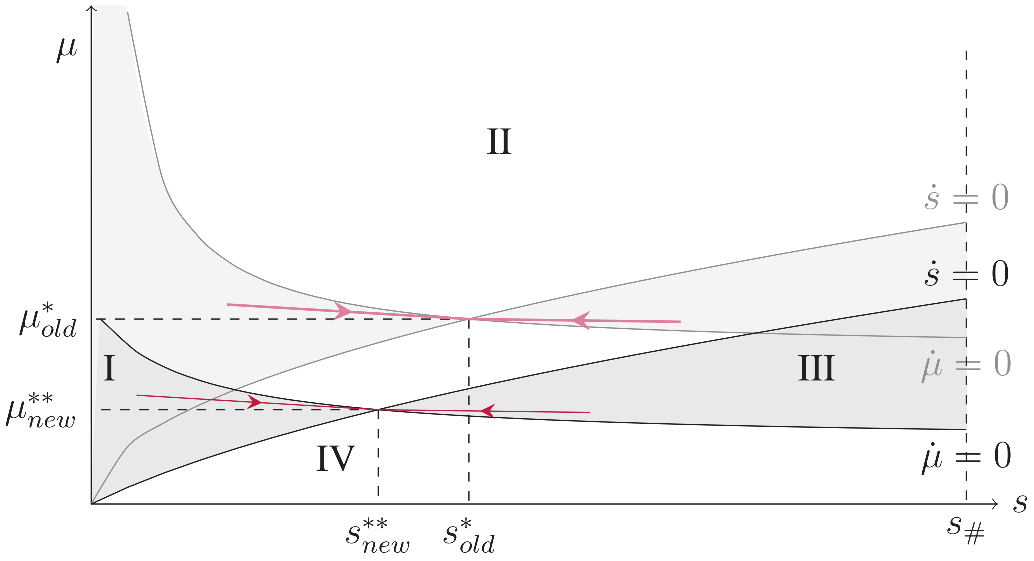
$$s = s(\overset{-}{c_m}, \overset{-}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{22}$$

*An increase in the cost associated with productive inputs, $c_m$,* leads to an expected decrease in productive inputs, and a decrease in soil quality and in soil marginal value at equilibrium. Since the value attributed to soil quality is lower, less investments are made in conservation practices.

*An increase in the cost associated with soil conservation and non-productive practices, $c_u$,* decreases the investment in soil conservation. As a consequence, soil quality at optimum is lower, and its associated marginal value increases. Since productive inputs and soil quality are cooperating, the use of productive inputs associated with a lower soil quality is smaller than in our original equilibrium.

*An increase in the crop yield price, $p$,* leads to an increase in soil quality and in productive inputs. Indeed, the farmer has the possibility to increase its production to attain an equilibrium where the marginal benefits of using more productive inputs are equal to the costs of these practices. Due to the cooperation between the two variables, soil quality at equilibrium also

Figure 3: Phase diagram and comparative statics: An increase in the cost of productive inputs $c_m$



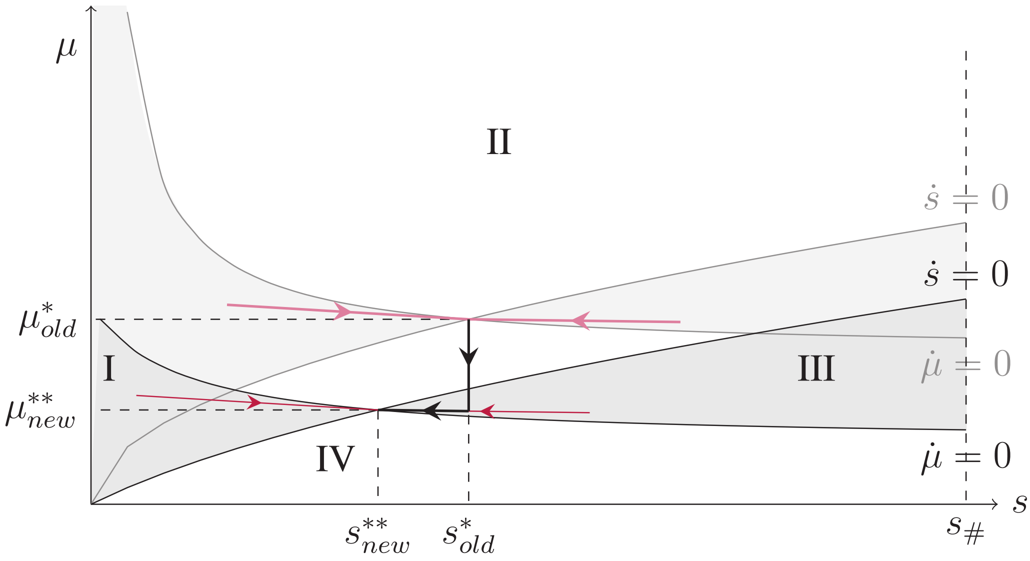

increases. To maintain this level of soil quality, the investment in soil conservation techniques is higher in this equilibrium. With a higher price and a higher productivity of soil quality at this optimum, the marginal soil quality is also higher.

*An increase in the discount rate, $r$,* means that the farmer values more present revenue than future ones. As a consequence, soil quality will be either more depleted or less restored by the farmer, who will be less willing to invest in soil conservation measures, since the marginal value attributed to soil quality has decreased in this equilibrium. The level of productive inputs also decreases, due to the cooperation relationship with soil quality: The loss in soil productivity seems to be compensated by lesser expenses in productive inputs. $r$ values range between $0$ and $1$. A $r$ equal to 0 may correspond to the time preference of a benevolent state, where future revenues are valuable as current ones. At the opposite, a $r$ equal to 1 can correspond to the time preference of a selfish short-termist private agent, who values only his/her present revenues.

Figure 3 gives a graphical representation of how the steady state equilibrium can be modified by an increase in the cost $c_m$ of productive inputs. In this example, the former optimal path which was located in region I, is now in region II. While in the old equilibrium (characterized by $s^*_{old}$ and $\mu^*_{old}$ in Figure 3) a farmer located on this path would have attained the steady state, now this farmer is in a situation of over-production with a total depletion of his/her soil quality. Conversely, farmers that in the previous situation were located in region IV, characterized by a non-sustainable under-production, may be either on the optimal path, or on the path not leading immediately to a situation of over- or under-production.

With comparative statics, we have investigated the change in the steady state as a response to a

Figure 4: Phase diagram and local comparative dynamics: An increase in the cost of productive inputs and the adjustment process




change in a given parameter. Making use of the steady state comparative statics results and of the phase diagram, it is possible to find the local comparative dynamics of the increase in the crop price.

### 3.2. Moving from one equilibrium to another: Transition paths

The local comparative dynamics illustrate the transition path from the old steady state to the new one. When considering local comparative dynamics, it is assumed that the economy observed is at rest at the old steady state. The local comparative dynamics, through the optimal transition path, describes how the economy comes to rest at the new steady state (Caputo, 2005) (see Figure 4).

Usually in an optimal control model, the state variable is considered as given at any moment in the planning horizon (Caputo, 2005). Hence, considering the old steady state as the initial condition, when a parameter of the model initially changes, the state variable will not at first. Nevertheless, it will eventually change. In the example of case 2 (see Figure 4), if there is an increase in productive inputs costs, it is the marginal value of soil quality that initially changes vertically. The marginal value of soil quality shifts downwards to zone III, where an optimal strategy leading to the new steady state can be reached (see Figure 4). Following this optimal strategy, soil quality decreases.

In addition to comparative statics and local comparative dynamics, comparative dynamics also provides interesting economic information as to how the cumulative discounted functions of our model can be impacted by changes in a given parameter (Caputo, 2005).

