ECONOMETRIC INSTITUTE

ON THE USE OF DESCRIPTIVE MEASURES FOR

CHAOS IN ECONOMIC TIME SERIES

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REPORT 9064/A

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by

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Abstract

The operational significance of the Lyapunov exponent and the correlation dimension for the measurement of chaos in economic time series of medium size length (200 observations) is investigated. In particular, models that are a mixture of a linear model, with a strong autoregressive component, and either a chaotic model or a white noise model are investigated. The empirical time series is the real exchange rate between Japan and the US. The results indicate that the implementation of the Lyapunov exponent for time series of 200 observations is not without problems and that for the JP/US real exchange rate an autoregressive model with white noise errors is more plausible than a model with chaotic disturbances according to the correlation dimension. However, the evidence in favor of the stochastic model is not very strong and a nonlinear component may be present in the data.

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July 1990

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I. Introduction

The observed cycles in the time series of several economic variables appear not as periodic and regular as the cycles that are implied by stochastic linear difference equations. In recent years several authors have proposed that the dynamics of a set of economic variables can be described by a set of *nonlinear deterministic* difference equations. It is nowadays wellknown that such a nonlinear dynamic model may not tend to stationarity or to a regular periodic behaviour as time tends to infinity; see, e.g., Devaney (1986). The study of nonlinear dynamic systems, in particular *chaotic* systems, has brought such terms as *orbit*, *strange attractor*, *fractal dimension*, *correlation dimension* and *Lyaponov exponent* (to name only a few). In this paper we describe these terms in an informal way and use the correlation dimension and the Lyapunov exponent to investigate the operational significance of the measurement of chaos in economic time series of medium size length, say 200 observations. Due to the advances in computing methods and machinery these measures are operational on long time series. Our application refers to time series of the real exchange rate between the US and Japan. Further, we make use of some simulated time series that are generated from a linear autoregressive model with chaotic disturbances and from one with white noise disturbances. We are interested in determining whether some economic time series can be described as a chaotic process or as a mixture of a linear autoregressive process and a chaotic one. This paper adds some experimental results to Granger (1990) and Frank and Stengos (1988). A more detailed analysis on the operational significance of simple measures for the determination of chaos that includes more experimental evidence will be reported in future work.

II. Orbit, strange attractor, chaos

Let x_t be a vector of observations on N economic variables at time t. Examples are prices of agricultural and financial commodities and macro-economic variables as gross national product and exchange rates. The process that generates these variables is supposed to have started at $t = 0$ and the subsequent values x_1, x_2, x_3,... are obtained in a sequential way by means of the difference equation

$$x_t = F(x_{t-1}) \qquad t = 1, 2, ... \qquad (1)$$

where F is a *nonlinear* function that maps x_t into x_{t+1}. Such a time series

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process $\{x_t\}$ is defined as the *forward orbit* of x_0 and is denoted as $O^+(x_0)$, where

$$O^+(x_0) = \{x_t | \; x_t = F(x_{t-1}), \; t = 1,2,...., \; x_0 \text{ is given}\} \qquad (2)$$

Of particular interest is the behaviour of the orbit when t tends to become large. Simple examples of orbits are the time series $\{p, \; p, \; p, \; ...\}$, where the starting value is p and equation (1) is such that $p = F(p)$, and periodic series $\{p, \; q, \; p, \; q, \; ...\}$, where p is the starting value and where $q = F(p)$ and $p = F(q)$. Other, more realistic examples of orbits and orbits with a nonperiodic character are presented in, e.g., Devaney (1986).

In practice one does not know the starting value x_0 exactly. Consider then a set of starting values close to x_0. As a consequence one has a set of orbits. This raises the following problem. Will the orbits with starting values close together stay close together as time progresses and what will happen asymptotically?

In this context one makes use of the term *attractor*. Suppose one has a set A, a subset of the N–dimensional real space (since we are interested in N economic variables), and an orbit $O^+(x_0)$. If the value x_t, for all x_0 near to A, tends to A when t tends to infinity, then A is defined as an attracting set. Suppose further that an orbit exists on the attracting set A that completely fills the set A, then A is defined as an *attractor*.

