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A BAYESIAN LEARNING PROCEDURE FOR THE
(s,Q) INVENTORY POLICY

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A BAYESIAN LEARNING PROCEDURE FOR THE (s,Q) INVENTORY POLICY.

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Abstract.

We present an asymptotically optimal Bayesian learning procedure for the (s,Q) inventory policy, for the case when the probability distribution of lead time demand is unknown. This distribution is not required to be a member of a certain family, and the maximal lead time demand is also allowed to be unknown.

The algorithm developed for this purpose is an extension of a standard iterative procedure, which in its original form -in spite of claims to the contrary- might produce solution values that are arbitrarily far away from the optimal one.

1. INTRODUCTION.

Consider the classical continuous review inventory model for which the procurement lead time L is nonzero and fixed. The cost of ordering y units of product is equal to $C+cy$, and the cost of holding y units in inventory is equal to hy per time unit. Demand occurring when the inventory level is zero is backlogged and eventually filled. The penalty cost incurred in such situations is stationary and determined by the size of the backlog just before the replenishment order arrives; it

is equal to π per unit of backlog. Let \underline{I} denote the total (integer) demand arising in a period of length L , and let us assume that \underline{I} is a random variable with a stationary probability distribution i.e., the probability of the event $\{\underline{I}=i\}$ is equal to

$$(1) \quad p(i) = \theta_i \quad (\theta_i > 0; i=0, \dots, m; \sum_{i=0}^m \theta_i = 1).$$

Then, if Q units are ordered at the moment that the inventory level drops to a reorder point s , the expected average cost per time unit is given in [Wagner 1975, Section 19.6] as

$$(2) \quad E(s, Q) = \frac{hQ}{2} + hs + \left(\frac{C}{QL} + \frac{c}{L} - h \right) \psi + \left(\frac{h}{2Q} + \frac{\pi}{QL} \right) \omega(s)$$

with

$$(3) \quad \psi = \sum_{i=0}^m i \theta_i,$$

$$(4) \quad \omega(s) = \sum_{i=0}^m \sum_{j=s}^m i(j-s) \theta_i \theta_j.$$

We are interested in values for s and Q that minimize $E(s, Q)$. A simple iterative procedure developed for this purpose, to which we return below, can be traced to [Hadley and Whitin 1963].

The standard model presented above suffers from a serious difficulty: in many practical situations the probability distribution of the lead time demand quantities is unknown. This is particularly true in the case of slow moving items with limited sales information. In [Silver 1965], a Bayesian estimation procedure is applied to cope with such situations. Under the assumption that the maximal demand m (i.e., the domain of the distribution) is known, the unknown probabilities $\theta_0, \dots, \theta_m$ are themselves regarded as random variables $\underline{\theta}_0, \dots, \underline{\theta}_m$, a priori taken to be distributed according to a Dirichlet distribution with parameters $\alpha_0, \dots, \alpha_m$. Under the additional assumption that demand quantities in different lead time periods are stochastically independent, it is a well known probabilistic result that, for n lead time demand observations, the number of times that demand is equal to i ($i=0, \dots, m$) follows a multinomial distribution with parameters $\theta_0, \dots, \theta_m$. Given a sample

(n_0, \dots, n_m) of n lead time observations ($\sum n_i = n$), the posterior expected value of θ_i is equal to

$$(5) \quad E(\theta_i | (n_0, \dots, n_m)) = \frac{n_i + \alpha_i}{n + \alpha(m)}, \quad \text{where } \alpha(m) = \sum_{i=0}^m \alpha_i.$$

In [Silver 1965] these expected posterior probabilities are used to select a reorder point s guaranteeing that the expected posterior probability of stock out is less than a prespecified value. The reorder quantity Q is then determined separately by balancing the ordering costs against the costs of carrying the cycle. Since the choice of reorder point is not based on cost considerations, there is no reason to assume that this procedure minimizes the expected average cost, as would be desirable.

