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Department of Agricultural Economics

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J. D. STEWART
Canterbury Agricultural College,
New Zealand,
and
University of Reading.

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CONTENTS

1. INTRODUCTION

Linear programming is a mathematical technique, based on the properties of matrices, for solving maximisation and minimisation problems. In the field of farm management it has been applied to two principal classes of problems: firstly to the maximising of a revenue function which is subject to limitations of farm resources, and secondly to the minimizing of a cost function subject to specified requirements as to the nature of the process. This paper is concerned with the first class of problem, namely, with the determination of the profit-maximising combination of activities on a farm which has an array of limiting resources. Examples of the second class are not as common as the first, but the technique has been used for determining the least cost mixture for feeding stuffs which have certain specified nutritional requirements. Illustrations of this application of linear programming may be found in Waugh (1951), Mackenzie and Godsell (1956) and Alexander and Hutton (1957). Heady and Candler (1958) have given a comprehensive description of linear programming methods in agricultural economics.

The profit maximisation problem will be recognized immediately as being one which is customarily solved in farm management work by comparative budgeting. In fact, linear programming is closely related to budgeting; it is based on the same assumptions in many respects and requires the same basic data although normally in a much more detailed form. It is claimed by the proponents of linear programming that it has one decided advantage. In comparative budgeting we have to select a certain number of possible programmes and budget these against each other. The number of programmes which can be selected is limited by the manpower and time available, and it is seldom that more than three or four combinations are compared. This may be, and often is, entirely satisfactory, especially when the experience and judgement of the person doing the budgets is substantial, and/or where the range of possible programmes for the farm being studied is very narrow. But if either or both of these conditions are not met the problem may be appropriate for linear programming. For the strength of linear programming is that provided the basic data are adequate, and the basic assumptions are not violated, it will produce the unique profit-maximising solution to the enterprise-combination problem.

The heart of the matter is this. In budgeting we rely on the ability of the human element to nominate the few alternatives which are crucial. In linear programming, if the above two provisions are satisfied we proceed automatically to the optimum solution.

It should be emphasised here that these two provisions are extremely important ones. By far the most important limitation to linear programming at present is the inadequacy of the basic input/output data which are an essential ingredient of the linear programming matrix. This is despite the fact that the resources devoted to the collection and analysis of farm statistics in this country, to outsiders, seem to be very great. Furthermore, despite the flood of literature on this subject from the U.S.A., it still seems that there is a continuing need to examine the appropriateness of the simple linear model for the sort of farm situations which are met in day to day advisory work. However, this is not the function of this particular paper. It is merely to present a further practical example of a linear programming problem, in the belief that there is a continuous need for some sort of communication of ideas and procedures between those interested in this work. It does not contain anything very new but it may be that something of practical value in the mechanical aspects will be found. While it is not likely that in the immediate future linear programming will be widely used in general farm management advisory work at the District Officer level, it may be that it will have an important role to play in basic farm management work in the Provincial Agricultural Economics Service, particularly when electronic computing facilities become more widely available and more resources are devoted to the collection of the necessary data.

The farm is now described briefly, then the steps in the construction of the model are discussed in some detail, and finally the solution is presented with some comment on its significance.

2. THE FARM

Area	375	acres.
Soil and Topography	120	acres of ploughable downs, clay with flints over chalk.
	160	acres flat.
	95	acres in permanent grass because of flooding risk.
	375	acres.
Buildings		Cow shed with 40 standings. Barn with accommodation for approximately 150 acres of corn at normal yields. Adequate range of sheds for rearing young stock, etc. and for equipment. Buildings generally fairly old.

Cropping 1959/60

Potatoes	12	acres
Corn	160	acres
Kale	30	acres
Leys	78	acres
	280	acres
Permanent grass ...	95	acres
	375	acres

Livestock

Dairy: ... ... 24 Friesian Cows in Milk
5 Dry Cows
24 Followers

Beef ... ... 30 A.A. Cows, and in-calf Heifers
5 Heifer Calves
4 Steer Calves
4 Friesian Steers.

Land Classification

It has been found that the following classification has been necessary for the design of rotations.

(1) Outlying arable ... ... 110 acres
(2) Area too distant for dairy herd to graze kale ... 90 acres
(3) Area sufficiently close for dairy herd to graze kale 80 acres
(4) Permanent grass ... ... 95 acres.

Other than this, despite the difference in topography the land is regarded as homogenous for the purposes of the linear programming approach. There does not appear to be any significant variation in yield or input requirements.

This farm for the previous year had been analysed using the conventional Blagburn (1957) system, and the following are some of the important indices, which will give an indication of the existing level of management and results. (Table 1).

TABLE 1

Efficiency Indicators 1958/1959

Index	Farm	Average	Best 25%
System Index	75	89	110
Yield Index	95	100	114
Feed Acres per Livestock Unit ...	2·9	3·1	2·2
Milk Yield per Cow (gals.) ...	1022	793	840
Milk Sales per Cow	£154	£120	£127
Net Output per Acre	£24–8s.	£30–13s.	£42–17s.
Expenses per Acre	£21–7s.	£23–19s.	£27–9s.
Profit per Acre	£3–1s.	£6–14s.	£15–8s.

These figures indicate an unsatisfactory result despite high milk sales per cow. The accompanying report stated that low output was the problem, two main factors being responsible:—

(1) the farming system was of low intensity as indicated by the system index.

(2) two enterprises, the arable crops and the cattle did not produce a satisfactory output.