One may conjecture that two orbits defined on an attractor with starting values close together will stay together. However this need not be the case. There exist attractors with orbits defined on them which start close together. However, each of these orbits follow a completely different time path and information from one orbit is totally irrelevant for the possible prediction of the time path of the other orbit. Such attractors are called *strange attractors* or *chaotic attractors* or less prosaic *aperiodic attractors*. A key feature of chaotic attractors is the sensitivity with respect to the initial condition.[1] A well known example is the time path of two orbits defined by the nonlinear equation $x_{t+1} = 4x_t(1 - x_t)$ with starting values, say, 0.10 and 0.10000001.

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[1] For a more formal definition of a chaotic attractor we refer to Devaney (1986).

III. MEASURING A CHAOTIC ATTRACTOR WITH THE LARGEST LYAPONOV EXPONENT AND THE CORRELATION DIMENSION

The behaviour of neighbouring orbits is of interest for the characterization of a chaotic attractor. Consider for convenience first the case where x_t is a one-dimensional vector. Let $O^+(x_0)$ and $O^+(y_0)$ be two orbits with starting values that are close together. Let d_t be the absolute value of the difference between the point x_t and the point y_t. Using equation (1), values of d_t are obtained as

$$d_t = |F(x_{t-1}) - F(y_{t-1})| \qquad t = 1, 2, \dots. \qquad (3)$$

In order to study the behaviour of neighbouring orbits we make use of the linear approximation of equation (1). Let d_0 and d_t be small and denote the derivative of the function $F(.)$ by $DF(.)$, then one can write the linear part of a Taylor expansion of $F(y_{t-1})$ in the point x_{t-1} as

$$\tilde{F}(y_{t-1}) = F(x_{t-1}) + DF(x_{t-1})(y_{t-1} - x_{t-1}) \qquad (4)$$

Take $\tilde{F}(y_{t-1})$ as an approximation of $F(y_{t-1})$ in (3), then the evolution of d_t in linear approximation can be written as

$$d_t = |DF(x_{t-1})| d_{t-1} \qquad t = 1, 2, 3, \dots \qquad (5)$$

If the derivative $DF(x_{t-1})$ is less than one in absolute value for all t then $d_t < d_{t-1}$ and the orbit $O^+(y_0)$ will tend to the orbit $O^+(x_0)$ as t tends to become large. Using (5) repeatedly, one can write

$$d_t = |DF(x_{t-1})| \, |DF(x_{t-2})| \, \dots |DF(x_0)| d_0 \qquad (6)$$

The ratio of d_t and d_0 is the product of the derivatives of F along the orbit of x_0. Define

$$\lambda(x_0) = \lim_{t \to \infty} \frac{1}{t} \ln \frac{d_t}{d_0} \qquad (7)$$

$$= \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln |DF(x_i)|$$

The left hand side of (7), $\lambda(x_0)$, is called the *Lyapunov exponent*.[2] It is the mean value of the logarithm of the absolute value of the derivative of the nonlinear function F along the orbit $O^+(x_0)$ and it gives a measure of the degree of attractiveness of an orbit. If the Lyapunov exponent is negative, i.e. the mean value of $\ln (d_t/d_0)$ is less than zero, then the orbit is stable. If the Lyapunov is positive then the orbit is unstable and the attractor is chaotic.

For an N-dimensional system one may use the definition of d_t given in (3) and interpret it as the Euclidean distance between two vectors x_t and y_t. Given the N dimensions one has N Lyapunov exponents, just as the $N \times N$ matrix of derivatives $DF(x_{t-1})$ has N eigenvalues. Note that in the definition (7) no specific direction is used. However, on a chaotic attractor, where all orbits are unstable, an arbitrarily chosen vector will become directed in the mean most expanding direction. The algorithm for the computation of the largest exponent, due to Wolfe et al (1985), is based on this idea and accordingly we refer to equation (7) as the definition of the largest Lyapunov exponent.