### 3.3. The impact of a change in parameters on the optimal paths

To conduct our comparative dynamics, we used the methodology proposed by Caputo (2005) *via* envelope methods. It is a general method of comparative dynamics that can be applied to any sufficiently smooth optimal control problem using a primal-dual approach (see Appendix B).

The primal form of our soil quality investment model is such that:

$$V(\alpha) \equiv \max_{m(.),u(.)} J\left[m(.),u(.),s(.)\right] \equiv \max_{m(.),u(.)} \int_0^T e^{-rt}\left[pf(s(t),m(t)) - c_m m(t) - c_u u(t)\right] dt \tag{23}$$

$$\begin{aligned} \text{s.t.} \quad & \dot{s}(t) = k(s(t),u(t)) = -\delta s(t) + g(u(t)) \ \text{ for case 1,} \\ & \dot{s}(t) = k(s(t),u(t)) = -\delta(m(t))s(t) + g(u(t)) \ \text{ for case 2,} \\ & s(0) = s_0, s(T) = s_T \end{aligned}$$

where $\alpha \equiv (p, c_m, c_u, r)$ is the vector of time-independent parameters. We denote by $z(t;\alpha)$, $v(t;\alpha)$ and $w(t;\alpha)$ the optimal paths of respectively soil quality, productive inputs, and investments in soil conservation practices. The farmer maximises the current value of his/her current and future profits $V(\alpha)$. The comparative dynamics analysis is conducted on the vector $\alpha \equiv (p, c_m, c_u, r)$ of parameters (see details in Appendix B).

Information obtained from the Dynamic Envelope Theorem of Caputo (2005) refer to the cumulative discounted profit and production functions.[4] The variation of $V(\alpha)$ (the maximised current value of farmer's current and future profits) with respect to crop price, productive inputs costs and soil investments is unambiguously determined:

$$V_p(\alpha) \equiv \int_0^T y(t;\alpha)e^{-rt}dt > 0 \tag{24}$$

$$V_{c_m}(\alpha) \equiv -\int_0^T v(t;\alpha)e^{-rt}dt < 0 \tag{25}$$

$$V_{c_u}(\alpha) \equiv -\int_0^T w(t;\alpha)e^{-rt}dt < 0 \tag{26}$$

According to the assumptions of our model, the production function cannot be negative, nor can be productive inputs or investments in soil quality conservation practices.

However, the variation of $V(\alpha)$ (the current value of farmer's current and future profits) with respect to the discount rate is ambiguous. For a discount rate $r$, applying Theorem 11.1 of Caputo (2005) yields for both cases:[5]

$$V_r(\alpha) \equiv -\int_0^T t\pi(t;\alpha)e^{-rt}dt \lesseqgtr 0 \tag{27}$$

[4] See Appendix B for details.
[5] See Appendix B for details.

where $\pi(t;\alpha) \equiv pf(z(t;\alpha), v(t;\alpha)) - c_m v(t;\alpha) - c_u w(t;\alpha)$ are the instantaneous profits along the optimal path, and $y(t;\alpha) \equiv f(z(t;\alpha), v(t;\alpha))$ is the value of the production function of the farm.

Although $V(\alpha) > 0$ must hold for the farm to be able to thrive in the market, instantaneous profits along the optimal path may be positive or negative at any given point. This could be the case when important investments in soil quality are made that are not generating productivity gains instantaneously. However, one could add a constraint holding instantaneous profits always positive, in which case $V_r(\alpha) < 0$.

In our model, the integrand function of the soil quality investment model is linear in $\gamma \equiv (p, c_m, c_u)$. Thus, the model satisfies the conditions of Corollary 11.2 in Caputo (2005).[6] It implies that the optimal value function $V(.)$ is locally convex in $\gamma$. Hence, when differentiating the first partial derivatives of $p$ and $c_u$, one can use the convexity of $V(.)$ to determine the signs of the second partial derivatives, and infer from these signs the own price effects:

$$V_{pp}(\alpha) \equiv \frac{\partial}{\partial p}\int_0^T y(t;\alpha)e^{-rt}dt = \int_0^T \frac{\partial y}{\partial p}(t;\alpha)e^{-rt}dt \geq 0 \tag{28}$$

$$V_{c_m c_m}(\alpha) \equiv -\frac{\partial}{\partial c_m}\int_0^T v(t;\alpha)e^{-rt}dt = -\int_0^T \frac{\partial v}{\partial c_m}(t;\alpha)e^{-rt}dt \geq 0 \tag{29}$$

$$V_{c_u c_u}(\alpha) \equiv -\frac{\partial}{\partial c_u}\int_0^T w(t;\alpha)e^{-rt}dt = -\int_0^T \frac{\partial w}{\partial c_u}(t;\alpha)e^{-rt}dt \geq 0 \tag{30}$$

Equation (28) shows that the cumulative discounted crop production is not decreasing in the crop price. Notice that it is the discounted production function slope, integrated over the entire planning horizon that is not decreasing. For a given and finite period of time, crop production could be decreasing, while the crop price has increased. While in the short run such a behaviour could appear as irrational, when equation (28) is verified over the entire planning horizon (*i.e.* in the long run), such a behaviour would be somehow rational. A similar reasoning can be applied to the effects of an increase in the cost of productive inputs and in the cost of conservation practices. Equations (29) and (30) demonstrate that the cumulative discounted use of productive inputs and the cumulative discounted investment in conservation practices are non-increasing in their own prices.

### 3.4. The case when marginal cooperating benefits in terms of productivity fall behind marginal damages on soil quality

As stated previously, the existence of a stable steady state equilibrium when $H_{ms} < 0$, *i.e.* when the cooperating benefits of productive inputs and soil quality are lower than the marginal

[6] Corollary 11.2 (Convexity of the Optimal Value Function): *For control problem (P), with assumptions (A.1) through (A.4) holding, if (i)* $g_\alpha(t, x, u; \alpha) \equiv 0_{N\times A}$ *and (ii)* $f(.)$ *is convex in* $\alpha$ *for all* $\beta \in B(\beta^\circ;\delta)$, *then* $V(.)$ *is convex in* $\alpha$ *for all* $\beta \in B(\beta^\circ;\delta)$.

damages of productive inputs use on soil quality, cannot be confirmed, because the determinant of the Jacobian matrix has an ambiguous sign.

Actually, some observations can be made with respect to such a situation. Irrespective of the value of $H_{ms}$, the trace of the Jacobian matrix of our problem is positive. Hence, for a saddle point to exist, the determinant of the matrix has to be negative. In addition, the slope of the curves are likely to be of similar sign, irrespective of the value of $H_{ms}$. The conditions under which such a situation can occur do not contradict one another. Thus, it is possible to conclude that a stable steady state equilibrium point can exist when $H_{ms} < 0$. However, such conditions are arithmetic and do not really allow for an economic interpretation.

Hence, when $H_{ms} < 0$, the existence of the equilibrium is not ensured. This does not necessarily mean that such an equilibrium does not exist, but that the existence of a stable steady state equilibrium point depends on the crop production and soil quality dynamics functions specification and calibration. In other words, it is a situation that requires an empirical analysis to determine if an equilibrium exists for a given situation.