The report added that a factor responsible for the comparatively low intensity was the absence of subsidiary livestock enterprises, e.g. pigs and poultry, which accounted for £11 per acre on the best farms, but recognised that the farmer might not be willing to incur the risks associated with moving into these enterprises. After dismissing other possibilities of adjustment the report concluded that further detailed work on the problem of the right balance between crops and the livestock enterprises was necessary. This is the point in farm management advisory work where linear programming, programme planning or budgeting comes in. In this case it was decided to try linear programming.

3. CONSTRUCTION OF THE MODEL

The various stages in the construction of a linear programming model are as follows:

(1) To list the possible activities which are to be programmed.

(2) To calculate the net revenue of each of the activities.

(3) To determine and enumerate the resource restrictions and other limitations which are to be imposed upon the activities. (These may be grouped under the single title—constraints).

(4) To detail the requirements of the activities for these resources.

1. Activities

These may be production activities such as cash crops and livestock enterprises, or they may be intermediate activities such as fodder crops and grazing. Intermediate activities do not enter directly into production, but contribute indirectly, normally through livestock enterprises. Other kinds of activities which can be included in linear programming matrices where necessary are buying and hiring activities. For example, if the purchase of hay would be possible in the profit maximising solution then we should include an activity called "Purchasing Hay". Or if we wished to allow for the possibility of hiring contractors for combining wheat then we should add an appropriate activity, e.g. "Hiring Combine Hours". Similarly we could allow for hiring extra casual labour at harvest time by defining an appropriate activity. It can be seen, therefore, that on a large and complex mixed farm it would soon be possible to have a very large number of activities to programme.

One of two general attitudes to the definition of activities may be adopted, depending on the purpose of the exercise. If we are attempting to establish an optimum modal programme which will be appropriate for a number of homogenous farms, as for example Barnard and Smith (1959) in their recent publication, then we must define the full range of practicable processes. This imposes important difficulties. In fact the ability of linear programming, or any other technique for that matter, to provide modal solutions of this kind requires further examination. Another situation where this sort of approach to the defining of activities might be appropriate would be where a farmer is beginning on a new farm, and where he has no preferences or prejudices as to what he is going to do.

But very often in farm advisory work the situation is that the farmer himself may put a limit on the number of enterprises to be considered because of personal likes and dislikes or abilities and disabilities. In these cases there is nothing to be gained by including in a linear programming matrix, activities which the farmer will not consider, other than perhaps to indicate the costs of these prejudices in terms of lower revenue. In the same sense farmers may also put certain minimum conditions upon any prospective plan, such as the carrying of a number of beef animals, which has a universal fascination wherever large scale farming is carried on. Linear programming in this respect is not nearly so inflexible as often seems to be imagined. These personal conditions can generally be incorporated in the basic matrix.

The most crucial problem in the defining of activities is to ensure that there is independence between them. It is unfortunate for the farm economist that so little seems to be known about what important, measurable inter-relationships between various enterprises on the farm in fact operate. We know that corn crops following leys which have been intensively managed and grazed will yield more heavily, or require less fertiliser, than corn following leys which have carried less stock. We know about certain residual effects e.g. those in corn following potatoes. We also know that disease and weed control require certain rotational patterns. These relationships are examples of the sort of complementarity which in theory should be avoided in the defining of activities for a linear programming matrix.

Two approaches have been used to minimise the effects of these relationships. One has been to group complementary activities into an aggregate activity such as a crop rotation. The second has been to ensure that rotational and fertility requirements will be met by specifying in the form of linear inequalities, certain requirements as to the proportional relationships which the activities must have to each other.

The first approach certainly avoids the problem of complementarity more successfully. But there are two, perhaps minor criticisms which may be levelled against it. In the first place the need to specify a small number of possible rotations may invite the same sort of criticisms which linear programmers level against comparative budgets; namely that we may be begging the question. In some classes of mixed farming the number of possible rotations containing significant variations in the proportions of the constituents is very large. In such cases, where we restrict our selection of rotations to say four or five, there may be no certainty that we have included the best. Secondly, there is always the possibility where we programme a number of rotations against each other that because of the continuous nature of the simplex solution we may finish up with some very small areas of some rotations. In fact where the land is homogenous in terms of soil, topography and access the farmer may question the practicability of more than one rotation. There have been examples where this sort of solution has been produced and where it has been adjusted subsequently to bring it into line with practical farming considerations, e.g. Peterson (1955). These two points may be by no means sufficiently strong to weaken the aggregative approach to the defining of activities. Indeed it may be argued that there are other reasons why the best that we can hope for from linear programming is a close approximation to the optimum combination of activities on a farm, and that frequently there will be a need for adjustments to the programmed solution.

The second alternative way of defining activities where rotational effects and requirements are important, namely the specification of maximum and minimum conditions as to the amount and proportions of each activity seems to have been first used successfully by Barnard and Smith (1959). It does permit greater flexibility in the combination of individual activities, but does not altogether overcome the complementarity problem. One other deficiency may be that in the process of rigorously defining the amounts and proportions of each activity, particularly on a small farm, the problem is put in such a straight-jacket that it is virtually solved before the programming begins. However this is the method used in this example, as it was found that the problem of defining aggregate activities in a comprehensive way was extraordinarily difficult.