A second way to measure a chaotic attractor is to make use of the notion of *dimension* of a set. The algebraic notion of a dimension is defined as the number of independent vectors necessary to describe a set. In such a case the dimension is a positive integer. In order to describe briefly how the concept of dimension is used in our context we consider, for convenience, a one-dimensional pure random system and a deterministic chaotic system. Let the random system be given as $x_{t+1} = \varepsilon_t$, where ε_t is a random variable that is uniformly distributed in the interval $[0, 1]$ and let the chaotic system be given by the logistic map $x_{t+1} = 4x_t(1 - x_t)$. If we display time series from both systems in one figure, then one will hardly see any difference between them. However, if we display the generated series in a scatter diagram, i.e., the graphs of (x_{t+1}, x_t) for both series, then the differences are obvious. In the case of the random system the whole square $[0,1] \times [0,1]$ will be filled and the system is characterized by dimension two. In the case of the chaotic sytem the points will lie on the graph of the function $x_{t+1} = 4x_t(1 - x_t)$ and the dimension is one. Suppose, as a further example, that we take the set of points (x_{t+2}, x_{t+1}, x_t), in the pure stochasic system, then the whole cube is filled while in the chaotic deterministic case on still gets a one dimensional graph. Summarizing, a pure stochastic system has as dimension the number of variables that one considers; a chaotic system has an attractor

[2] Under some general conditions the Lyapunov exponent is independent of the initial value x_0; see Brock (1986).

with dimension less than the number of variables. The dimension of a chaotic attractor may not be a natural number but a rational one, a *fractal*.

In order to measure the difference between a stochastic and a deterministic system one may consider the dimension of the attractor. A related concept is the notion of *correlation dimension*; see Grassberger and Proccacia (1983) and Brock (1986). The intuitive idea is to measure the distribution of points of some orbit on an attractor. Suppose $O^+(x_0)$ is an orbit defined on the attractor set A, a subset of $\mathbb{R}^N$. For each value x_i on the orbit we count the number of times that an other value x_j lies in an ε-neighborhood of x_i, where ε is an arbitrary positive number. That is, we check for each point x_i whether $\left| x_i - x_j \right| < \varepsilon$ for all $j = 1,...,t$. For convenience, we make use of the mathematical notation with a Heaviside function. Consider the function $H(.)$ with argument $\varepsilon - \left| x_i - x_j \right|$. Let $H(.) = 0$ if the argument $\varepsilon - \left| x_i - x_j \right|$ is nonpositive and $H(.) = 1$ if the argument is positive. Then the number of times that $\left| x_i - x_j \right| < \varepsilon$ for each value of x_i, $i = 0, 1, 2, ...$can be denoted as $\sum_{j=1}^{t} H(\varepsilon - \left| x_i - x_j \right|)$. Take the sum of this quantity over x_i, $i = 0, 1, 2,...$ with $i \neq j$ and scale the result with t^2 and define

$$C_t(\varepsilon) = \frac{\sum_{i=1}^{t} \sum_{j=1}^{t} H(\varepsilon - \left| x_i - x_j \right|)}{t^2}$$

The correlation dimension D is defined as

$$D = \limsup_{\varepsilon \to 0} \frac{\lim_{t \to \infty} \ln C_t(\varepsilon)}{\ln \varepsilon} \qquad (8)$$

The definition of the correlation dimension involves two limits. In practice, one has a finite number of observations in economic time series, which we denote , henceforth, by T. With respect to the order of magnitude of ε we note that one should experiment with different values and check the stability of the results. A reasonable procedure is to compute the correlation dimension from a regression of $\ln C_T(\varepsilon)$ on a constant and $\ln(\varepsilon)$; see, e.g., Brock (1986).