## 4. Theory *versus* reality: A comparison between 2006 and 2010 in France

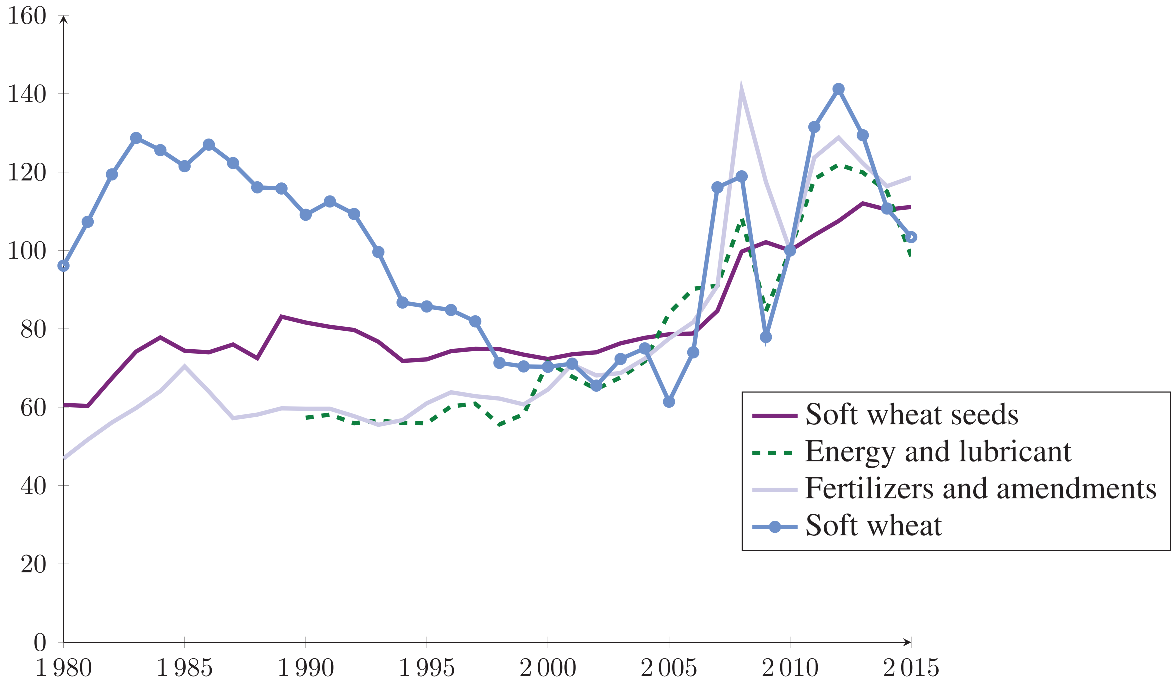
In this section are presented some stylized facts established from French national statistics on farm product prices, purchase prices of agricultural production factors and inputs, and surveys of observed farming practices. The objective here is to illustrate that the theoretical predictions of our simplified optimal control models are consistent with what can be observed in reality.

In particular, we focus on the results of the static comparative analysis of our equilibrium state. We compare farming practices in soft wheat production in France in 2006 and 2011, with respect to changes in soft wheat price, and in the price of fertilizers and amendments.

To do so, we compare the evolution of the Farm Product Price Index (FPPI) for soft wheat to the evolution of the purchase price indices of agricultural production factors and inputs (soft wheat seeds, energy and lubricant, and fertilizers) in France. Purchase price indices include seasonal adjustments. These are national data. Information related to farming practices are obtained from two surveys on farming practices conducted by the French office of statistics and forecasting. They are available online, for 2006 and 2011 (DISAR website platform, Agreste website). These data are available as aggregated means for twenty-two French regions.

Since the 1980's, the prices of agricultural production factors and inputs followed an increasing trend. Differently, the farm price of soft wheat decreased until the early 2000's (for almost twenty years) (see Figure 5). Since then, soft wheat prices followed an increasing trend, marked by strong volatility.

Figure 5: Long term price evolutions for soft wheat and its main production inputs




Source: Insee statistics. Note: The figure plots price indices in base 2010.

Let us assume that in 2006, at a national scale, we were at an equilibrium point. This point is characterized by a crop price index of 116.1 and a fertilizers cost index of 91.0. That year, tillage (with plow-down) was used on 55.6% of the French parcels, and other sorts of tillage (without plow-down, superficial tillage) were employed on 43.6% of parcels. In 2006, 98% of the French soft wheat surface was N fertilized, with an average dose of 165 kg per hectare on treated parcels.

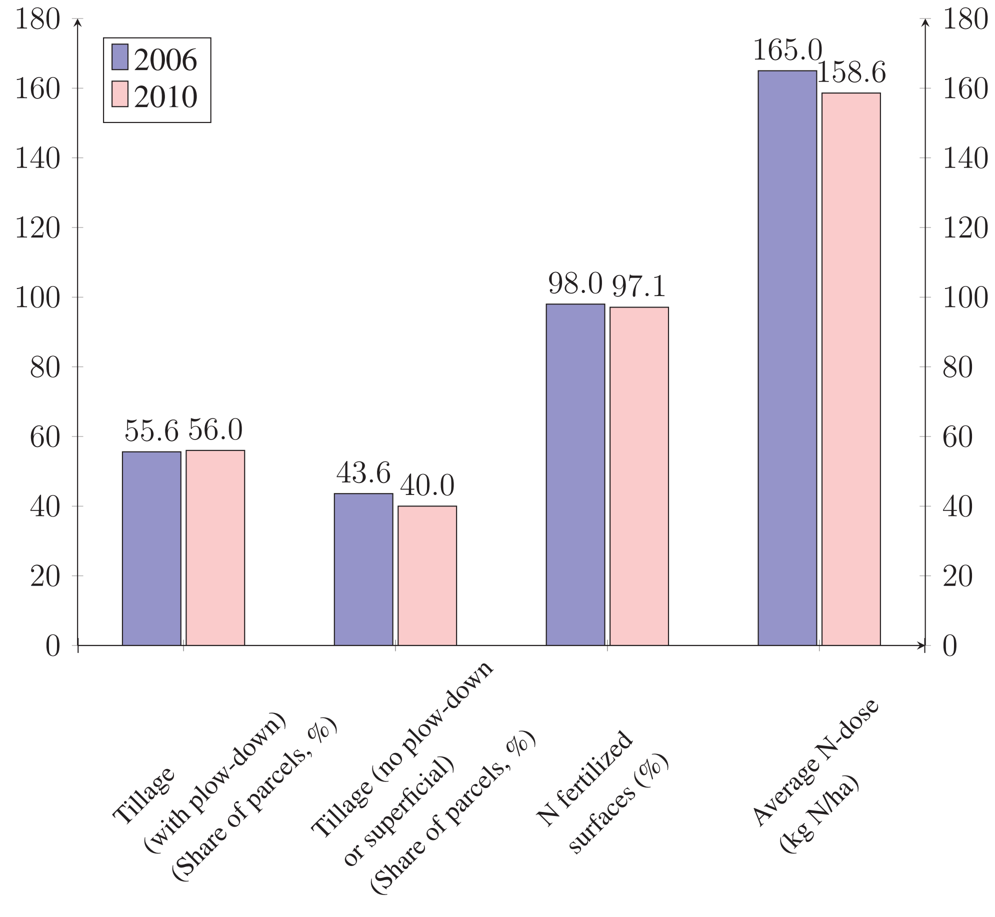
In 2011, the price indexes of soft wheat and fertilizers have increased to attain respectively 141.2 points and 128.8 points. From our static comparatives, an increase in crop price and in fertilizer costs (assimilated to productive inputs costs $c_m$) have antagonistic effects. An increase in crop price increase the optimal use of productive inputs, as well as the optimal amount of soil investments, while an increase in fertilizer costs have the opposite effect.

Nonetheless, since the soft wheat price has increased by 21.6% while fertilizers prices have increased by 41.5%, we conclude that the rise in fertilizer prices outweighs the increase in crop price.

From our static comparatives, when the costs of productive inputs increase, the use of productive inputs decreases, along with investment of soil quality, because soil quality is a production factor cooperating with productive inputs. Actually, in such a situation, the farmer will use his/her soil quality capital and spend less in more expensive productive inputs. In our example, productive inputs are fertilizers, and the costs of productive inputs are reflected by the fertilizers and amendments price index. Soil quality investments can be represented by a lesser use of tillage with plow-down, and an increased use of no-plow-down tillage or the use of superficial tillage.