A further small complication in the defining of activities arises where we have to distinguish between classes of land. We may have to distinguish land classes on the basis of soil or topography, location and access, or even on the basis of time. In this example we have defined four land classes (page 3) on the basis of variations in location and access. We shall in what follows refer to these as Land (1), Land (2), Land (3), and Land (4) respectively.

The activities included in the model are:—

Cash Crop Activities		*Unit*
P_1	Wheat (1)	1 acre
P_2	Wheat (2)	,,
P_3	Wheat (3)	,,

Cash Crop Activities		*Unit*
P_4	Barley (1)	1 acre
P_5	Barley (2)	,,
P_6	Barley (3)	,,
P_7	Oats (1)	,,
P_8	Oats (2)	,,
P_9	Oats (3)	,,
P_{10}	Potatoes	,,
Livestock Activities		
P_{11}	Single-suckling beef	1 cow+followers
P_{12}	Dairy	1 cow+followers
Intermediate Activities		
P_{13}	Kale for Dairy Cows	1 acre
P_{14}	Kale for Beef Cows	,,
P_{15}	Hay (1)	,,
P_{16}	Hay (2)	,,
P_{17}	Hay (3)	,,
P_{18}	Hay (4)	,,
P_{19}	Grazing (1)	,,
P_{20}	Grazing (2)	,,
P_{21}	Grazing (3)	,,
P_{22}	Grazing (4)	,,
Purchasing Activity		
P_{23}	Purchase straw	1 ton
Overtime Activities		
P_{24}	February Overtime	1 hour
P_{25}	March ,, ...	,,
P_{26}	April ,, ...	,,
P_{27}	May ,, ...	,,
P_{28}	August ,, ...	,,
P_{29}	September ,, ...	,,

Notes on the activities

1. The wheat, oats and barley may be grown on any of the three land classes, (1), (2) and (3). The oats and a portion of the barley are used for feed, but these have been entered at their market price, and the livestock enterprises "pay" this price.
2. Potatoes may be grown on Land (2) only.
3. Kale for dairy and beef herds is grown on Land (3) and Land (1) respectively.
4. Hay and Grazing may be on any of the four land classes.
5. Straw is provided by the corn crops and required by the two livestock activities, but a purchasing activity is included in case this is needed.

6. The beef enterprise consists of Aberdeen Angus cows and replacement heifers, the sales being weaner calves and cull cows.
7. The dairy enterprise consists of Friesian cows and followers, the sales being milk, surplus calves and cull cows.
8. After an examination of the overtime records, overtime activities for the months in which they were likely to be needed were included, namely February, March, April, May, August and September.

2. Calculation of the Net Revenues

In the calculation of the net revenues we make the significant assumption that for the duration of the plan fixed costs will not be influenced by the level at which the activities are carried on. Net revenue per unit of an activity is therefore calculated by deducting from the gross revenue per unit, the direct variable costs per unit. It is, of course, the characteristic assumption of linear programming that this net revenue per unit of an activity is constant however many units of the activity may be included. By maximising the revenue based on this calculation we are maximising the returns to the fixed resources. In symbolic form we are maximising the function:

$$f = \sum_{j=1}^{n} c_j P_j$$

where P_j states the level at which each activity is to be carried on, where there are n different activities, and where c_j states the net revenue per unit of each activity.

The unit of an activity may be taken at any convenient level, 1 acre or 10 acres or 100 acres, 1 cow or 100 cows. The aim should be to define the unit so that the coefficients in the basic matrix have a minimum range in their numerical values. This may avoid the need for scaling the matrix later when it is being prepared for a computer, or it will simplify the arithmetic if the problem is being solved on a desk machine.

The costs and returns used in the example were those ruling in 1958/59, although wherever it seemed appropriate these were adjusted to eliminate obvious abnormalities. The general picture on this farm is one of relatively low corn yields (wheat and barley, 22 cwts., oats, 20 cwts.) and high milk sales per cow. On the basis of recent results the gross output of potatoes was put at £90 per acre. A summary of the net revenue calculations is given in Table 2. Lack of more precise information precluded the possibility of allowing for differences in costs and returns on each of the land classes. Where a classification of land has to be based on important differences of soil it would, of course, be imperative to define activities more specifically in relation to each land class, with appropriate variations in net revenues.

TABLE 2

Net Revenues

1 Activity	Unit	2 Gross Revenue per Unit (£)	3 Assignable Costs per Unit (£)	4 Net Revenue per Unit (£)
Wheat (P_{1-3}) ...	1 acre	33·5	12·3	21·2
Barley (P_{4-6}) ...	1 ,,	31·9	12·8	19·1
Oats (P_{7-9})	1 ,,	27·5	12·4	15·1
Potatoes (P_{10}) ...	1 ,,	90·0	43·5	46·5
Beef (P_{11})	1 cow	44·0	13·5	30·5
Dairy (P_{12}) ...	1 cow	166·0	70·6	95·4
Kale (P_{13-14}) ...	1 acre	—	6·5	−6·5
Hay (P_{15-18}) ...	1 ,,	—	4·0	−4·0
Grazing (P_{19-22}) ...	1 ,,	—	5·0	−5·0
Straw (P_{23}) ...	1 ton	—	2·0	−2·0
Overtime (P_{24-29}) ...	1 hour	—	0·25	−0·25

3. Determination of the Constraints

The constraints take the form of a series of linear inequalities. This is merely to say, that in the final solution, the total requirements for any resource must be equal to or less than the total amount of that resource which is available.