In this paper we develop a Bayesian procedure that truly minimizes posterior expected average costs and that applies to an extension of the original model in which uncertainty is also allowed about the domain of the unknown probability distribution i.e., about the maximal demand m . Thus, in our procedure the user can specify a prior distribution for the maximal demand \underline{M} , and, for each realization m of \underline{M} , a prior must be specified for the probabilities, denoted as $\theta_0(m), \dots, \theta_m(m)$. Then, given the observed demand quantities n_0, \dots, n_m , the expected value of the average cost $E(s, Q)$ (2) is computed with respect to the posterior distribution of the unknowns. Next, we compute the values of s and Q which minimize this posterior expected average cost per time unit.

We note that if m is not treated as a random variable but fixed a priori, then its specified value plays an important role in the analysis. If it is chosen too large, then inventory will be kept needlessly high, and the computation of the optimal policy will be unnecessarily time consuming. If, on the other hand, m is chosen smaller than the true maximal demand, then with positive probability demand quantities greater than m will be observed, which cannot be used for updating the posterior distribution, and which show that the underlying model is incorrect.

We do acknowledge that the posterior expected average cost is a function of s and Q which is updated as the result of newly observed lead time

demands, while formula (2) is correct only when s and Q remain fixed. However, the posterior distribution is asymptotically equal to the true values $m, \theta_0(m), \dots, \theta_m(m)$ with probability 1. Hence, when the number of observed lead time demands n grows, the values of s and Q which minimize the posterior expected average cost, will converge to their truly optimal values.

Finally, we recognize the fact that [Wagner 1975, Section 19.6], the assumptions underlying formula (2) of the average cost per time unit logically imply that the lead time demand distribution must be Poisson. However, according to Wagner "this is much too restrictive a conclusion in reality" and the above model "is often an excellent approximation".

Since the functional form of the posterior expectation of $E(s, Q)$ is similar to formula (2), it is tempting to use the traditional procedure referred to above to seek the values for s and Q that minimize this cost function. However, we shall see that this procedure is not as reliable as has been assumed so far in the literature. Even for the original model, it can produce values for s and Q that are arbitrarily far away from the optimal ones, as indicated by a simple numerical example. Closer inspection reveals that the iterative procedure is no more than a local search procedure that always produces a pair (s, Q) satisfying local, but not necessarily global, first order conditions for optimality. We shall see how systematic application of this procedure to appropriate starting points can produce globally optimal values for s and Q with respect to the original model and the extension considered here.

The outline of this paper is as follows. In Section 2 we derive the formula of the posterior expected average cost of an (s, Q) policy. In Section 3 we develop the extended procedure to find the (s, Q) policy which globally minimizes posterior expected average cost. Section 4 concludes the paper with some computational experience.

2. THE POSTERIOR EXPECTED AVERAGE COST.

Our statistical model for lead time demand is an arbitrary discrete probability distribution i.e., the probability that lead time demand is equal to i ($i=0, \dots, m$) is given by

$$(6) \quad p(i) = \theta_i(m) \quad (\theta_i(m) > 0; i=0, \dots, m; \sum_{i=0}^m \theta_i(m) = 1).$$

In the Bayesian analysis the unknowns $m, \theta_0(m), \dots, \theta_m(m)$ are themselves assumed to be random variables for which a prior distribution can be specified. For the maximum lead time demand M the user may specify an arbitrary discrete prior distribution i.e., the probability that $M=m$ is equal to an arbitrary constant τ_m satisfying $\sum \tau_m = 1$. For each realization m of M the probabilities $\theta_0(m), \dots, \theta_m(m)$ are assumed to follow a Dirichlet distribution with parameters $\alpha_0(m), \dots, \alpha_m(m)$ ($\alpha_0(m) + \dots + \alpha_m(m) = \alpha(m)$). Note that this implies that the prior expected value and variance of the demand probability $\theta_i(m)$ are respectively equal to $\alpha_i(m)/\alpha(m)$ and $[\alpha_i(m)/\alpha(m)] \cdot \{[\alpha_i(m)+1]/[\alpha(m)+1] - \alpha_i(m)/\alpha(m)\}$ (see e.g. [Wilks 1962]). In the sequel we assume that $\alpha_i(m) = \alpha_i$ for all possible values of m and all $i \leq m$. Note that this assumption for all i and j implies that the ratio of the expected value of $\theta_i(m)$ and $\theta_j(m)$ is independent of m .