Interestingly, we observe that between 2006 and 2011, the percentage of N-fertilized surface and the average dose applied have decreased (with respectively 0.9% and 3.9 %), while the use of tillage (with plow-down) has slightly increased (by 0.7%) and the use of other sorts of tillage have decreased by 8.2% (see Figure 6). This is a simple illustration, where only few aspects are taken into account. However, such observations are consistent with the theoretical predictions of our model.

Figure 6: Evolution of farming practices in France between 2006 and 2010 (regional values)




Source: DISAR

## 5. Conclusion

This article examines whether farmers have a private interest in maintaining or increasing soil quality. It explores and discusses the different optimal strategies to achieve a long term equilibrium. In addition, the dynamic elements of the soil resource management problem have been characterized. The importance of the cooperation relationship between soil quality and productive inputs is also demonstrated.

The proposed investment model highlights some situations the maintenance and enhancement of soil quality. The model shows the importance of the cooperation between the two production factors (soil quality and productive inputs). When the marginal cooperative productivity is higher than the marginal damages of productive inputs on soil quality, there exists a long-term optimal equilibrium with optimal strategies that can be followed by the farmer to reach the optimum. However, when the marginal productivity of the cooperating inputs are lower than the marginal and detrimental impact of productive inputs on soil quality, one cannot conclude about the existence of an equilibrium. It covers situations where the increase in production does not cover the costs of organizing this cooperation in terms of the shadow value of soil quality.

These ambiguities show that we are indeed facing empirical questions that depend on technical interactions that are difficult to know and control. Furthermore, such simplified theoretical models offer interesting analysis and diagnostic perspectives for farm advisory services. These theoretical models can produce interesting qualitative analyses. Besides, an empirical modelling can provide both qualitative and quantitative analysis. This may serve for private farm advisory services, such as those provided by cooperatives. It might also serve as a tool to design public and private incentives for farmers to implement or maintain a sustainable and yet productive use of their soils, while preserving the common good.

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## A. Appendix A: To consider or not to consider soil quality dynamics?

When farmers do not consider soil quality dynamics, the first order conditions of our problem can be rewritten such as:

$$H_m = pf_m - c_m = 0 \tag{31}$$
$$H_u = -c_u = 0 \tag{32}$$

Since the farmer does not consider soil quality dynamic,s nor the detrimental impact of productive inputs on soil quality, he/she does not internalize the additional cost of using productive inputs in terms of soil quality marginal value.

However, according to condition (32), the optimal use of conservation practices is such that the investment is equal to zero at any point in time. That is to say, when not considering the dynamics of soil quality, no soil conservation investment are made. Hence, we are always in a situation of under-investment in soil quality.

One can still expect that soil quality will attain a long-term equilibrium (Smith *et al.*, 2000), such that:

$$\dot{s} = -\delta(m)s + g(u) = 0 \Leftrightarrow s^S = \frac{g(0)}{\delta(m^S)} \tag{33}$$

We compare the long term soil quality equilibrium ($s^S$) when the farmer does not consider soil quality dynamics, and the optimum soil quality level ($s^*$) when the farmer considers soil quality dynamics:

$$s^S = \frac{g(0)}{\delta(m^S)} \quad \text{and} \quad s^* = \frac{g(u^*)}{\delta(m^*)} \tag{34}$$

In addition, from the conditions (10) and (31) of the two optimization problems, at any point of time, and in particular for bundles $(m^*, s^*)$ and $(m^S, s^S)$, we have:

$$pf_m(m^*, s^*) - c_m - \mu\delta_m(m^*)s^* = 0 \quad \text{and} \quad pf_m(m^S, s^S) - c_m = 0 \tag{35}$$
$$\Leftrightarrow pf_m(m^*, s^*) - c_m - \mu\delta_m(m^*)s^* = pf_m(m^S, s^S) - c_m \tag{36}$$
$$\Leftrightarrow f_m(m^*, s^*) - \frac{\mu}{p}\delta_m(m^*)s^* = f_m(m^S, s^S) \tag{37}$$

There is a more complex relationship between productive inputs $m$ and soil quality $s$. In addition to the cooperation effect in terms of production, the detrimental impact of productive inputs on soil quality dynamics needs to be considered. In this case, several situations are plausible, depending on the initial soil quality.

**case** $(1)$**:** $m^* < m^S$ and $s^* > s^S$

From the assumptions of our model:

$$m^* < m^S \Rightarrow \delta(m^*) < \delta(m^S) \tag{38}$$

$$g(u^*) > g(0) \Rightarrow \frac{g(u^*)}{\delta(m^*)} > \frac{g(0)}{\delta(s^S)} \tag{39}$$

$$s^* > s^S \tag{40}$$

**case** $(2)$**:** $m^* = m^S$ and $s^* > s^S$

From the assumptions of our model:

$$m^* = m^S \Rightarrow \delta(m^*) = \delta(m^S) \tag{41}$$

$$g(u^*) > g(0) \Rightarrow \frac{g(u^*)}{\delta(m^*)} > \frac{g(0)}{\delta(s^S)} \tag{42}$$

$$s^* > s^S \tag{43}$$

These two cases are consistent with (37). The interpretation is fairly intuitive: These are situations when the farmer does not consider soil quality dynamics, and uses productive inputs without compensating for the damages caused to soil quality, thereby degrading the quality of soil below the optimum. There is an over-use of productive inputs and an under-use of soil quality investments.

**case** $(3)$**:** $m^* > m^S$

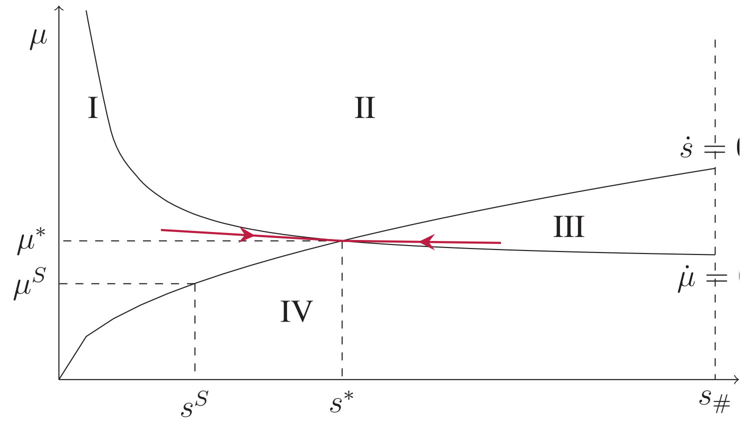
$$m^* > m^S \Rightarrow \delta(m^*) > \delta(m^S) \tag{44}$$

$$g(u^*) > g(0) \Rightarrow \frac{g(u^*)}{\delta(m^*)} \gtreqless \frac{g(0)}{\delta(s^S)} \tag{45}$$

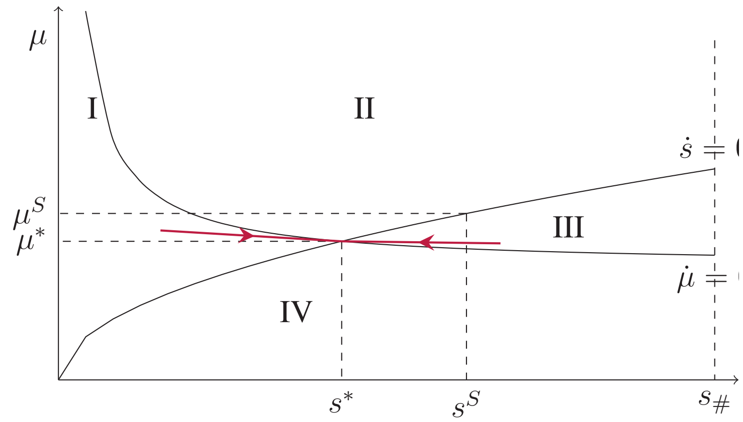
$$s^* \gtreqless s^S \tag{46}$$

This can correspond to different situations. One situation is when the initial soil quality is above the optimum, and sufficiently high for long-term soil quality to stabilize above the optimum, even when the farmer does not compensate for the impact of productive inputs on soil. This could be possible since the farmer also use less productive inputs than optimum, and thus causes lesser damages. The other situation is when the initial soil quality is not sufficiently high for the lower use of productive inputs to compensate for the lack of investment in soil quality.