This may be expressed as follows:—

$$b_i \geqslant \sum_{j=1}^{n} r_{ij} P_j \qquad (i = 1, 2, \ldots, m.)$$

where b_i states the level of availability of the i^{th} resource, where there are m different resources, P_j states the level at which each activity is to be carried on, where there are n possible activities, and r_{ij} indicates the per unit requirement of the j^{th} activity for the i^{th} resource.

In the computation process the inequalities are converted to equalities by introducing activities which allow the possibility of some units of the resource being un-used.

The inequality above converted to the corresponding equality becomes:—

$$b_i = \sum r_{ij} P_j + P_{n+i}$$

where P_{n+i} is the activity allowing for the non-use of some or all of the i^{th} resource.

The assumption of linearity in the equations expressing the relationships between activities and limiting resources is of crucial importance. This assumption implies that at no matter what scale an activity is carried on its requirements of a particular resource per unit will be the same; that is, returns to scale are linear. This assumption may trouble some people. While it has been said that there is not a great deal of statistical evidence for the existence of scale economies or diseconomies in agricultural technological relationships it is clear that it takes less time per cow to do all the work associated with 50 cows than it does with 20 cows, and that it takes less hours per acre to cultivate or harvest large fields than it does small fields. These may not be important examples from the point of view of the validity of a linear programming solution, but they do at least indicate the possibility of scale effects.*

As will be shown in the details of the construction of the model, constraints are not restricted to those which are imposed by limitations of physical resources such as land, labour and capital. It may be necessary to put restrictions on the proportions of various crops from the point of view of fertility and disease risk and also to reconcile the feed supplied by crop activities with the feed requirements of the livestock enterprises. In general, all these kinds of restrictions and assumptions which are used in budgeting may be translated into appropriate linear constraints for a linear programming matrix.

In the example the resources available and the constraints they impose upon the level of the activities were divided as follows:—

A. Land.

B. Buildings.

C. Labour.

D. Crop restrictions.

E. Feed reconciliation.

Operating capital is not included as a constraint in the model. The farm is freehold, and it was stated that there would be adequate working

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* However where scale effects are evident and definite it is possible to handle them in linear programming work. The major scale problem is one where resource supplies do not occur in continuous amounts, the familiar problem of economic theory, indivisibility. The handling of this problem awaits further developments in the technique of integer programming.

But where our problem is simply one of diminishing returns to scale, we can deal with it by defining linear segments on the appropriate curve, and naming these as separate activities, with appropriate lower and upper constraints on their level. The diminishing returns to scale situation will ensure that one of these activities will not come into the solution before the one preceding it on the scale is at its maximum level.

Where there are increasing returns to scale, and this will be more common, the problem is more complex and will require additional computations after an initial solution has been reached with the coefficients assumed at one level. A discussion on these issues can be seen in Giaever and Seagraves (1960). The crucial point is that linear segments of an increasing returns curve do not meet the mathematical requirement of convexity. (See Hicks 1960).

capital for any feasible programme in the short term (*Stewart*). Similarly the plant and equipment was considered to be adequate for any feasible programme.

Land

There are four land classes, (1) to (4), the areas being 110 acres, 90 acres, 80 acres and 95 acres respectively. Thus the relationship for Land (1) is expressed as follows:—

$$110 \geqslant P_1 + P_4 + P_7 + P_{12} + P_{15} + P_{19} \qquad \ldots\ldots 1$$

and for Land (2) as

$$90 \geqslant P_2 + P_5 + P_8 + P_{10} + P_{16} + P_{20} \qquad \ldots\ldots 2$$

In 1., P_1, P_4, P_7 represent the corn crops which may be grown on Land (1). P_{12} is the kale for the beef enterprise which is best suited to the outlying arable, P_{15} and P_{19} represent Hay and Grazing respectively on this area. In 2. we note that the kale activities P_{11} and P_{12} are not to use Land (2) but potatoes, P_{10} may. The dairy herd kale, P_{11} must be grown on Land (3), because of its proximity to the milking shed. Thus the relationship for Land (3) is:—

$$80 \geqslant P_3 + P_6 + P_9 + P_{11} + P_{17} + P_{21} \qquad \ldots\ldots 3$$

Land (4), in permanent grass, is not to be cropped because of flood risk, but may occasionally be ploughed and direct reseeded to maintain a good sward. Thus the only activities which may use Land (4) are P_{18}, Hay (4) and P_{22}, Grazing (4), and we have:—

$$95 \geqslant P_{18} + P_{22} \qquad \ldots\ldots 4$$

Buildings

The number of standings in the cow shed is 40, imposing a restriction of that number on the size of the dairy herd. Thus we have:—

$$40 \geqslant P_{14} \qquad \ldots\ldots 5$$

There is grain storage for approximately 150 acres of corn at normal yields, but a restriction was not imposed because of this, as corn may be sold ex field. Strictly it would be more accurate to include an additional activity for selling corn ex field, as the revenue per unit may be different. However a conservative revenue figure for the corn activities as a whole has been adopted and ex-field corn has been grouped with stored corn. It will be observed that this degree of accuracy would result in an additional nine activities.

Labour

The fixed labour available is six men, and the working hours per month have been taken as 200 per man. Twelve monthly periods were taken as the basis for estimating the labour requirements of the activities, and corresponding labour profiles were estimated on the basis of man-hours per unit of activity. This approach suffers from some deficiencies. In the first place these profiles do not account for the labour involved in those jobs on the farm which are not directly associated with the activities. Overhead labour of this nature may account for 10–15% of the total labour requirements. The assumption in this model is that these overheads will be met in those months when the labour hours are not fully utilised directly by the activities. This is probably a reasonable assumption, for in fact these jobs are normally those done during slack periods. Simpson (1960) has handled this problem by including an additional restriction related to the labour available over the whole year, a deduction of 13% being made for the overhead work.