Before one can apply the Lyapunov exponent and the correlation dimension, one has to face a measurement problem. The orbit of a vector x_0 defines a theoretical time path of this set of variables. In practice, one observes a time series, denoted by a_t, $t = 1,...,T$. (Just like a stochastic process of a

variable may be different from the measured time series.) In the literature of nonlinear deterministic analysis one assumes that the observed series a_t is related in an unknown deterministic way to the orbit $O^+(x_0)$. Then the problem arises: can one reconstruct the attractor of the original system F from the signal a_t? If the time series a_t has a deterministic explanation, then one can reconstruct the original attractor as follows. Define an m-dimensional vector a_t^M as

$$a_t^M = (a_t, \ a_{t-1}, \ a_{t-2}, \ \ldots \ , a_{t-M+1}), \ t = M, \ \ldots \ , T$$

where M is called the *embedding* dimension. According to a theorem by Takens (1981), if M is greater or equal to $2N + 1$ with N the dimension of the system, the original attractor is represented by the series a_t^M. Given that in practice we do not know the dimension N of a theoretical system, how do we choose a value of the embedding dimension M? A practical procedure is to compute the correlation dimension and the Lyapunov exponent for different values of M. For chaotic processes the correlation dimension stabilizes at some value for sufficiently large embedding dimension; for stochastic processes it will increase with the embedding dimension. Similarly, at a proper embedding dimension the Lyapunov exponent stabilizes for chaotic processes. We emphasize that the Lyapunov exponent is not defined for a stochastic process since the derivative in x_t does not exist; see equation (4). A priori one does not know whether a time series is stochastic or chaotic and one may yet report the Lyapunov exponent but the results have to be interpreted with care.

IV. Results

Our data series are monthly observations on the natural logarithm of dollar/yen real exchange rate for the period December 1972 to June 1988. The data are referred to as JPUS and are shown in Figure 1 (top level). For more details on the data we refer to Schotman and Van Dijk (1990) who fitted a linear autoregressive model of order one, $x_t = \alpha x_{t-1} + \varepsilon_t$ and obtained as least squares estimate for the autoregressive parameter α a value of 0.982 (s.e. 0.014). This motivated us to consider some experiments where the data are generated according to a linear autoregressive process of order one with disturbances that are either white noise or generated according to the logistic map $4\varepsilon_t(1 - \varepsilon_t)$, which is chaotic. More specifically, consider models A and B with
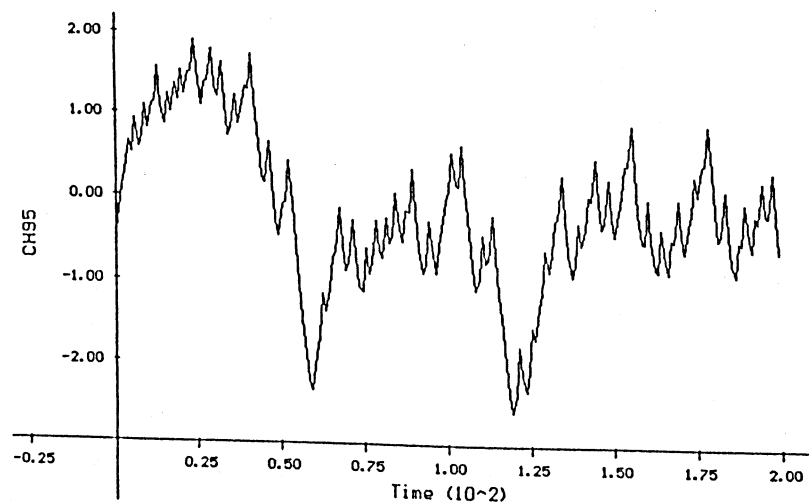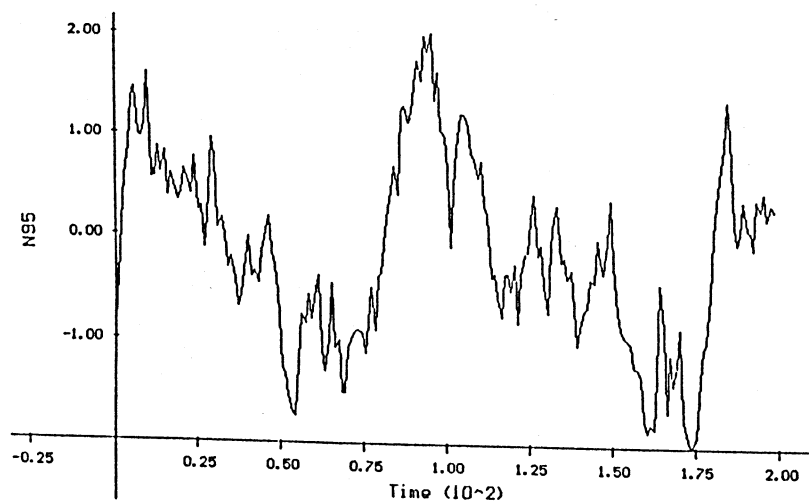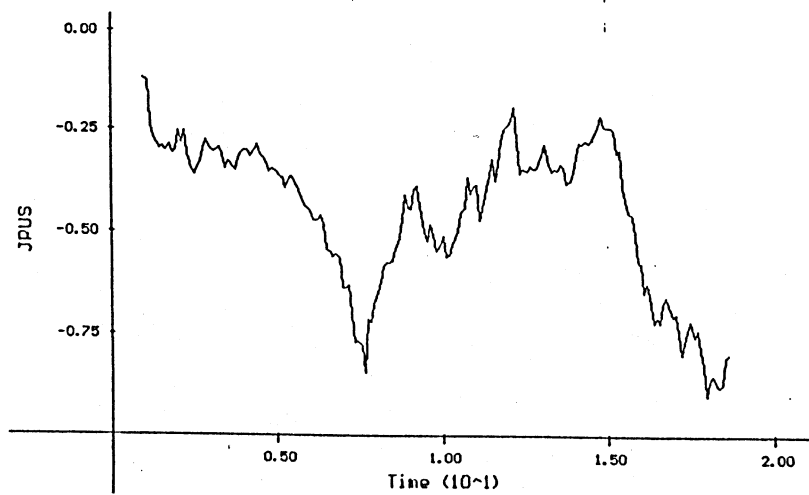
Figure 1. Data series