Figure 7: Phase diagram without considering soil quality dynamics in the farmer's decision making process




(a) $s^S < s^*$




(b) $s^S > s^*$

In most cases, not considering soil quality dynamics leads to a long-term equilibrium level of soil quality lower than the optimal. This can be observed when the farmer makes an over-use or an under-use of productive inputs, compared to the case when the farmer considers soil quality. Indeed, in all cases, no investments are made in soil quality. The damages, whether natural or caused by the use of productive inputs, are not compensated for. In one of the situations described, a sufficiently high initial soil quality level can still lead to a long term equilibrium of soil quality higher than the optimal. This case corresponds to a situation where the cooperation relationship between soil quality and productive inputs use is under-used.

The problem is that in all cases, the long term equilibrium of soil quality is not a stable one: These are situations that cross the $\dot{s} = 0$ locus, so that the strategies followed by the farmer remain non-optimal strategies with an under-investment in soil quality leading to a depletion of the resource (See Figure 7).

# B. Appendix B: Computations of the soil investment model

## B.1. Phase diagram and stability properties of the optimization problem: Ambiguity due to the prevalence of the cooperating benefits over the marginal damages on soil quality

The long-run or steady state equilibrium of the optimal control problem is determined by the intersection of the ($\dot{\mu} = 0$) and ($\dot{s} = 0$) demarcation curves which are such that:

$$\begin{aligned} & A(s,\mu) = \dot{\mu} \gtreqless 0 \\ & \text{if} \;\; \mu(r + \delta(m(s,\mu))) - pf_s(m(s,\mu),s) \gtreqless 0 \end{aligned} \tag{47}$$

$$\begin{aligned} & B(s,\mu) = \dot{s} \gtreqless 0 \\ & \text{if} \;\; -\delta(m(s,\mu))s + g(u(s,\mu)) \gtreqless 0 \end{aligned} \tag{48}$$

The slopes of the stationary loci are given by:

$$\left.\frac{d\mu}{ds}\right|_{B=\dot{s}=0} = -\frac{\partial H_\mu/\partial s}{\partial H_\mu/\partial \mu} = -\frac{-\delta_m m_s s - \delta(m) + g_u u_s}{-\delta_m m_\mu s + g_u u_\mu} = -\frac{-\delta_m m_s s - \delta(m)}{-\delta_m m_\mu s + g_u u_\mu} \tag{49}$$

$$\begin{aligned} \left.\frac{d\mu}{ds}\right|_{A=\dot{\mu}=0} &= -\frac{\partial(\mu r - H_s)/\partial s}{\partial(\mu r - H_s)/\partial \mu} = -\frac{\partial(\mu(r+\delta(m(s,\mu))) - pf_s(s,m(s,\mu)))/\partial s}{\partial(\mu(r+\delta(m(s,\mu))) - pf_s(s,m(s,\mu)))/\partial \mu} \\ &= -\frac{\delta_m m_s \mu - pf_{ss} - pf_{sm} m_s}{r + \delta_m m_\mu \mu + \delta(m) - pf_{sm} m_\mu} \end{aligned} \tag{50}$$

To determine the stability properties of our problem, *i.e.* whether all solutions converge towards the steady state, one can evaluate the Jacobian matrix:

$$J = \begin{bmatrix} \partial\dot{s}/\partial s & \partial\dot{s}/\partial\mu \\ \partial\dot{\mu}/\partial s & \partial\dot{\mu}/\partial\mu \end{bmatrix} = \begin{bmatrix} H_{\mu s} & H_{\mu\mu} \\ -H_{ss} & r - H_{s\mu} \end{bmatrix} = \begin{bmatrix} \underset{(?)}{-\delta_m m_s s - \delta} & \underset{(+)}{-\delta_m m_\mu s + g_u u_\mu} \\ \underset{(?)}{m_s(-H_{ms}) - pf_{ss}} & \underset{(?)}{r + m_\mu(-H_{ms}) + \delta} \end{bmatrix} \tag{51}$$

at the steady sate $(s^*, \mu^*)$. Computing the trace of the Jacobian matrix, it appears that:

$$tr[J] = -\delta_m m_s s - \delta + r + m_\mu(-H_{ms}) + \delta = -m_s(\delta_m s - \delta_m s) + r = r > 0 \tag{52}$$

Since the eigenvalues of the Jacobian matrix are equal to its trace, it means that at least one eigenvalue is positive. This implies that the fixed point (here the intersection point of the ($\dot{\mu} = 0$) and ($\dot{s} = 0$) demarcation curves) is not locally asymptotically stable (Caputo, 2005). If the determinant of the Jacobian matrix is negative, then the steady state is a local saddle point (Hediger, 2003; Narain and Fisher, 2006). Otherwise, if the determinant of the Jacobian matrix is positive, the steady state is an unstable node, or at the center of an unstable spiral (Caputo, 2005), so that the system is not converging towards the steady state.

With a general form of the problem, that is without specifying the functional forms of the different functions considered, the existence of an equilibrium can be found in the case when $H_{ms} > 0$. However, no conclusion can be made in the case when $H_{ms} < 0$.

## B.2. When the marginal cooperating benefits are higher than the marginal damages on soil quality: Phase diagram and stability properties of the optimization problem

In the case where $H_{ms} > 0$, which corresponds to the case where the marginal benefits of using productive inputs in terms of revenues are higher than the damages in terms of soil quality marginal value, there is a steady state equilibrium, since the Jacobian matrix is such that:

$$\begin{aligned}
\det J &= \begin{vmatrix} H_{\mu s} & H_{\mu\mu} \\ -H_{ss} & r - H_{s\mu} \end{vmatrix} = H_{\mu s}(r - H_{s\mu}) - H_{\mu\mu}(-H_{ss}) \\
&= (-\delta_m m_s s - \delta(m) + u_s)(r + \delta_m m_\mu \mu + \delta(m) - pf_{sm} m_\mu) \\
&\quad -(-\delta_m m_\mu s + u_\mu g_u)(\delta_m m_s \mu - pf_{ss} - pf_{sm} m_s) \\
&= (-\delta_m m_s s - \delta(m))(r + m_\mu(-H_{sm}) + \delta(m)) \\
&\quad -(-\delta_m m_\mu s + u_\mu g_u)(m_s(-H_{ms}) - pf_{ss}) \\
&= (-\delta_m m_s s - \delta(m))(r + m_\mu(-H_{sm}) + \delta(m)) \\
&\quad -(-\delta_m m_\mu s + u_\mu g_u)\left(\left(-\frac{H_{ms}}{H_{mm}}\right)(-H_{ms}) - pf_{ss}\right) \\
&= (-\delta_m m_s s - \delta(m))(r + m_\mu(-H_{sm}) + \delta(m)) \\
&\quad -(-\delta_m m_\mu s + u_\mu g_u)\left(\frac{H_{ms}^2 - pf_{ss}H_{mm}}{H_{mm}}\right) \\
&= (-\delta_m m_s s - \delta(m))(r + m_\mu(-H_{sm}) + \delta(m)) \\
&\quad -(-\delta_m m_\mu s + u_\mu g_u)\left(\frac{p^2(f_{ms}^2 - f_{ss}f_{mm}) + \mu\delta_m(\mu\delta_m - 2pf_{sm}) + pf_{ss}\mu\delta_{mm}s}{H_{mm}}\right) \\
&< 0
\end{aligned}$$

From conditions (2), (3) and (5), and equations (15) - (16), given that $r$ and $p$ are positive, and assuming that $H_{ms} > 0$, then we obtain: $H_{\mu s} < 0$, $r - H_{s\mu} > 0$, $H_{\mu\mu} > 0$ and $H_{\mu\mu}(-H_{ss}) > 0$. From results permit we conclude that the determinant of the Jacobian matrix is negative.