Next, suppose a sample (n_0, \dots, n_m) of n independent lead time observations is taken, where n_i is the number of times that the value i is sampled. It is well known that the sampling distribution is multinomial for a given value of m i.e., the probability that the sample (n_0, \dots, n_m) occurs is equal to

$$(7) \quad n! \cdot \prod_{i=0}^m \theta_i(m)^{n_i} / n_i! = n! \cdot \prod_{i=0}^{\max} \theta_i(m)^{n_i} / n_i!$$

where \max is the maximum observed lead time demand in the sample (that is, $n_{\max+1} = \dots = n_m = 0$). Due to the latter equality we denote the sample (n_0, \dots, n_m) in the sequel as (n_0, \dots, n_{\max}) .

Given the above sampling distribution and the prior distributions for the unknown maximum and probabilities, it follows easily from Bayes Theorem that the posterior distribution of the unknowns is equal to:

$$(8) \quad \Pr\{ \underline{M}=m, (\underline{\theta}_0(m) \in d\theta_0, \dots, \underline{\theta}_m(m) \in d\theta_m(m)) \mid (n_0, \dots, n_{\max}) \}$$

$$\propto \tau_m \cdot [\alpha(m)-1]! \cdot \prod_{i=0}^{\max} \theta_i(m)^{n_i} \cdot \prod_{j=0}^m \theta_j(m)^{\alpha_j-1} / (\alpha_j-1)!,$$

where α denotes proportionality.

Given the sample and the prior distribution for the unknowns, we now wish to compute the policy (s, Q) which minimizes the expected value of the average cost per time unit given by (2), with respect to the posterior distribution (8).

The two terms in (2) involving the unknowns are the random variables

$$(9) \quad \underline{\Psi} = \sum_{i=0}^{\underline{M}} i \underline{\theta}_i(\underline{M}) \quad \text{and} \quad \underline{\Omega}(s) = \sum_{i=0}^{\underline{M}} \sum_{j=s}^{\underline{M}} i(j-s) \underline{\theta}_i(\underline{M}) \cdot \underline{\theta}_j(\underline{M}).$$

We recall that the expected value of a sum of, not necessarily independent, random variables is equal to the sum of the expected values. Hence, the expectation of $E(s, Q)$ with respect to the posterior distribution (8) is given by

$$(10) \quad E(\underline{E}(s, Q) \mid (n_0, \dots, n_{\max})) = \frac{hQ}{2} + hs \\ + \left(\frac{C}{QL} + \frac{C}{L} - h \right) E(\underline{\Psi} \mid (n_0, \dots, n_{\max})) \\ + \left(\frac{h}{2Q} + \frac{\pi}{QL} \right) E(\underline{\Omega}(s) \mid (n_0, \dots, n_{\max})).$$

Hence, the posterior expected average cost per time unit can be obtained by substituting in (10) the posterior expected value of the random variables defined in (9). In [Boender et al 1985] it is shown that these posterior expectations equal respectively to

$$(11) \quad E(\underline{\Psi} \mid (n_0, \dots, n_{\max})) = \\ = \sum_{i=0}^{\infty} \sum_{m=\max\{i, \max\}}^{\infty} \frac{i[n_i + \alpha_i]}{n + \alpha(m)} \cdot \sum_{i=\max}^{\infty} \frac{\tau_m[\alpha(m)-1]! / [n + \alpha(m)-1]!}{\tau_i[\alpha(i)-1]! / [n + \alpha(i)-1]!}.$$

$$(12) \quad E(\underline{\Omega}(s) \mid (n_0, \dots, n_{\max})) =$$

$$= \sum_{i=0}^{\infty} \sum_{\substack{j=s \\ j \neq i}}^{\infty} \sum_{m=\max\{i, j, \max\}}^{\infty} \frac{i[j-s][n_i+\alpha_i][n_j+\alpha_j]}{[n+\alpha(m)][n+\alpha(m)+1]} \cdot \frac{\tau_t[\alpha(m)-1]!/[n+\alpha(m)-1]!}{\sum_{i=\max}^{\infty} \tau_i[\alpha(i)-1]!/[n+\alpha(i)-1]!}$$

$$+ \sum_{i=s}^{\infty} \sum_{m=\max\{i, \max\}}^{\infty} \frac{i[i-s][n_i+\alpha_i][n_i+\alpha_i+1]}{[n+\alpha(m)][n+\alpha(m)+1]} \cdot \frac{\tau_t[\alpha(m)-1]!/[n+\alpha(m)-1]!}{\sum_{i=\max}^{\infty} \tau_i[\alpha(i)-1]!/[n+\alpha(i)-1]!}$$