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$$A: x_t = \alpha x_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2) \qquad t=1,2,3,\ldots$$

$$B: x_t = \alpha x_{t-1} + (\varepsilon_t - 0.5), \quad \varepsilon_t = 4\varepsilon_t(1 - \varepsilon_t), \quad t=1,2,3,\ldots$$

The series generated according to the logistic map has mean 0.5 and variance 0.125; see Holden (1987) and we took in model A for σ^2 the same variance. For both models we generated a time series of 200 observations ($x_0 = 0$, $\varepsilon_0 = 0.2$, and $\alpha = 0.95$). The series are shown in the middle and the bottom of Figure 1 and are denoted by $N95$ and $CH95$, respectively.

In Figure 2 scatter diagrams of (x_t, x_{t+1}) $t = 1,\ldots,199$ are shown on the left hand side, while scatter diagrams of the residuals from a least squares regression of x_{t+1} on x_t are shown on the right hand side. A graphical analysis indicates that the data series of the real exchange rate are to some extent comparable to the data series from *both* models A and B. The analysis of the residuals casts, however, strong doubts on this. The residuals are more in accordance with white noise than with chaotic disturbances.

A less superficial analysis involves the computation of the largest Lyapunov exponent and the correlation dimension for embedding dimension 1 to 10. (For the simulated series from model B the correct embedding dimension is 3.) The results are shown in Tables 1 and 2 and in Figure 3. Apart from the three series mentioned before and the residuals of the series we added a pure chaotic process and a pure white noise process. That is, we consider as two separate cases the true disturbance processes of models A and B, denoted by N and CH, respectively. Thus we have a total of 8 series.