The slopes of the stationary loci are given by:

$$\left.\frac{d\mu}{ds}\right|_{B=0} = -\frac{\partial H_\mu/\partial s}{\partial H_\mu/\partial \mu} = -\frac{H_{\mu s}}{H_{\mu\mu}} > 0 \tag{53}$$

$$\left.\frac{d\mu}{ds}\right|_{A=0} = -\frac{\partial(\mu r - H_s)/\partial s}{\partial(\mu r - H_s)/\partial \mu} = -\frac{-H_{ss}}{r - H_{s\mu}} < 0 \tag{54}$$

From conditions (2), (3) and (5), and equations (15) - (16), given that $r$ and $p$ are positive, and assuming that $H_{ms} > 0$, the gradient of the ($\dot{s} = 0$)-curve is positive, while the gradient of the ($\dot{\mu} = 0$)-curve is negative.

In addition, the slope of the trajectories in the $(s, \mu)$ space are such that:

$$\frac{d\mu}{ds} = \left(\frac{d\mu}{dt}\right) \cdot \left(\frac{dt}{ds}\right) = \frac{\dot{\mu}}{\dot{s}} \tag{55}$$

Hence, when a trajectory goes through a locus where $\dot{\mu} = 0$, it has a slope zero, and when it goes through a locus where $\dot{s} = 0$, it has an infinite slope (a vertical line).

Furthermore, when $\dot{s} = 0$ and $\dot{\mu} = 0$ and in the case where the steady state is a local saddle point (which is the case when $H_{ms} > 0$), we have:

$$\left[\underbrace{\frac{\partial \dot{s}}{\partial s}}_{-}\underbrace{\frac{\partial \dot{\mu}}{\partial \mu}}_{+} - \underbrace{\frac{\partial \dot{\mu}}{\partial s}}_{+}\underbrace{\frac{\partial \dot{s}}{\partial \mu}}_{+}\right] < 0 \Leftrightarrow \frac{-\partial \dot{s}/\partial s}{\partial \dot{s}/\partial \mu} > \frac{-\partial \dot{\mu}/\partial s}{\partial \dot{\mu}/\partial \mu} \tag{56}$$

From (56), one can conclude that the slope of the $\dot{s} = 0$ isocline is greater than the slope of the $\dot{\mu} = 0$ isocline in a neighborhood of the steady state. This is true if and only if the steady state is a local saddle point (Caputo, 2005).

### B.3. Comparative statics for the case when $H_{ms} > 0$

Here, we aim at estimating the impact of a change in parameters, $c_m$, $c_u$, $p$ or $r$, *i.e.* in the costs associated to the soil degrading practices $m$, the costs associated with soil quality investment or conservation practices, the crop price and, respectively, the discount rate. When one parameter changes, all variables change. However, the other parameters remain fixed and have a zero differential. To study this change, we evaluate the total differentials at the original equilibrium, that is the total differentials of the first order conditions (FOCs) when $\dot{\mu} = \dot{s} = 0$.

The FOCs at equilibrium are such that:

$$\tilde{H}_m = pf_m - c_m - \mu\delta_m s = 0 \tag{57}$$
$$\tilde{H}_u = -c_u + \mu g_u = 0 \tag{58}$$
$$\tilde{H}_\mu = -\delta(m)s + g(u) = 0 \tag{59}$$
$$\dot{\mu} - r\mu = -\tilde{H}_s \Leftrightarrow \dot{\mu} = r\mu - pf_s + \delta(m)\mu = \mu(r + \delta(m)) - pf_s = 0 \tag{60}$$

The total differentials of the system are such that:

$$\begin{aligned}(pf_{mm} - \mu\delta_{mm}s)dm + 0du - \delta_m s d\mu + (pf_{ms} - \mu\delta_m)ds & \\ +f_m dp - dc_m + 0dc_u + 0dr &= 0 \quad (61)\\ 0dm + \mu g_{uu}du + g_u d\mu + 0ds + 0dp + 0dc_m - dc_u + 0dr &= 0 \quad (62)\\ -\delta_m sdm + g_u du + 0d\mu - \delta(m)ds + 0dp + 0dc_m + 0dc_u + 0dr &= 0 \quad (63)\\ (\mu\delta_m - pf_{sm})dm + 0du + (r + \delta(m))d\mu - pf_{ss}ds & \\ -f_s dp + 0dc_m + 0dc_u + \mu dr &= 0 \quad (64)\end{aligned}$$

The determinant of the matrix of the system, denoted as $B$, is positive:

$$\begin{aligned}|B| &= \begin{vmatrix} pf_{mm} - \mu\delta_{mm}s & 0 & -\delta_m s & pf_{ms} - \mu\delta_m \\ 0 & \mu g_{uu} & g_u & 0 \\ -\delta_m s & g_u & 0 & -\delta \\ \mu\delta_m - pf_{sm} & 0 & r+\delta & -pf_{ss} \end{vmatrix} \\ &= \mu g_{uu} \begin{vmatrix} H_{mm} & -\delta_m s & H_{ms} \\ -\delta_m s & 0 & -\delta \\ -H_{ms} & r+\delta & -pf_{ss} \end{vmatrix} - g_u \begin{vmatrix} H_{mm} & -\delta_m s & H_{ms} \\ 0 & g_u & 0 \\ -H_{ms} & r+\delta & -pf_{ss} \end{vmatrix} \\ &= \mu g_{uu}\left(\delta_m s \begin{vmatrix} -\delta_m s & -\delta \\ -H_{ms} & -pf_{ss} \end{vmatrix} - (r+\delta) \begin{vmatrix} H_{mm} & H_{ms} \\ -\delta_m s & -\delta \end{vmatrix}\right) - g_u\left(g_u \begin{vmatrix} H_{mm} & H_{ms} \\ -H_{ms} & -pf_{ss} \end{vmatrix}\right) \\ &= \mu g_{uu}\left(\delta m_s(\delta_m spf_{ss} - H_{ms}\delta) - (r+\delta)(-H_{mm}\delta + H_{ms}\delta_m s)\right) - g_u^2\left(-H_{mm}H_{ss} + H_{ms}^2\right) \\ &= \mu g_{uu}\left((\delta_m s)^2 H_{ss} - \delta_m s\delta H_{ms} + (r+\delta)H_{mm}\delta - (r+\delta)H_{ms}\delta_m s\right) \\ &\quad -g_u^2\left(-H_{mm}H_{ss} + H_{ms}^2\right) > 0 \end{aligned} \tag{65}$$

Applying Cramer's rule, we obtain the following comparative statics for the case where the damages caused by the use of productive inputs are overcompensated by its cooperating benefits with soil quality in terms of revenue ($H_{ms} > 0$):

$$m = m(\overset{-}{c_m}, \overset{-}{c_u}, \overset{+}{p}, \overset{?}{r}) \tag{66}$$

$$u = u(\overset{-}{c_m}, \overset{-}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{67}$$

$$\mu = \mu(\overset{-}{c_m}, \overset{+}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{68}$$

$$s = s(\overset{-}{c_m}, \overset{-}{c_u}, \overset{?}{p}, \overset{?}{r}) \tag{69}$$