As a special case we may choose the prior to be non-informative i.e, for the maximum value each positive integer is regarded equiprobable ($\tau_m=1$ for all nonnegative m), and given $\underline{M}=m$ the probabilities $\theta_0(m), \dots, \theta_m(m)$ are uniformly distributed on the unit simplex in R^m ($\alpha_i=1$; $i=0, \dots, m$). Then

$$(13) \quad E(\underline{\Psi} \mid (n_0, \dots, n_{\max})) =$$

$$= \frac{[n-1][\max+1][\max+2] - [n-2][\max+1] + [n-2][n-1] \sum_{i=0}^{\max} i[n_i+1]}{[n-2]n[n+\max]}$$

For $s \leq \max$:

$$(14) \quad E(\underline{\Omega}(s) \mid (n_0, \dots, n_{\max})) = \frac{[n-1][\max+1][\max+2]}{[n-1][n+\max][n+\max+1]}$$

$$\cdot \left\{ \frac{\sum_{i=0}^{\max} \sum_{\substack{j=s \\ j \neq i}}^{\max} i[j-s][n_i+1][n_j+1]}{[\max+1][\max+2]} + \frac{\sum_{i=s}^{\max} i[i-s][n_i+1][n_i+2]}{[\max+1][\max+2]} \right.$$

$$- \frac{[s+1] \sum_{i=0}^{\max} i[n_i+1]}{n[\max+2]} + \frac{\sum_{i=0}^{\max} i[n_i+1]}{n-1} + \frac{\sum_{i=s}^{\max} [i-s][n_i+1]}{n-1}$$

$$- \frac{\sum_{i=s}^{\max} [i-s][n_i+1]}{n[\max+2]} + \frac{[\max+3][\max+4]}{[n-1][n-3]} - \frac{[s+3][\max+3]}{[n-1][n-2]}$$

$$- \frac{[\max+3]}{n[n-2]} + \frac{s+2}{n[n-1]} + \frac{2[\max+3]}{n-2} - \frac{2[s+3]}{n-1} + \frac{2[s+1]}{n[\max+2]}$$

$$+ \frac{[\max+3][\max+4]}{[n-1][n-3]} - \frac{3[\max+3]}{[n-1][n-2]} - \frac{[s+1][\max+3]}{n[n-2]} + \frac{2[s+1]}{n[n-1]} \Big\}$$

and, for $s > \max$:

$$\begin{aligned}
 (15) \quad E(\underline{Q}(s) \mid (n_0, \dots, n_{\max})) &= \\
 &= \frac{1}{[n+1]n[n-2][n-3]} \cdot \frac{[\max+n-1]![s+1]!}{\max![n+s+1]!} \cdot \\
 &\quad \cdot \{ [n/2][n+1]s^3 + [n^3/2+7n^2/2+2n+3]s^2 \\
 &\quad + [5n^3/2+6n^2+21n/2+3+[n-2][n-3] \sum_{i=0}^{\max} in_i]s \\
 &\quad + [2n[n+1][n+3] + [n+1][n-2][n-3] \sum_{i=0}^{\max} in_i] \}.
 \end{aligned}$$

3. THE EXTENDED ITERATIVE PROCEDURE.

An iterative procedure to minimize the average cost $E(s, Q)$ given in (2) was originally proposed in [Hadley and Whitin 1963]. The procedure is based on first order conditions that can be obtained for s and Q , under the assumption that Q can be treated as a continuous variable. Since it can be shown that the optimal integer value for Q is always the integer rounddown or roundup of the optimal continuous value (cf. [de Wit 1983]), this assumption is not a serious one.

Based on that assumption we find that an optimal (s, Q) policy must satisfy the first order conditions:

$$(16) \quad Q = Q^*(s) = \left\{ \frac{2C}{hL} \psi + \left[1 + \frac{2\pi}{hL} \right] \omega(s) \right\}^{1/2};$$

$$(17) \quad s = s^*(Q) = \min \{ f \mid f \in \mathbb{N}^+, \omega(f+1) - \omega(f) < \frac{-hQ}{h/2 + \pi/L} \}.$$

The following iterative procedure is now a natural one to consider (cf. [Wagner 1975], Section 19.6):

Step 1. $s^{(1)} = \infty$; $Q^{(1)} = Q^*(s^{(1)})$; $k=1$.