Secondly the simple monthly profiles do not meet the possibility of the overlapping of work from one month to another, for example at harvest time, nor of the need to provide holidays at certain times for the farm workers. Simpson has met these difficulties by making appropriate groupings of some months, e.g. August and September, September and October, and August, September and October, to allow for the possible overlapping of the corn harvest, potato lifting, manure loading and straw baling. This seems to be a particularly successful approach to the problem of stating the labour requirements of each activity, and is certainly an improvement upon the method generally used.

The actual coefficients, the per unit labour requirements of the activities, were built up from information gathered on the farm, and where necessary this was supplemented by other data, such as enterprise costings, where these seemed appropriate. The establishment of the labour co-efficients may be found to be the most troublesome feature of linear programming work. There may be constant concern at the possibility of small changes in these coefficients, which are often not accurately known and which are subject to considerable variance, having a significant impact on the final solution. Moreover there is the problem of scale effects which has been discussed above. These are fundamental points, the significance or insignificance of which seem to require further examination under specific conditions. Certainly a modified simplex method has been suggested by Heady and Candler (1958) for allowing input coefficients to vary, but the situation in practice is that there may be no reason why one or two coefficients may vary rather than ten or twenty. These possibilities occasionally lead one to the view that the sort of judgements often made in budgeting work which impose broad restrictions on the principal activities such as combining, hay baling and cultivation may provide us with as close an approximation to the real limitation as we may get by using more detailed but perhaps spurious coefficients of labour requirements. However it is considered more satisfactory and more in keeping with the precision of the linear programming model to attempt to state the requirements of labour per unit of activity, particularly at what are likely to be labour bottlenecks, in a precise way. Relationships 6 to 17 in Table 3 therefore express the labour requirements of the activities by months from January to December inclusive. For example, for January labour the relationship is:—

$$1200 \geqslant 22{\cdot}0\ P_{10} + 2{\cdot}5\ P_{13} + 13{\cdot}8\ P_{14} \qquad \ldots\ldots 6$$

Where overtime work is introduced the relationship takes the following form,

August labour:—

$$1200 \geqslant 3{\cdot}2 \sum_{j=1}^{9} P_j + {\cdot}5\ P_{13} + 11{\cdot}0\ P_{14} - 1{\cdot}0\ P_{27} \qquad \ldots\ldots 13$$

Crop Restrictions

For reasons of fertility and crop health, acreage restrictions have been imposed on the total corn which may be grown on each of the land classes (1) to (3). For example this limit is put at 70 acres of Land (1), giving the relationship:—

$$70 \geqslant 1{\cdot}0\ P_1 + 1{\cdot}0\ P_4 + 1{\cdot}0\ P_7 \qquad \ldots\ldots 18$$

Similarly, limits of 55 and 40 acres are imposed on Land (2) and Land (3) respectively, giving us rows 19 and 20. A limit of 20 acres is placed upon potatoes. This is not because of the quota, although it does happen to coincide, but because the farmer considers this the maximum proportion of the suitable potato land, Land (2), which should be in this crop. This gives row 21:—

$$20 \geqslant 1{\cdot}0\ P_{10} \qquad \ldots\ldots 21$$

Rows 22 and 23 prescribe maximum and minimum proportions of wheat and oats respectively, for husbandry reasons. For example, the condition for wheat is that it must not exceed $\frac{1}{4}$ of the total area of corn. Thus:—

$$\sum_{j=1}^{3} P_j \leqslant {\cdot}25 \sum_{j=1}^{9} P_j$$

$$\therefore \quad 0 \leqslant {\cdot}25 \sum_{j=4}^{9} P_j - {\cdot}75 \sum_{j=1}^{3} P_j$$

$$\text{or} \quad 0 \geqslant - {\cdot}25 \sum_{j=4}^{9} P_j + {\cdot}75 \sum_{j=1}^{3} P_j \qquad \ldots\ldots 22$$

The area of oats must at least be equal to $\frac{1}{8}$ of the total corn area for rotational requirements.

$$\sum_{j=7}^{9} P_j \geqslant {\cdot}125 \sum_{j=1}^{9} P_j$$

$$\therefore \quad 0 \geqslant {\cdot}125 \sum_{j=1}^{6} P_j - {\cdot}875 \sum_{j=7}^{9} P_j \qquad \ldots\ldots 23$$

Feed Reconciliation

The requirements of kale in the anticipated feeding programme are $\frac{1}{2}$ acre per unit of the beef and dairy enterprises. (It will be recalled that a unit of these enterprises consists of a cow plus followers). Thus to reconcile the dairy herd with the dairy kale it is required that:—

$$1{\cdot}0\ P_{13} \geqslant {\cdot}5\ P_{12}$$

$$\therefore \quad o \quad \geqslant {\cdot}5\ P_{12} - 1{\cdot}0\ P_{13} \qquad \ldots\ldots 24$$

and for the beef herd:—

$$1{\cdot}0\ P_{14} \geqslant {\cdot}5\ P_{11}$$

$$\therefore \quad o \quad \geqslant {\cdot}5\ P_{11} - 1{\cdot}0\ P_{14} \qquad \ldots\ldots 25$$