Using this method, some impacts have an ambiguous sign. We use an alternative methodology to determine the sign of the impact of a change in the discount rate $r$ and in the crop price $p$ on the steady state. Indeed, it is not the FOCs that are taken into account, but only the ($\dot{s} = 0$) and ($\dot{\mu} = 0$) equations, together with the expressions of $m$ and $u$ as implicit functions of soil quality $s$ and marginal soil quality $\mu$. We reach the following set of equations:

$$\dot{s} = H_\mu = -\delta(m^*(s,\mu)) + g(u^*(s^*,\mu^*)) = 0 \tag{70}$$

$$\dot{\mu} = r\mu - H_s = \mu^*(r + \delta(m^*(s,\mu))) - pf_s(m^*(s,\mu), s) = 0 \tag{71}$$

Differentiating the system with respect to $s$, $\mu$, $p$ and $r$ yields:

$$(-\delta_m m_s - \delta(m) + g_u u_s)ds + (g_u u_\mu - \delta_m m_\mu s)d\mu + 0dr + 0dp = 0 \tag{72}$$

$$(\mu\delta_m m_s - pf_{sm}m_s - pf_{ss})ds + (r + \delta(m) + \mu\delta_m m_\mu - pf_{sm}m_\mu)d\mu + \mu dr - f_s dp = 0 \tag{73}$$

Considering only changes in $r$ gives the following system:

$$\begin{bmatrix} -\delta_m m_s - \delta(m) & g_u u_\mu - \delta_m m_\mu s \\ -H_{ms}m_s - pf_{ss} & r + \delta - m_\mu H_{ms} \end{bmatrix} \begin{bmatrix} ds/dr \\ d\mu/dr \end{bmatrix} = \begin{bmatrix} 0 \\ -\mu \end{bmatrix} \tag{74}$$

Applying Cramer's rule yields the following results:

$$\frac{ds}{dr} = \frac{\begin{vmatrix} 0 & g_u u_\mu - \delta_m m_\mu s \\ -\mu & r + \delta(m) - m_\mu H_{ms} \end{vmatrix}}{|J|} = \frac{\mu(g_u u_\mu - \delta_m m_\mu s)}{|J|} < 0 \tag{75}$$

$$\frac{d\mu}{dr} = \frac{\begin{vmatrix} -\delta_m m_s - \delta(m) & 0 \\ -H_{ms}m_s - pf_{ss} & -\mu \end{vmatrix}}{|J|} = \frac{\mu(\delta_m m_s s + \delta(m))}{|J|} < 0 \tag{76}$$

Similarly, considering only changes in $p$, we obtain the following system:

$$\begin{bmatrix} -\delta_m m_s - \delta(m) & g_u u_\mu - \delta_m m_\mu s \\ -H_{ms}m_s - pf_{ss} & r + \delta - m_\mu H_{ms} \end{bmatrix} \begin{bmatrix} ds/dr \\ d\mu/dr \end{bmatrix} = \begin{bmatrix} 0 \\ f_s \end{bmatrix} \tag{77}$$

Applying Cramer's rule yields the following results:

$$\frac{ds}{dp} = \frac{\begin{vmatrix} 0 & g_u u_\mu - \delta_m m_\mu s \\ f_s & r + \delta(m) - m_\mu H_{ms} \end{vmatrix}}{|J|} = \frac{-f_s(g_u u_\mu - \delta_m m_\mu s)}{|J|} > 0 \tag{78}$$

$$\frac{d\mu}{dp} = \frac{\begin{vmatrix} -\delta_m m_s - \delta(m) & 0 \\ -H_{ms} m_s - p f_{ss} & f_s \end{vmatrix}}{|J|} = \frac{f_s(\delta_m m_s s + \delta(m))}{|J|} > 0 \tag{79}$$

The comparative statics for the case where the damages caused by the use of productive inputs are overcompensated by the cooperating benefits with soil quality in terms of revenue ($H_{ms} > 0$) are the following:

$$m = m(\overset{-}{c_m}, \overset{-}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{80}$$

$$u = u(\overset{-}{c_m}, \overset{-}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{81}$$

$$\mu = \mu(\overset{-}{c_m}, \overset{+}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{82}$$

$$s = s(\overset{-}{c_m}, \overset{-}{c_u}, \overset{+}{p}, \overset{-}{r}) \tag{83}$$

### B.4. When the marginal cooperating benefits are higher than the marginal damages on soil quality: Comparative dynamics

To conduct our comparative dynamics, we use the methodology proposed by Caputo (2005) *via* envelope methods. This is a general method of comparative dynamics that can be applied to any sufficiently smooth optimal control problem using a primal-dual approach (see Caputo (2005) - chapter 11). The conditions necessary to use the theorems or corollary proposed by Caputo (2005) have to be verified for the specific set-up of our soil quality investment model when farming practices impact soil quality, both positively and negatively.

The primal form of our soil quality investment model is such that:

$$V(\alpha) \equiv \max_{m(.),u(.)} J[m(.), u(.), s(.)] \equiv \max_{m(.),u(.)} \int_0^T e^{-rt}[pf(s(t), m(t)) - c_m m(t) - c_u u(t)]dt \tag{84}$$

$$\text{s.t.} \quad \dot{s}(t) = k(m(t), s(t), u(t)) = -\delta(m(t))s(t) + g(u(t)), \tag{85}$$

$$s(0) = s_0, s(T) = s_T \tag{86}$$

where $\alpha \equiv (p, c_m, c_u, r)$ is the vector of time-independent parameters. We denote $z(t;\alpha)$, $v(t;\alpha)$ and $w(t;\alpha)$ the optimal paths of respectively soil quality, soil degrading practices, and investments in soil conservation practices. The comparative dynamics analysis is conducted on the vector $\alpha \equiv (p, c_m, c_u, r)$ of parameters.

We use the Dynamic Envelope Theorem proposed in (Caputo, 2005). According to the theorem, when the assumptions (A.1) through (A.4) hold, the partial derivative of the optimal value function with respect to a parameter can be obtained by differentiating the Hamiltonian of the optimal control problem, then evaluating it along the optimal paths (that is for $s(t) = z(t;\alpha)$, $m(t) = v(t;\alpha)$ and $u(t) = w(t;\alpha)$), and finally integrating the result over the planning horizon.

Before doing so, let us verify that assumptions (A.1) to (A.4) hold for our soil quality investment problem. The assumptions mentioned in Caputo (2005) (page 288) and applied to our case are the following:

**(A.1)** $f(.) \in C^{(2)}$ and $k(.) \in C^{(2)}$ on their respective domains,

**(A.2)** There exists a unique optimal solution to problem (P) for each $\beta \in B(\beta^\circ;\delta)$, which we denote by the quadruplet $(z(t;\alpha), v(t;\alpha), w(t;\alpha), \lambda(t;\alpha)$ where $B(\beta^\circ;\delta)$ is an open $2+2N+A$ - ball centered at the given value of the parameter $\beta^\circ$ of radius $\delta > 0$.

**(A.3)** The vector-valued functions $z(.), v(.), w(.), \lambda(.)$ are $C^{(1)}$ in $(t;\beta)$, $\forall\ (t;\beta) \in [t_0^\circ, t_1^\circ] \times B(\beta^\circ;\delta)$.