Step 2. IF $s^{(k)} = s^*(Q^{(k)})$ STOP.

Step 3 Set $s^{(k+1)} = s^*(Q^{(k)})$; $Q^{(k+1)} = Q^*(s^{(k+1)})$; $k=k+1$; repeat Step 2.

It is not difficult to prove by induction that $Q^{(k+1)} \geq Q^{(k)}$ and that

$s(k+1) \leq s(k) - 1$, so that the successive values for s form a strictly decreasing sequence of integers which terminates with the first pair $(s(k), Q(k))$ encountered such that $s(k) = s^*(Q(k))$ and $Q(k) = Q^*(s(k))$; since $s^*(Q) = 1$ for all Q sufficiently large, this must occur in a finite number of steps.

We now consider the important question if the necessary conditions (16) and (17) are sufficient for global optimality. In [Wagner 1975], Section 19.6, one finds that "the terminating value for s may not be globally optimal; at worst, a neighboring value of s and the associated optimal Q may have lower expected cost". Unfortunately, the situation can be much worse.

Consider Figure 1, illustrating the curves defined by (16) and (17) for the following choice of parameters: $h=1$; $L=1$; $C=2.3075$; $c=0$; $(\theta_0, \dots, \theta_9) = (0.23, 0.12, 0.23, 0.0125, 0.0125, 0.12, 0.0125, 0.0125, 0.12, 0.13)$. In this example the first order conditions are seen to be simultaneously satisfied for three (s, Q) policies, corresponding to $s=2$, $s=5$, and $s=8$. The above procedure will produce the value $s=8$. However, the global minimum arises for $s=5$.

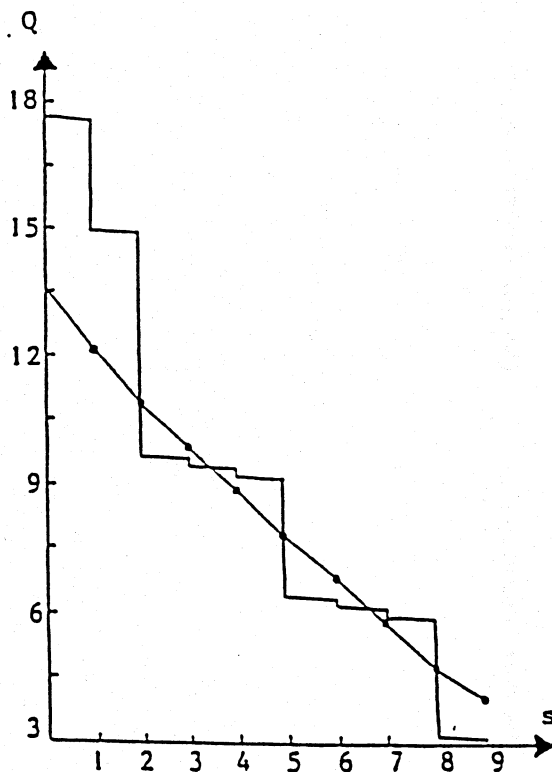


Figure 1

It follows that the iterative procedure can be viewed as no more than a local search procedure, arriving at a local, not necessarily global optimum. To find a global optimum, the original procedure has to be applied repeatedly.

Let us denote the original iterative procedure, with initial value s by $A(s)$. We have seen that for the original model $A(\infty)$ converges to a local optimum (s^u, Q^u) that is not necessarily global. We do know, however, that the globally optimal value s^* will not be larger than s^u . Similarly, we can bound s^* from below by applying $A(0)$; it is easy to verify that starting at this s value, an increasing sequence of s values will be generated that terminates in a local optimum (s^l, Q^l) in a finite number of steps. If $s^l = s^u$ or $s^l = s^u - 1$, we are finished; if not, the interval $[s^{l+1}, s^u - 1]$ is systematically explored for other local optima by restarting the algorithm A in a yet unexplored neighbour of one of the observed local optima.