Reconciling the hay based on a requirement of $\frac{1}{4}$ acre per beef unit, and $\frac{3}{4}$ acre per dairy unit (at yields of $1\frac{3}{4}$ tons) gives row 26.

$$1{\cdot}0 \sum_{j=15}^{18} P_j \geqslant {\cdot}25\ P_{11} + {\cdot}75\ P_{12}$$

$$\therefore \quad o \quad \geqslant {\cdot}25\ P_{11} + {\cdot}75\ P_{12} - 1{\cdot}0 \sum_{j=15}^{18} P_j \qquad \ldots\ldots 26$$

For reasons of minimum quality of hay a restriction is imposed on the amount which can be saved from the permanent grass, Land (4), viz. 20 acres.

$$20 \geqslant 1{\cdot}0\ P_{18} \qquad \ldots\ldots 27$$

The yield of straw from wheat and barley is estimated at 1·2 tons and from oats 1 ton, while the requirements per unit of the dairy and beef activities respectively are 0·75 tons and 0·25 tons, and of one acre of potatoes, 1 ton.

$$1{\cdot}2 \sum_{j=1}^{6} P_j + 1{\cdot}0 \sum_{j=7}^{9} P_j + 1{\cdot}0\ P_{23} \geqslant {\cdot}25\ P_{11} + {\cdot}75\ P_{12} + 1{\cdot}0\ P_{10}$$

$$\therefore \quad o \quad \geqslant {\cdot}25\ P_{11} + {\cdot}75\ P_{12} + 1{\cdot}0 P_{10} - 1{\cdot}2 \sum_{j=1}^{6} P_j - 1{\cdot}0 \sum_{j=7}^{9} P_j - 1{\cdot}0\ P_{23} \qquad \ldots\ldots 28$$

The grazing requirements of the cows were worked out on the basis of the existing carrying capacities, the available grazing being allocated on the basis of stock units.

The figures were 2·6 acres per dairy unit and 1·8 acres per beef unit, giving the relationship:—

$$1{\cdot}0 \sum_{j=19}^{22} P_j \geqslant 1{\cdot}8\ P_{11} + 2{\cdot}6\ P_{12}$$

$$\therefore \quad o \quad \geqslant 1{\cdot}8\ P_{11} + 2{\cdot}6\ P_{12} - 1{\cdot}0 \sum_{j=19}^{22} P_j \qquad \ldots\ldots 29$$

The initial matrix is presented as Table 3. The activities allowing for the non-use of resources are omitted.

4. THE SOLUTION

The initial simplex solution,* is presented in Table 4. It is feasible in terms of practical farming conditions, which means merely that the constraints imposed are comprehensive. The absence of grazing on Land (2) may cause some concern, but need not necessarily do so. There will be

* Ferranti Pegasus Computer using Simpfix Mk. 6.

TABLE 3

Basic Matrix

Limitations	Availability	Relation-ship	P1 Wheat (1)	P2 Wheat (2)	P3 Wheat (3)	P4 Barley (1)	P5 Barley (2)	P6 Barley (3)	P7 Oats (1)	P8 Oats (2)	P9 Oats (3)	P10 Potatoes	P11 S.S. Beef	P12 Dairy	P13 Dairy Kale	P14 Beef Kale	P15 Hay (1)	P16 Hay (2)	P17 Hay (3)	P18 Hay (4)	P19 Grazing (1)	P20 Grazing (2)	P21 Grazing (3)	P22 Grazing (4)	P23 Purchase Straw	P24 February Overtime	P25 March Overtime	P26 April Overtime	P27 May Overtime	P28 August Overtime	P29 September Overtime
Net Revenue (£) ...			21·2	21·2	21·2	19·1	19·1	19·1	15·1	15·1	15·1	46·5	30·5	95·4	−6·5	−6·5	−4·0	−4·0	−4·0	−4·0	−5·0	−5·0	−5·0	−5·0	−2·0	−0·25	−0·25	−0·25	−0·25	−0·25	−0·25
1. Land (1)	110 acres	⩾	1·0			1·0			1·0							1·0	1·0				1·0										
2. Land (2)	90 ,,	⩾		1·0			1·0			1·0		1·0						1·0				1·0									
3. Land (3)	80 ,,	⩾			1·0			1·0			1·0				1·0				1·0				1·0								
4. Land (4)	95 ,,	⩾																		1·0				1·0							
5. Cow Standings ...	40	⩾												1·0																	
6. January Labour	1200 hours	⩾										22·0	2·5	13·8																	
7. February ,,	1200 ,,	⩾				1·8	1·8	1·8	1·8	1·8	1·8	22·0	2·5	13·8												−1·0					
8. March ,,	1200 ,,	⩾	0·6	0·6	0·6	1·4	1·4	1·4	1·4	1·4	1·4	2·5	2·5	12·8			0·7	0·7	0·7	0·7							−1·0				
9. April ,,	1200 ,,	⩾	0·8	0·8	0·8	2·9	2·9	2·9	2·9	2·9	2·9	15·6	2·5	11·8			1·4	1·4	1·4	1·4	0·5	0·5	0·5	0·5				−1·0			
10. May ,,	1200 ,,	⩾	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	2·9	4·0	11·0	3·2	3·2													−1·0		
11. June ,,	1200 ,,	⩾										3·0	0·5	11·0	3·4	3·4	10·0	10·0	10·0	10·0											
12. July ,,	1200 ,,	⩾											0·5	11·0	4·5	4·5															
13. August ,,	1200 ,,	⩾	3·2	3·2	3·2	3·2	3·2	3·2	3·2	3·2	3·2		0·5	11·0																−1·0	
14. September ,,	1200 ,,	⩾	5·9	5·9	5·9	3·2	3·2	3·2	3·2	3·2	3·2	10·0	0·5	12·3																	−1·0
15. October ,,	1200 ,,	⩾	4·6	4·6	4·6							12·0	0·5	12·8																	
16. November ,,	1200 ,,	⩾											2·5	13·8	5·4	5·4															
17. December ,,	1200 ,,	⩾				2·7	2·7	2·7	2·7	2·7	2·7	2·7	2·5	13·8																	
18. Maximum Corn (1)	70 acres	⩾	1·0			1·0			1·0																						
19. Maximum Corn (2)	55 ,,	⩾		1·0			1·0			1·0																					
20. Maximum Corn (3)	40 ,,	⩾			1·0			1·0			1·0																				
21. Maximum Potatoes	20 ,,	⩾										1·0																			
22. Maximum Wheat	0	⩾	0·75	0·75	0·75	−0·25	−0·25	−0·25	−0·25	−0·25	−0·25																				
23. Minimum Oats ...	0	⩾	0·125	0·125	0·125	0·125	0·125	0·125	−0·875	−0·875	−0·875																				
24. Dairy Kale ...	0	⩾												0·5	−1·0																
25. Beef Kale ...	0	⩾											0·5			−1·0															
26. Hay	0	⩾											0·25	0·75			−1·0	−1·0	−1·0	−1·0											
27. Maximum Hay (4)	20 acres	⩾																		1·0											
28. Straw	0	⩾	−1·2	−1·2	−1·2	−1·2	−1·2	−1·2	−1·0	−1·0	−1·0	1·0	0·25	0·75											−1·0						
29. Grazing	0	⩾											1·8	2·6							−1·0	−1·0	−1·0	−1·0							