**(A.4)** $V(.) \in C^{(2)}$ in $\beta$ for all $\beta \in B(\beta^\circ;\delta)$.

(A.1) holds because of the assumptions made for the production function and the soil quality dynamics function. In addition, from the Mangasarian Sufficient Conditions theorem, since the Hamiltonian $\tilde{H}$ of our problem is strictly concave in $m$, $u$, and $s$ when $\mu$ is the costate variable, there is a unique global maximum of $J[.]$.[7] (A.3) and (A.4) are assumed to hold.

[7] The Hessian matrix $\mathcal{H}$ of the Hamiltonian $\tilde{H}$ when examining the concavity of $\tilde{H}$ is such that:

$$\mathcal{H}(m,u,s) = \begin{bmatrix} H_{mm} & H_{mu} & H_{ms} \\ H_{um} & H_{uu} & H_{us} \\ H_{sm} & H_{su} & H_{ss} \end{bmatrix} = \begin{bmatrix} H_{mm} & 0 & H_{ms} \\ 0 & H_{uu} & 0 \\ H_{sm} & & H_{ss} \end{bmatrix}$$

Note that $\mathcal{H}$ is a square symmetric matrix of order $n = 3$. If the $n = 3$ leading principal minors $D_k$ (*i.e.* the determinants of the $(k \times k)$ matrix obtained by eliminating the $n-k$ last rows and $n-k$ last columns of the matrix) are alternatively $< 0$ ($k$ odd) and $> 0$ ($k$ even), then $\mathcal{H}$ is negative-definite.
Here we have, in the case where $H_{ms} > 0$:

$$\begin{aligned} D_1 &= H_{mm} < 0 \\ D_2 &= H_{mm}H_{uu} - (H_{mu})^2 = H_{mm}H_{uu} > 0 \\ D_3 &= H_{uu}(H_{mm}H_{ss} - (Hms)^2 = H_{uu}(-\mu s\delta_{mm}spf_{ss} + p^2(f_{mm}f_{ss} - (f_{sm})^2) \\ &\quad +\mu\delta_m(2pf_{sm} - \mu\delta_m)) < 0 \end{aligned}$$

Hence $\mathcal{H}$ is negative-definite. If the Hessian matrix of a function $f$ is negative-definite $\forall x \in \mathbb{R}^n$, then $f$ is strictly concave. Then, we can conclude that $\tilde{H}$ is strictly concave in $m$, $u$ and $s$, when $\mu$ is the costate variable.

Hence, applying Theorem 11.1 yields:

$$V_p(\alpha) \equiv \int_0^T y(t;\alpha)e^{-rt}dt > 0 \tag{87}$$

$$V_{c_m}(\alpha) \equiv -\int_0^T v(t;\alpha)e^{-rt}dt < 0 \tag{88}$$

$$V_{c_u}(\alpha) \equiv -\int_0^T w(t;\alpha)e^{-rt}dt < 0 \tag{89}$$

$$V_r(\alpha) \equiv -\int_0^T t\pi(t;\alpha)e^{-rt}dt \lesseqgtr 0 \tag{90}$$

where $y(t;\alpha) \equiv f(z(t;\alpha), v(t;\alpha))$ is the value of the production function of the farm, and $\pi(t;\alpha) \equiv pf(z(t;\alpha), v(t;\alpha)) - c_m v(t;\alpha) - c_u w(t;\alpha)$ is the instantaneous profits along the optimal path.

Information obtained from the dynamic envelope theorem refer to cumulative discounted profits and production functions. The signs of equations (87), (88) and (89) are unambiguously determined: According to the assumptions of our model, the production function cannot be negative, nor can be the productive inputs or the investment in soil quality conservation practices. However, the sign of equation (90) is ambiguous. Indeed, although $V(\alpha) > 0$ must hold for the farm to be able to survive in the market, instantaneous profits along the optimal path may be positive or negative at any given point. This could be the case when important investment in soil quality are made that do not yield productivity gains instantaneously. However, one could add a constraint where instantaneous profits have to be positive, in which case $V_r(\alpha) < 0$.

In our model, the integrand function of the soil quality investment model is linear in $\gamma \equiv (p, c_m, c_u)$. Thus, the model satisfies the conditions of Corollary 11.2 (Caputo, 2005). It implies that the optimal value function $V(.)$ is locally convex in $\gamma$. Hence, when differentiating equations (87) to (89), one can use the convexity of $V(.)$ to determine the signs of the second partial derivatives, and infer from those signs the own-price effects:

$$V_{pp}(\alpha) \equiv \frac{\partial}{\partial p}\int_0^T y(t;\alpha)e^{-rt}dt = \int_0^T \frac{\partial y}{\partial p}(t;\alpha)e^{-rt}dt \geq 0 \tag{91}$$

$$V_{c_m c_m}(\alpha) \equiv -\frac{\partial}{\partial c_m}\int_0^T v(t;\alpha)e^{-rt}dt = -\int_0^T \frac{\partial v}{\partial c_m}(t;\alpha)e^{-rt}dt \geq 0 \tag{92}$$

$$V_{c_u c_u}(\alpha) \equiv -\frac{\partial}{\partial c_u}\int_0^T w(t;\alpha)e^{-rt}dt = -\int_0^T \frac{\partial w}{\partial c_u}(t;\alpha)e^{-rt}dt \geq 0 \tag{93}$$

Equation (28) shows that cumulative discounted crop production is not decreasing in crop price. One can notice that it is the discounted production function slope, integrated over the entire planning horizon that is not decreasing. For a given finite period of time, crop production could be decreasing, while the crop price has increased. While on the short run such a behaviour could appear as irrational, on the long run, *i.e.* when equation (91) is verified over the entire planning horizon, such a behaviour would be somehow rational. A similar reasoning can be followed for the impact of an increase in the cost of soil degrading practices and in the cost of conservation practices. Equations (92) and (93) demonstrate that the cumulative discounted use of soil degrading practices and the cumulative discounted investment in conservation practices are non-increasing in their own prices.

The comparative dynamics of the discount rate $r$ cannot be derived through the use of Corollary 11.2, since the integrand function $F(.)$ of our soil quality investment model:

$$F(t, m, u, s; \alpha) \equiv [pf(s, m) - c_m m - c_u u]e^{-rt} \tag{94}$$

is not convex in the discount rate $r$. Hence, to conduct the comparative dynamics of the discount rate, we rely on the Theorem 11.2 of Caputo (2005).

From Theorem 11.2, with $\alpha \equiv (p, c_m, c_u, r)$, so that the discount rate $r$ is the fourth element of the parameter vector $\alpha$, and since $L_{\alpha\alpha}(\beta)$ is a negative semi-definite matrix, we have:

$$\begin{aligned} L_{rr}(\beta) = &- \int_0^T [F_{rs}(t, z(t;\alpha), v(t;\alpha);\alpha)\frac{\partial z}{\partial r}(t;\alpha) + F_{rm}(t, z(t;\alpha), v(t;\alpha;\alpha)\frac{\partial v}{\partial r}(t;\alpha) \\ &+ F_{ru}(t, z(t;\alpha), v(t;\alpha);\alpha)\frac{\partial w}{\partial r}(t;\alpha)]dt \qquad (95) \\ = &- \int_0^T [pf_s(t, z(t;\alpha), v(t;\alpha))\frac{\partial z}{\partial r}(t;\alpha) + [pf_m(z(t;\alpha), v(t;\alpha))\frac{\partial v}{\partial r}(t;\alpha) - c_m] \\ &- c_u\frac{\partial w}{\partial r}(t;\alpha)]te^{-rt}dt \leq 0 \qquad (96) \end{aligned}$$

Similarly to the previous equations describing the comparative dynamics of our model, equation (96) describes the impact of a change in the discount rate on the soil quality investment model over the entire planning horizon. However, the comparative dynamics of the discount rate are not easy to interpret, contrary to the comparative dynamics of the crop price and the costs of soil degrading practices and soil conservation practices.

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