It should be obvious that such an extended procedure terminates with the correct global minimum value in a finite number of steps. In practice, the algorithm will skip over a large number of irrelevant s -values and will require only slightly more computational effort than the original procedure.

Finally, we observe that the posterior expected average cost function (10) is of the same form as the original average cost function (2) (for details we refer again to [Boender et al 1985]). Hence, when the above extended iterative procedure is applied to the posterior expected average cost function (10), it will arrive at the globally optimal Bayesian (s, Q) policy in a finite number of steps.

4. COMPUTATIONAL EXPERIENCE.

Throughout this section we consider an example which is proposed in [Wagner 1975], Section 19.6, for which $h=1$, $\pi=9$, $C=32$, $c=0$ and $L=1$, and with lead time demand distribution where each integer on the interval $[0,25]$ is equiprobable.

Then the policy which globally minimize the average cost per time unit given in (2) is equal to $(s^*, Q^*) = (18, 32.5)$ with $E(s^*, Q^*) = 37.992$. The neighbouring values of the global optimum are depicted in the table below.

s	$Q^*(s)$	$E(s, Q^*(s))$
16	34.8	38.300
17	33.6	38.098
18	32.5	37.992
19	31.5	37.993
20	30.6	38.110
21	29.8	38.355

Table 1

We simulated the behaviour of a user applying our Bayesian learning procedure, while observing lead time demands generated by the unknown distribution, and assuming the non-informative prior distribution i.e., for maximal demand \underline{M} each nonnegative integer is equiprobable and given $\underline{M}=m$, for the probabilities $(\theta_0, \dots, \theta_m)$ each possible combination has equal density. We proceeded by computing the optimal Bayesian policy for 25 sequences of 100 subsequent observations from the true lead time demand distribution. In the table below we depicted the average results and standard deviations of the 25 trials for different values of the number n of lead time demand observations. The column $E(s, Q^*(s))$ refers to the expected cost of the optimal Bayesian policies with respect to the true lead time demand distribution.

n	Average s^*	Standard- deviation	Average Q^*	Standard- deviation	Average $E(s^*, Q^*)$	Standard- deviation
4	18.600	3.359	57.637	7.633	44.708	3.443
5	19.400	3.085	42.109	4.292	40.355	1.751
6	19.520	2.670	37.427	2.765	39.223	1.170
7	19.360	3.097	35.403	2.202	39.095	0.760
8	18.920	2.869	34.140	2.294	38.853	0.698
9	18.640	3.135	33.297	2.084	38.906	0.869
10	18.680	2.894	32.997	1.945	38.790	0.999
11	18.720	2.807	32.539	1.672	38.695	0.959
12	18.560	2.772	32.392	1.525	38.674	1.154
13	18.800	2.117	32.137	1.624	38.426	0.429
14	18.760	1.945	32.189	1.392	38.337	0.363
15	18.520	1.962	32.106	1.328	38.337	0.354
16	18.320	1.891	32.089	1.479	38.328	0.391
17	18.280	1.755	32.143	1.306	38.270	0.360
18	18.320	1.690	31.955	1.442	38.256	0.348
19	18.560	1.551	31.574	1.491	38.228	0.350
20	18.520	1.473	31.457	1.561	38.213	0.342
30	18.640	1.694	31.332	1.323	38.239	0.298
40	18.440	1.388	31.753	0.843	38.138	0.249
50	18.720	1.457	31.738	0.746	38.129	0.179
60	18.760	1.365	31.678	0.675	38.112	0.173
70	18.640	1.382	31.758	0.718	38.112	0.174
80	18.560	1.098	31.760	0.667	38.070	0.147
90	18.520	1.204	31.834	0.723	38.082	0.150
100	18.600	1.200	31.735	0.667	38.083	0.153

Table 2

Taking into account that the cost function is extremely flat in the neighbourhood of the globally optimal policy (cf. Table 1), we conclude from Table 2 that the Bayesian policy can properly be applied to determine the (s, Q) policy in situations where the lead time demand distribution is not (yet) known. A final observation which supports this recommendation is that asymptotically the posterior distribution of the the probabilities $\theta_0, \dots, \theta_m$ degenerates at the true value of these quantities, so that for n sufficiently large, the Bayesian policy will equal the truly optimal policy with probability 1.

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