TABLE 4

Initial Solution

A. LAND UTILIZATION (Acres)

	Grazing	Wheat	Barley	Oats	Potatoes	Kale	Hay
Land (1)	32·9	41·2	28·8			7·1	
Land (2)			34·4	20·6	20·0		15·0
Land (3)	20·0		40·0			20·0	
Land (4)	76·5						18·5
TOTAL	129·4	41·2	103·2	20·6	20·0	27·1	33·5

B. LIVESTOCK (nearest whole number)

Dairy Cows	40
Beef Cows	14

C. OVERTIME HOURS

February	50
April	144
September	138

D. NET REVENUE £7,292

some aftermath grazing, and also the possibility of spreading muck for fertility maintenance.

In relation to the existing programme, which has fluctuated slightly in recent years, the main adjustments suggested involve adjustments in the potato and corn acreages, an increase in the dairy herd to the maximum permitted by the cow standings, and a reduction in the size of the beef herd. Although these changes may appear small, they are important to the extent of increasing the returns to the fixed resources by about 15%.

In addition to the real activities listed in Table 4, the simplex solution provides a list containing the amounts of resources unused. In this example these are mainly unused labour hours, which could be expected to be quite substantial in view of the seasonal nature of the farm programme, and also because no allowance has been made for labour overheads. The total labour surplus to the direct requirements of the activities amounts to about 18% in this example.

Finally, the simplex solution provides us with estimates of the marginal product values of the limiting factors, and with an indication of the change in the net revenue which would be required to bring an excluded activity into the programme. The marginal product values of the important limiting factors are given in Table 5.

TABLE 5

Marginal Product Values

Initial Solution

Limitations	Units	Marginal Product Values per Unit
		£
Land	Acres	5·9
Dairy Standings	—	43·2
Corn (maximum)	Acres	11·3
Wheat (maximum)	Acres	2·4
Oats (minimum)	Acres	4·0*
Potatoes (maximum)	Acres	28·6
February, April and September Labour	Hours	0·25
Grazing	Acres	11·1
Hay	Acres	10·3

* This figure may be interpreted as the increase in net revenue which would arise from a decrease of one acre in oats.

It is interesting to note that the estimate for the marginal product of land from this model is markedly lower than other estimates from linear programming solutions recently published. For example Barnard and Smith (1959) arrived at a figure of £22·41 for an East Anglian dairy farm, Simpson (1960) £23·1 for arable acres on a Yorkshire farm, and Tyler (1960), £28·4 for a mixed farm in Kent. These are very high estimates in relation to rents, and of course imply that with other resources remaining the same, profits could be substantially increased by acquiring additional acres of land. The lower figure produced by the present model, is an indication that with the existing constraints on cow numbers, cash crop areas etc. additional acres would yield virtually only their rent. But of course if at the same time adjustments were made to these constraints, then the impact of additional acres on the revenue would differ from this figure. Theoretically the very high marginal product values in the other examples cited indicate a lack of balance in the resource structure of the farms.

The figure of £0·25 for February, April and September labour merely indicates that extra hours of labour available in these months would reduce overtime hours correspondingly and hence, costs.

It is evident from these marginal product values that the most important limitation to the expansion of profits, at the existing level of management is the restriction on cow numbers by the number of standings. (The potato acreage is considered to be restricted by the area of suitable soils). The guide which these residual figures give to the relative importance of the constraints on the particular farm is an important by-product of a linear programming solution. But their use should finish at this point. Further programming would be necessary to study the range over which these values hold good, before any planning of expansion could be based on them.