We note that the results on the Lyapunov exponent are scaled by $\ln(2)$ since the theoretical value of this exponent is $\ln(2)$ for the series CH, the pure chaotic one. From the results, reported in the second column of Table 1, one may conclude that the Lyapunov exponent can find a pure chaotic process for time series with only 200 observations. For the pure white noise process (see column denoted N) one finds at dimension one a large value of the Lyapunov exponent, which tends to zero at a fast rate when the embedding dimension becomes larger. The same holds for the columns $CH95$ and $N95$. It appears that the Lyapunov exponent is not a useful measure in a mixed process with only 200 observations (Maybe this result is due to the way the Lyapunov exponent is computed. This is a topic outside the scope of the present paper). If one compares the results of columns CH95R and N95R with those of columns CH and N, it is seen that the use of the least squares regression filter is helpful in discriminating between a chaotic and a stochastic

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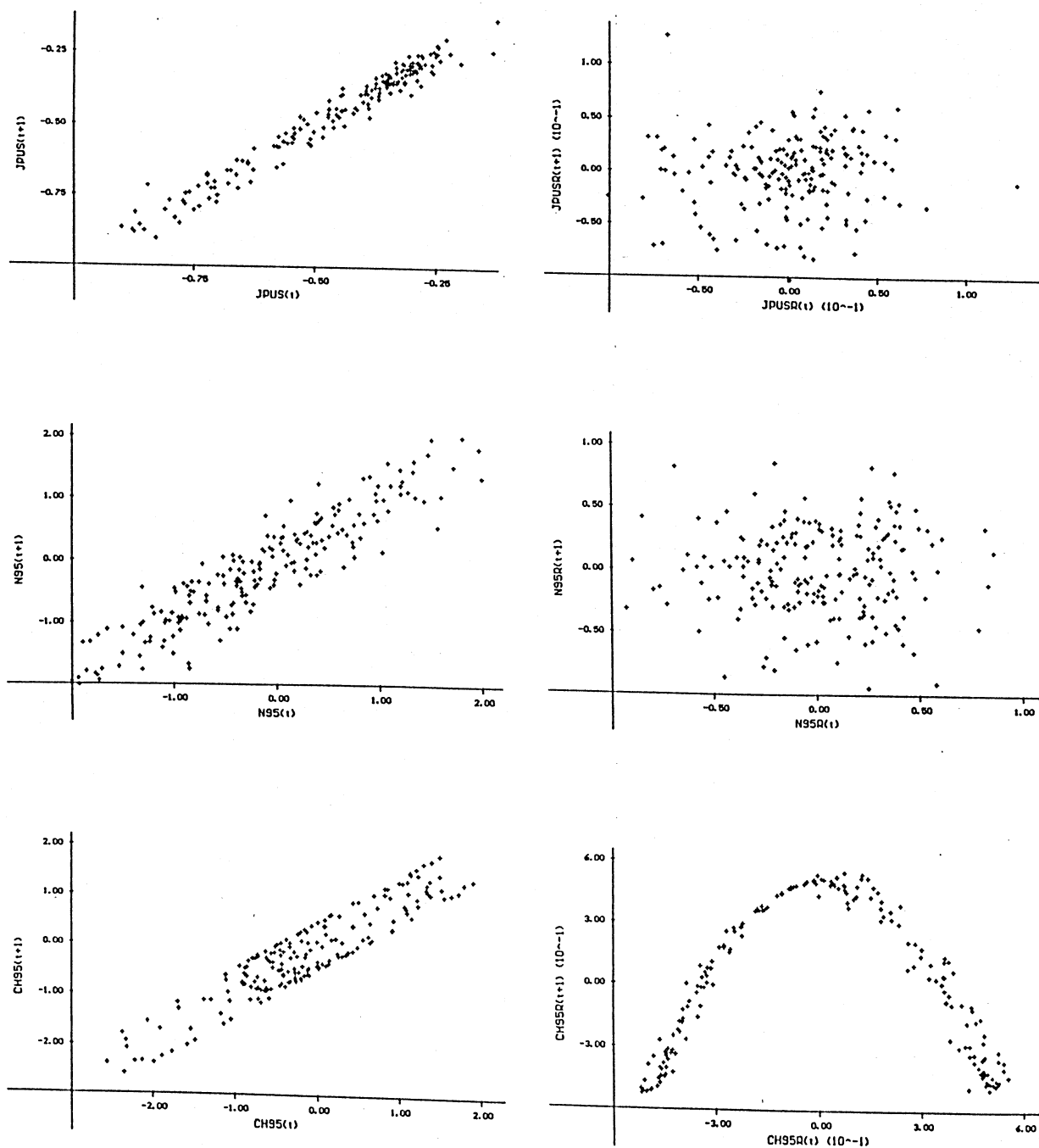
Data

Residuals

Figure 2. Scatter diagrams

process. However, the results of columns CH95 and CH95R are rather different. This is contrary to Brock's (asymptotic) analysis; see Brock (1986). Whether our results are only due to a small sample effect is a topic for further research. The results of the columns JPUS and JPUSR indicate that the Lyapunov decreases at relatively fast but different rates. This difference is again contrary to Brock's asymptotic analysis. Finally we note that the Lyapunov exponent is always positive.

Table 1

Largest Lyapunov exponent

dim.	CH	CH95R	CH95	JPUSR	JPUS	N95	N95R	N
1	0.98	1.55	2.46	4.31	1.21	3.19	5.19	5.08
2	0.96	0.97	0.61	0.80	0.31	0.73	1.21	1.40
3	0.97	0.90	0.46	0.56	0.23	0.47	0.77	0.71
4	0.88	0.85	0.38	0.38	0.21	0.34	0.57	0.55
5	0.93	0.82	0.28	0.34	0.17	0.26	0.42	0.46
6	0.91	0.74	0.29	0.25	0.15	0.22	0.31	0.32
7	0.84	0.62	0.19	0.18	0.14	0.17	0.21	0.22
8	0.75	0.49	0.18	0.13	0.13	0.15	0.12	0.15
9	0.67	0.35	0.14	0.10	0.12	0.14	0.09	0.12
10	0.47	0.31	0.11	0.07	0.09	0.12	0.06	0.08