In this example it would be useful to carry the analysis a stage further, and programme the problem with the limitation on cow numbers omitted. This will give an indication of the kind of adjustments which would then be necessary to continue to maximise profits, and more important, whether the cost of providing additional milking facilities would be justified on the basis of increased profits. This additional solution is given in Table 6, and the corresponding marginal product values in Table 7.

TABLE 6

Solution Omitting Limitation on Dairy Cow Numbers

A. LAND UTILIZATION (Acres)

	Grazing	Wheat	Barley	Oats	Potatoes	Kale	Hay
Land (1) ...	40·0	1·3	68·7				
Land (2) ...	13·3		34·4	20·6	20·0		1·7
Land (3) ...		40·0				24·6	15·3
Land (4) ...	75·0						20·0
TOTAL ...	128·3	41·3	103·1	20·6	20·0	24·6	37·0

B. LIVESTOCK (nearest whole Number)

Dairy Cows	49
Beef Cows	—

C. OVERTIME HOURS

February	144
April	202
September	266

D. NET REVENUE ... £7,696

The revised solution contains only small adjustments in the crop activities (as well as some reallocation of activities between land classes) and the replacement of the beef herd by additional dairy cows, bringing the dairy herd size to 49. The most interesting aspect is that land has now become the crucial limitation, yielding a much higher marginal product value of £17·2 and similarly high figures for grazing and hay of £22·3 and £21·5 per acre respectively.

The increase in estimated net revenue of £400 ($5\frac{1}{2}$%) in the second plan would barely justify a heavy outlay on the redesigning or rebuilding

TABLE 7

Marginal Product Values from

Model Omitting Limitation on Dair Cow Numbers

Limitations	Units	Marginal Product Values per Unit
		£
Land	Acres	17·2
Corn (maximum)	Acres	0·06
Wheat (maximum)	Acres	2·4
Potatoes (maximum)	Acres	17·4
Oats (minimum)	Acres	4·0
February, April and September Labour	Hours	0·25
Grazing	Acres	22·3
Hay	Acres	21·5

of the cowshed, particularly bearing in mind the expected variability of the revenue and the farmer's discounting of future income.

By further analysis of the final matrix of the simplex solution it is possible to obtain a guide to the stability of the optimum solution in relation to changes in the net revenues of the activities. For example, it is possible to determine the range over which the net revenues of individual included activities may vary, other revenues remaining the same, without causing a change in the optimum solution. Also, the increase in the net revenue of an excluded activity which is necessary to bring it into the solution, again other revenues remaining the same, is available. For example, in the second solution an increase in the revenue of single suckle beef of £16·8 per unit would bring this enterprise back into the solution. However the changes which are generally of interest involve simultaneous changes in the revenues of a number of products and this is a much more complex matter which merits further research. A discussion on the analysis of the final matrix is given in Puterbaugh, Kehrberg and Dunbar (1959).

5. COMMENTS

In linear programming, as in budgeting, a certain level of managerial skill and technique is assumed. In the example input/output coefficients have been incorporated reflecting the existing level of performance, e.g. high milk sales per cow, but low yields of corn and low carrying capacities on grass. Whether management advice should be based primarily on the possibility of reshuffling activities at existing technical levels, or primarily on the scope for improving technical levels is a matter of some importance.

There are some grounds for the assertion that what is important is not so much "what you do, but how you do it". While debate on this point is not within the scope of the present paper, what may be emphasised is that in respect to variations in technical levels linear programming is no more inflexible than budgeting and other forms of farm planning, provided there is access to a digital computer to permit quick and accurate adjustments to be made. A modified simplex method, allowing for variations in input/output relationships has been explained by Heady and Candler (1958) but it is questionable whether this is sufficiently versatile to be useful in those situations where we wish to consider significant and substantial changes to managerial practices. In the present example for instance, it may be desirable to examine the possible outcome of such changes as pasture improvement through direct reseeding, increased top dressing and improved stocking, with adjustments to stock management and feeding policy. An experienced advisory officer would see these empirically as being fundamental to any improvement to the earning capacity of the farm. However, if this is the case, provided the input/output data corresponding to the new techniques and level of management can be forecast with some confidence, then it may still prove valuable to programme. After all we construct budgets which may be no more than conjectural in similar circumstances. Nevertheless it is a matter of some fascination that a high proportion of the effective farm management advisory work in this country and elsewhere continues to be at the empirical and technical level. It would be tragic if in our preoccupation with the so called "new tools" of farm management we lost touch with the art of farm management.

No amount of refinement of mathematical models is a substitute for the technical and psychological attributes which distinguish the more successful farm management advisory officers. Without this equipment, or access to it, the linear programmer is likely to be no more successful than the economic forecaster who is not prepared to be involved with technicians.

Equally, such mathematical refinement is handicapped without data of a corresponding degree of refinement with which to operate. Data processing by digital computers is highly interesting, even exciting, so that there may be a danger of this becoming an end in itself, rather than an easy and quick way of doing arithmetic.

However these points having been emphasised it is clear that for some important classes of farm management problems, linear programming is a much more powerful and certain technique than other forms of farm planning. Where its power is diminished by lack of data this is an indication of the need to devote more resources to the accumulation of such data, not of the futility of using the technique. Such data furthermore will be vital to any form of detailed farm planning.

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