Table 2

Correlation dimension

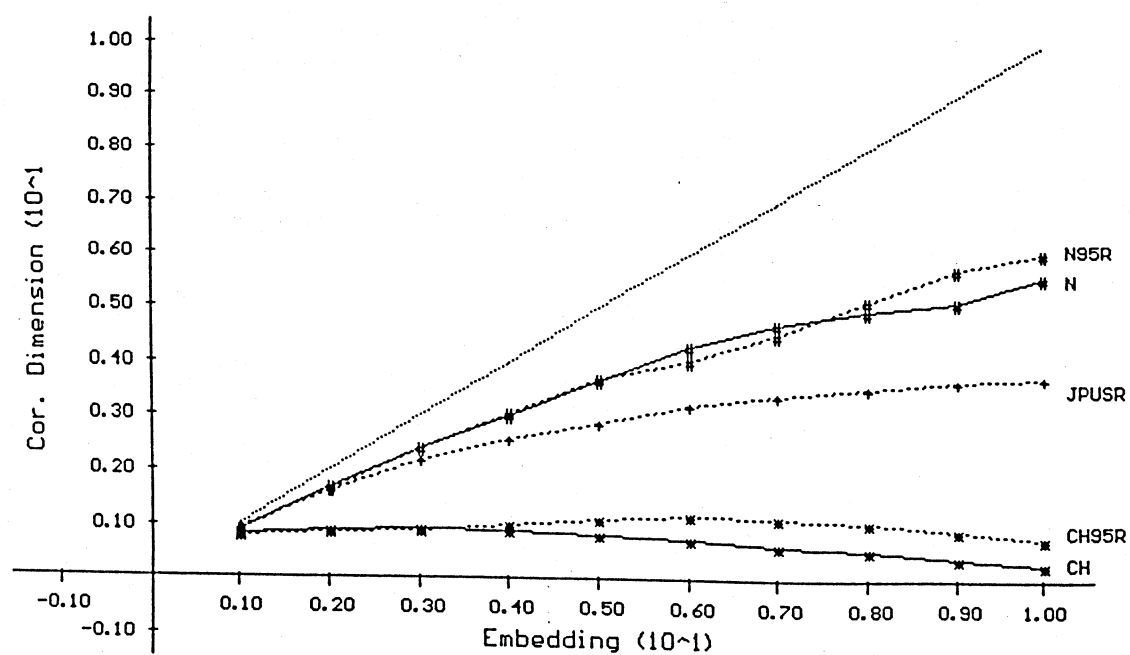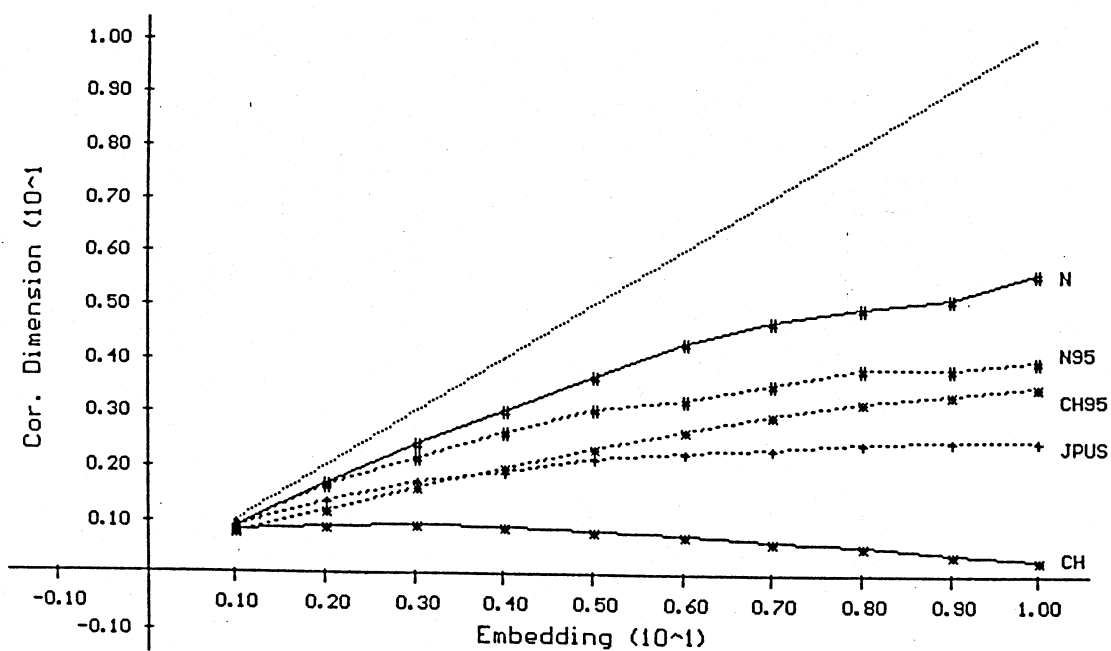| dim.	CH	CH95R	CH95	JPUSR	JPUS	N95	N95R	N
1	0.83	0.77	0.79	0.92	0.88	0.89	0.90	0.87
2	0.90	0.84	1.18	1.61	1.37	1.64	1.64	1.67
3	0.93	0.89	1.62	2.17	1.72	2.14	2.40	2.40
4	0.88	1.00	1.96	2.57	1.91	2.61	3.04	3.01
5	0.83	1.09	2.33	2.88	2.15	3.05	3.69	3.65
6	0.77	1.17	2.67	3.18	2.26	3.23	4.03	4.28
7	0.65	1.15	2.96	3.39	2.34	3.51	4.49	4.66
8	0.61	1.08	3.18	3.53	2.44	3.81	5.12	4.94
9	0.52	0.93	3.35	3.68	2.48	3.82	5.71	5.11
10	0.41	0.78	3.50	3.74	2.47	3.95	6.04	5.59

Figure 3. Graphs of the correlation dimension of data (upper figure) and residuals (lower figure).

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